1. *$\mathscr{F}$ is anti-monotonic.*
2. *The Barcan formula is valid in every model based on $\mathscr{F}$.*
3. $\Diamond \mathsf{E}(x) \supset \mathsf{E}(x)$ *is valid in every normal model based on $\mathscr{F}$.*

Exercises

Exercise 12.8.1 Show the equivalence of items 1 and 3 in Theorem 12.8.1.

Exercise 12.8.2 Give a varying domain tableau proof of the following.

$$(\forall x)\Box\mathsf{E}(x) \supset [\Box(\forall x)\Phi(x) \supset (\forall x)\Box\Phi(x)]$$

Exercise 12.8.3 Show the equivalence of items 1 and 3 in Theorem 12.8.2.

12.9 Using Validities in Tableaus

Almost all tableau proofs we have given have been direct—they did not make use of assumptions. In Section 7.3 we did discuss how to use local and global assumptions in propositional tableau proofs, but for the most part this has played little role so far. Now such ideas become particularly useful because, as we have seen in the previous section, there are formulas that capture monotonicity and anti-monotonicity, and these are important semantic notions. We only need *global* assumptions in tableaus now, but we will need to make use of open formulas, which hitherto have played no role in tableaus. For convenience, we start from the beginning here—review of Section 7.3 would be nice, but is not necessary.

Definition 12.9.1 (Using a Closed Global Assumption) *To use a* closed *formula Φ as a global assumption in a tableau proof the rule is: $\sigma\,\Phi$ can be added to any branch, for any prefix σ that occurs on the branch.*

This means we can have a version of tableaus incorporating *monotonicity* by using the varying domain rules and taking $(\forall x)\Box\mathsf{E}(x)$ as a global assumption.

Example 12.9.2 Here is a tableau proof, using the varying domain **K** rules, of $\Box(\exists x)\Diamond A(x) \supset \Box\Diamond(\exists x)A(x)$, *taking* $(\forall x)\Box\mathsf{E}(x)$ *as a global assumption*. Thus in effect it is a monotonic proof of the formula.

$$
\begin{array}{lll}
1 & \neg[\Box(\exists x)\Diamond A(x) \supset \Box\Diamond(\exists x)A(x)] & 1. \\
1 & \Box(\exists x)\Diamond A(x) & 2. \\
1 & \neg\Box\Diamond(\exists x)A(x) & 3. \\
1.1 & \neg\Diamond(\exists x)A(x) & 4. \\
1.1 & (\exists x)\Diamond A(x) & 5. \\
1.1 & \Diamond A(p_{1.1}) & 6. \\
1.1.1 & A(p_{1.1}) & 7. \\
1.1.1 & \neg(\exists x)A(x) & 8. \\
1.1 & (\forall x)\Box \mathsf{E}(x) & 9. \\
1.1 & \Box \mathsf{E}(p_{1.1}) & 10. \\
1.1.1 & \mathsf{E}(p_{1.1}) & 11. \\
1.1.1 & \neg A(p_{1.1}) & 12.
\end{array}
$$

Items 2 and 3 are from 1 by a Conjunctive Rule; 4 is from 3 by a Possibility Rule; 5 is from 2 by a Necessity Rule; 6 is from 5 by an Existential Rule; 7 is from 6 by a Possibility Rule; 8 is from 4 by a Necessity Rule; 9 is our global assumption; 10 is from 9 by a Universal Rule; 11 is from 10 by a Necessity Rule; and 12 is from 8 and 11 by the derived Parameter Existence Rule, Definition 12.2.2.

Using an *open* formula as a global assumption is almost as easy. Remember, a free variable can represent anything in the domain of the model.

Definition 12.9.3 (Using an Open Global Assumption) *To use an open formula $\Phi(x)$ as a global assumption in a tableau proof the rule is: $\sigma\,\Phi(p_\tau)$ can be added to any branch, for any prefix σ and any parameter p_τ that occur on the branch.*

Now we can get the effect of anti-monotonicity by using the varying domain rules and taking $\Diamond \mathsf{E}(x) \supset \mathsf{E}(x)$ as an assumption.

Example 12.9.4 Here is a tableau proof, using the varying domain **K** rules, of an instance of the Barcan formula, $(\forall x)\Box A(x) \supset \Box(\forall x)A(x)$, taking $\Diamond \mathsf{E}(x) \supset \mathsf{E}(x)$ as an open global assumption. In effect, it is a tableau verification of part of Theorem 12.8.2.

$$
\begin{array}{lll}
1 & \neg[(\forall x)\Box A(x) \supset \Box(\forall x)A(x)] & 1. \\
1 & (\forall x)\Box A(x) & 2. \\
1 & \neg\Box(\forall x)A(x) & 3. \\
1.1 & \neg(\forall x)A(x) & 4. \\
1.1 & \neg A(p_{1.1}) & 5. \\
1 & \Diamond \mathsf{E}(p_{1.1}) \supset \mathsf{E}(p_{1.1}) & 6.
\end{array}
$$

Items 2 and 3 are from 1 by a Conjunctive Rule; 4 is from 3 by a Possibility Rule; 5 is from 4 by an Existential Rule; 6 is by our global assumption;

At this point the tableau branches, using item 6. The left branch contains $1\,\neg\Diamond \mathsf{E}(p_{1.1})$, from which we get $1.1\,\neg \mathsf{E}(p_{1.1})$, and this branch closes immediately using the Parameter NonExistence derived rule, Definition 12.2.3. The right branch continues as follows.

$$
\begin{array}{lll}
1 & \mathsf{E}(p_{1.1}) & 7. \\
1 & \Box A(p_{1.1}) & 8. \\
1.1 & A(p_{1.1}) & 9.
\end{array}
$$

Item 8 follows from items 2 and 7 using the Parameter Existence derived rule, Definition 12.2.2; 9 follows from 8 by a Necessity Rule.

Finally, by using both $(\forall x)\Box\mathsf{E}(x)$ (or $\mathsf{E}(x) \supset \Box\mathsf{E}(x)$) and $\Diamond\mathsf{E}(x) \supset \mathsf{E}(x)$ as global assumptions, varying domain tableau machinery allows us to construct what are, in effect, constant domain proofs. We do not recommend this—the constant domain tableau rules we gave earlier are simpler to use. The point is simply that varying domain rules can be made to do constant domain work, just as constant domain rules can simulate varying domain arguments. On purely formal grounds, neither version has primacy.

Exercises

Exercise 12.9.1 Give a varying domain **K** proof of

$$\Box(\forall x)(\exists y)\Diamond R(x,y) \supset (\forall x)\Box\Diamond(\exists y)R(x,y)$$

using $(\forall x)\Box\mathsf{E}(x)$ as a global assumption.

12.10 Tableaus Imitate Tableaus

Using the machinery given in the previous section, 12.9, varying domain tableaus can be used to establish constant domain validities. We simply need to take the closed formula $(\forall x)\Box\mathsf{E}(x)$ and the open formula $\Diamond\mathsf{E}(x) \supset \mathsf{E}(x)$ as global assumptions in our varying domain tableaus; Definitions 12.9.1 and 12.9.3. As it happens, the other way around is also possible; that is, one can use constant domain tableaus to establish varying domain validities. Here's how.

In Section 8.9 the relativization of a formula to an existence predicate was introduced, Definition 8.9.1, and it was established in Proposition 8.9.2 that this turned a varying domain validity problem into a constant domain one. Based on that Proposition, we can use constant domain tableaus to establish varying domain validities. First, we must work with formulas relativized to a primitive, not defined, existence predicate, $\mathscr{E}$, and second, we must somehow guarantee that this existence predicate 'thinks' each possible world is non-empty. We can do this easily by adopting the following rule.

Definition 12.10.1 (Non-Empty Domains Rule) *$\sigma\,(\exists x)\mathscr{E}(x)$ can be added to the end of any tableau branch on which the prefix σ occurs.*

We present some examples showing how this works.

Example 12.10.2 The formula

$$[(\forall x)\Diamond P(x) \wedge (\exists x)\Box Q(x)] \supset (\exists x)\Diamond[P(x) \wedge Q(x)]$$

is valid in varying domain **K**. Of course it trivially follows that it is valid in constant domain **K**, and hence provable using the constant domain rules. This is not very interesting. We leave it to you to establish the *varying* domain validity by constructing the relativization of this formula with $\mathscr{E}$, and then proving the result using constant domains tableaus. As it happens, the Non-Empty Domains Rule isn't needed.

What is more interesting than constructing proofs is extracting counter-examples from failed proof attempts. It is possible to extract *varying* domain counter-examples from failed constant domain tableau constructions for *relativized* formulas. We give an example, a version of the Barcan formula.

Example 12.10.3 $\Diamond(\exists x)P(x) \supset (\exists x)\Diamond P(x)$. The relativization of this is

$$\Diamond(\exists x)[\mathscr{E}(x) \wedge P(x)] \supset (\exists x)[\mathscr{E}(x) \wedge \Diamond P(x)].$$

A failed proof attempt using constant domain **K** rules, and the Non-Empty Domain Rule, is displayed in Figure 12.1. Only item 12 really needs specific comment, but here are all the steps, for the record. 2 and 3 are from 1 by an Implication Rule, 4 is from 2 by a Possibility Rule, 5 is from 5 by an Existential Rule, 6 and 7 are from 5 by a Conjunction Rule, and 8 is from 3 by a Universal Rule. Next, 9 and 10 are from 8 by a Disjunctive Rule, 11 is from 10 by a Necessity Rule, and the right branch is closed by 7 and 11. On the left branch, we have that something exists with prefix 1.1 by 6 (and so in the model, the corresponding possible world domain will be non-empty). There is nothing similar for prefix 1, so 12 is added using the Non-Empty Domain Rule, then 13 follows by an Existential Rule, and 14 follows from 3 by a Universal Rule. Finally, 15 and 16 are from 14 by a Disjunctive Rule, and 17 is from 16 by a Necessity Rule.

The leftmost branch is closed by 13 and 15, but the center branch is not closed. We use it to construct a varying domain counter-model, which is shown in Figure 12.2. First, the possible worlds are the prefixes appearing on the branch, 1 and 1.1, with 1.1 accessible from 1. The domain of world 1 is $\{b\}$ because of 13, and of world 1.1 is $\{a\}$ by 6. Finally, the interpretation of P at world 1 is empty, and at world 1.1 is $\{a\}$.

12.11 On Symmetry

This section consists of a small remark, but it is of some technical interest. If the accessibility relation of a frame is symmetric, monotonicity and anti-monotonicity are equivalent—either implies the other. To say monotonicity and anti-monotonicity are equivalent is to say each instance of the Barcan formula is derivable from some

$1 \quad \neg\{\Diamond(\exists x)[\mathcal{E}(x) \wedge P(x)] \supset (\exists x)[\mathcal{E}(x) \wedge \Diamond P(x)]\} \quad 1.$
$1 \quad \Diamond(\exists x)[\mathcal{E}(x) \wedge P(x)] \quad 2.$
$1 \quad \neg(\exists x)[\mathcal{E}(x) \wedge \Diamond P(x)] \quad 3.$
$1.1 \quad (\exists x)[\mathcal{E}(x) \wedge P(x)] \quad 4.$
$1.1 \quad \mathcal{E}(a) \wedge P(a) \quad 5.$
$1.1 \quad \mathcal{E}(a) \quad 6.$
$1.1 \quad P(a) \quad 7.$
$1 \quad \neg[\mathcal{E}(a) \wedge \Diamond P(a)] \quad 8.$

Left branch:
$1 \, \neg\mathcal{E}(a) \quad 9.$
$1 \quad (\exists x)\mathcal{E}(x) \quad 12.$
$1 \quad \mathcal{E}(b) \quad 13.$
$1 \, \neg[\mathcal{E}(b) \wedge \Diamond P(b)] \quad 14.$

Left sub-branch:
$1 \, \neg\mathcal{E}(b) \quad 15.$

Right sub-branch:
$1 \quad \neg\Diamond P(b) \quad 16.$
$1.1 \, \neg P(b) \quad 17.$

Right branch:
$1 \quad \neg\Diamond P(a) \quad 10.$
$1.1 \, \neg P(a) \quad 11.$

Fig. 12.1: Failed Constant Domain Tableau Proof

instances of the Converse Barcan formula, and each instance of the Converse Barcan formula is derivable from some instances of the Barcan formula.

If we use the Barcan/Converse Barcan equivalents involving $\mathsf{E}(x)$, the equivalence is straightforward to verify. Recall that to say a frame has a symmetric accessibility relation is equivalent to saying that all instances of the schema $\Phi \supset \Box\Diamond\Phi$ are valid in it, or equivalently, all instances of $\Diamond\Box\Phi \supset \Phi$ are valid. Now we have the following informal argument (formalizable axiomaticallly, however).

1. $\Diamond\mathsf{E}(x) \supset \mathsf{E}(x)$	anti-monotonicity
2. $\Box\Diamond\mathsf{E}(x) \supset \Box\mathsf{E}(x)$	necessitation
3. $\mathsf{E}(x) \supset \Box\mathsf{E}(x)$	using $\Phi \supset \Box\Diamond\Phi$

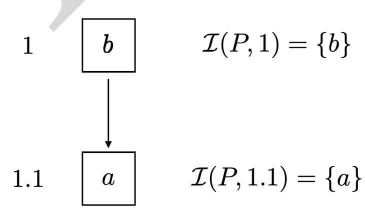

Fig. 12.2: A Varying Domain Counter Model