Modal Logic

Answers to
Sample Questions
May 12, 2023

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1. The following is a tableau proof in $\mathsf{S5}$ of $\Diamond(P \wedge \Diamond(Q \wedge \Box R)) \supset (\Diamond(P \wedge R) \wedge \Diamond(Q \wedge R))$.

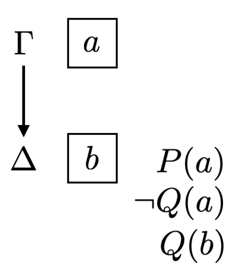
$$
\begin{array}{lll}
1 & \neg[\Diamond(P \wedge \Diamond(Q \wedge \Box R)) \supset (\Diamond(P \wedge R) \wedge \Diamond(Q \wedge R))] & 1. \\
1 & \Diamond(P \wedge \Diamond(Q \wedge \Box R)) & 2. \\
1 & \neg[(\Diamond(P \wedge R) \wedge \Diamond(Q \wedge R))] & 3. \\
1.1 & P \wedge \Diamond(Q \wedge \Box R) & 4. \\
1.1 & P & 5. \\
1.1 & \Diamond(Q \wedge \Box R) & 6. \\
1.1.1 & Q \wedge \Box R & 7. \\
1.1.1 & Q & 8. \\
1.1.1 & \Box R & 9.
\end{array}
$$

$$
\begin{array}{c|c}
\begin{array}{lll} 1 & \neg\Diamond(P \wedge R) & 10. \\ 1.1 & \neg(P \wedge R) & 12. \end{array} & \begin{array}{lll} 1 & \neg\Diamond(Q \wedge R) & 11. \\ 1.1 & \neg(Q \wedge R) & 16. \end{array} \\
\begin{array}{l|l} 1.1\ \neg P \quad 13. & \begin{array}{l} 1.1\ \neg R \quad 14. \\ 1.1\ \ R \quad 15. \end{array} \end{array} & \begin{array}{l|l} 1.1\ \neg Q \quad 17. & \begin{array}{l} 1.1\ \neg R \quad 18. \\ 1.1\ \ R \quad 19. \end{array} \end{array}
\end{array}
$$

$1 \to 2,3$; $2 \to 4$; $4 \to 5,6$; $6 \to 7$; $7 \to 8,9$; $3 \to 10,11$; $10 \to 12$; $12 \to 13,14$; $9 \to 15$ (KB rule); $11 \to 16$; $16 \to 17,18$; $18 \to 19$ (KB rule).

2. For the formula $[(\exists x)\Diamond P(x) \wedge \Box(\forall x)(P(x) \supset Q(x))] \supset (\exists x)\Diamond Q(x)$:

   (a) Varying domain **K** counter-model. The following model will serve. Two possible worlds, Γ and Δ, with Δ accessible from Γ. The domain of Γ is $\{a\}$, and of Δ is $\{b\}$, where a and b are different. At Δ, $P(a)$ and $Q(b)$ are true while $Q(a)$ is false, and other atomic formulas won't matter. In this model (left to you to verify), the formula is false at Γ. Here is the model schematically.



   (b) Constant domain **K** proof.

$$
\begin{array}{ll}
1 & \neg\{[(\exists x)\Diamond P(x) \wedge \Box(\forall x)(P(x) \supset Q(x))] \supset (\exists x)\Diamond Q(x)\} \quad 1.\\
1 & \quad (\exists x)\Diamond P(x) \wedge \Box(\forall x)(P(x) \supset Q(x)) \quad 2.\\
1 & \neg(\exists x)\Diamond Q(x) \quad 3.\\
1 & \quad (\exists x)\Diamond P(x) \quad 4.\\
1 & \quad \Box(\forall x)(P(x) \supset Q(x)) \quad 5.\\
1 & \quad \Diamond P(a) \quad 6.\\
1 & \neg\Diamond Q(a) \quad 7.\\
1.1 & \quad P(a) \quad 8.\\
1.1 & \quad (\forall x)(P(x) \supset Q(x)) \quad 9.\\
1.1 & \neg Q(a) \quad 10.\\
1.1 & \quad P(a) \supset Q(a) \quad 11.
\end{array}
$$

$$
1.1\, \neg P(a) \quad 12. \qquad\qquad 1.1\, Q(a) \quad 13.
$$

$1 \to 2, 3$; $2 \to 4, 5$; $4 \to 6$; $3 \to 7$; $6 \to 8$; $5 \to 9$; $7 \to 10$; $9 \to 11$; $11 \to 12, 13$.

3. Here is a tableau proof in constant domain K, under the CA assumptions of $\langle \lambda y.\Box\langle \lambda x.x = y\rangle(c)\rangle(c) \supset [\langle \lambda x.\Box\varphi(x)\rangle(c) \supset \Box\langle \lambda x.\varphi(x)\rangle(c)]$.

$$
\begin{array}{ll}
1 & \neg\{\langle \lambda y.\Box\langle \lambda x.x = y\rangle(c)\rangle(c) \supset [\langle \lambda x.\Box\varphi(x)\rangle(c) \supset \Box\langle \lambda x.\varphi(x)\rangle(c)]\} \quad 1.\\
1 & \quad \langle \lambda y.\Box\langle \lambda x.x = y\rangle(c)\rangle(c) \quad 2.\\
1 & \neg[\langle \lambda x.\Box\varphi(x)\rangle(c) \supset \Box\langle \lambda x.\varphi(x)\rangle(c)]\} \quad 3.\\
1 & \quad \langle \lambda x.\Box\varphi(x)\rangle(c) \quad 4.\\
1 & \neg\Box\langle \lambda x.\varphi(x)\rangle(c) \quad 5.\\
1 & \quad \Box\langle \lambda x.x = c^1\rangle(c) \quad 6.\\
1 & \quad \Box\varphi(c^1) \quad 7.\\
1.1 & \neg\langle \lambda x.\varphi(x)\rangle(c) \quad 8.\\
1.1 & \neg\varphi(c^{1.1}) \quad 9.\\
1.1 & \quad \langle \lambda x.x = c^1\rangle(c) \quad 10.\\
1.1 & \quad c^{1.1} = c^1 \quad 11.\\
1.1 & \quad \varphi(c^1) \quad 12.\\
1.1 & \neg\varphi(c^1) \quad 13.
\end{array}
$$

In this: $1 \to 2, 3$; $3 \to 4, 5$; $2 \to 6$; $4 \to 7$; $5 \to 8$; $8 \to 9$; $6 \to 10$; $10 \to 11$; $7 \to 12$; $9, 11 \to 13$.

4. The following is for a *CN* seting, and the choice of logic doesn't matter.

   (a) There are many models showing that $\langle \lambda x.P(x)\rangle(\imath x.P(x))$ is not valid. Here is one. There is one possible world. The domain of that world has two objects in it. At that world, P is true of both objects. Since P is true of two things, $\imath x.P(x)$ doesn't designate, so $\langle \lambda x.P(x)\rangle(\imath x.P(x))$ is not true.

   (b) For the formula $\langle \lambda x.\psi(x)\rangle(\imath x.\varphi(x)) \supset \mathsf{D}(\imath x.\varphi(x))$.

      i. This is valid. The following is sufficient to establish this. Take any model and any possible world Γ in it. If $\langle \lambda x.\psi(x)\rangle(\imath x.\varphi(x))$ is not true at Γ it follows that $\langle \lambda x.\psi(x)\rangle(\imath x.\varphi(x)) \supset X$ is true for *any* formula X. If $\langle \lambda x.\psi(x)\rangle(\imath x.\varphi(x))$ is true at Γ then $\imath x.\varphi(x)$ must designate (this is part of the definition of truth for predicate abstracts). Since what it designates is self-identical, $\langle \lambda y.y = y\rangle(\imath x.\varphi(x))$ is true and so again the implication is true. Since Γ was arbitrary we have validity.

ii. Here is a tableau proof. The choice of logic and conditions won't matter.

$$
\begin{array}{ll}
1\,\neg[\langle\lambda x.\psi(x)\rangle(\imath x.\varphi(x)) \supset \mathsf{D}(\imath x.\varphi(x))] & 1.\\
1\quad \langle\lambda x.\psi(x)\rangle(\imath x.\varphi(x)) & 2.\\
1\,\neg\mathsf{D}(\imath x.\varphi(x)) & 3.\\
1\,\neg\langle\lambda y.y = y\rangle(\imath x.\varphi(x)) & 4.\\
1\,\neg[[\imath x.\varphi(x)]^1 = [\imath x.\varphi(x)]^1] & 5.\\
1\quad [[\imath x.\varphi(x)]^1 = [\imath x.\varphi(x)]^1] & 6.
\end{array}
$$

$1 \to 2, 3$; 4 is 3 unabbreviated; $4 \to 5$ since the definite description is positively generated on line 2; 6 is by reflexivity. The branch is closed.

(c) This question has a typo in it. I wrote $\mathsf{D}(\imath x.\varphi(x)) \supset \langle\lambda x.\psi(x)\rangle(\imath x.\varphi(x))$, and this is not valid. It is easily shown. For example, just take $\psi(x)$ to be a formula that is always false, but $\varphi(x)$ to be such that exactly one thing makes it true, so that $\mathsf{D}(\imath x.\varphi(x))$ is true.

What I meant to write was the formula $\mathsf{D}(\imath x.\varphi(x)) \supset \langle\lambda x.\varphi(x)\rangle(\imath x.\varphi(x))$. I'll give answers for this one.

i. For validity. Take any model and any possible world Γ in it. If $\mathsf{D}(\imath x.\varphi(x))$ is not true at Γ the implication is true. Now assume $\mathsf{D}(\imath x.\varphi(x))$ is true at Γ. That is, $\langle\lambda y.y = y\rangle(\imath x.\varphi(x))$ is true at Γ. Then from the truth definition for predicate abstracts, $\imath x.\varphi(x)$ must designate at Γ, and what it designates must make $\varphi(x)$ true. This is what is needed for $\langle\lambda x.\varphi(x)\rangle(\imath x.\varphi(x))$ to be true at Γ, so again the implication is true.

ii. Here is a tableau proof.

$$
\begin{array}{ll}
1\,\neg[\mathsf{D}(\imath x.\varphi(x)) \supset \langle\lambda x.\varphi(x)\rangle(\imath x.\varphi(x))] & 1.\\
1\quad \mathsf{D}(\imath x.\varphi(x)) & 2.\\
1\,\neg\langle\lambda x.\varphi(x)\rangle(\imath x.\varphi(x)) & 3.\\
1\quad \langle\lambda y.y = y\rangle(\imath x.\varphi(x)) & 4.\\
1\quad [[\imath x.\varphi(x)]^1 = [\imath x.\varphi(x)]^1] & 5.\\
1\,\neg[[\imath x.\varphi(x)]^1 = [\imath x.\varphi(x)]^1] & 6.
\end{array}
$$

$1 \to 2, 3$; 4 is 2 unabbreviated; $4 \to 5$; $3 \to 6$ (the definite description is positively generated because of 4).