Modal Logics Between Propositional and First Order

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Abstract

One can add the machinery of relation symbols and terms to a propositional modal logic without adding quantifiers. Ordinarily this is no extension beyond the propositional. But if terms are allowed to be non-rigid, a scoping mechanism (usually written using lambda abstraction) must also be introduced to avoid ambiguity. Since quantifiers are not present, this is not really a first-order logic, but it is not exactly propositional either. For propositional logics such as **K**, **T** and **D**, adding such machinery produces a decidable logic, but adding it to **S5** produces an undecidable one. Further, if an equality symbol is in the language, and interpreted by the equality relation, logics from **K4** to **S5** yield undecidable versions. (Thus transitivity is the villain here.) The proof of undecidability consists in showing that classical first-order logic can be embedded.

1 Introduction

The best known propositional modal logics are decidable—something that can be shown by filtration to produce finite models, or by using special characteristics of a proof procedure such as tableaus. Of course first-order versions are undecidable, since they conservatively extend classical logic. But there are modal logics that are, in a sense, intermediate between first-order and propositional, between the dark and the daylight, so to speak. For these decidability is not obvious, and in fact it fails for a whole range of them. I will state the results of this paper properly after appropriate machinery has been introduced.

The syntax of a first-order logic involves more than simply admitting quantifiers—one also needs relation symbols, variables, and perhaps constant and function symbols as well. In a modal setting these are all rather straightforward, unless the semantics allows terms to be *non-rigid*, taking on different values at different possible worlds. If this happens, ambiguity results. The standard example is $\Diamond P(c)$, where the constant symbol c is interpreted non-rigidly. To say this is true at possible world Γ could mean that $P(c)$ is true at an alternative world, meaning that P holds of what c designates at that alternative world. Or we could say $\Diamond P(c)$ is true at Γ if whatever c designates at Γ has the "possible-P" property, meaning that at an alternative world P holds of what c designated back at Γ. If c is non-rigid, these can lead to different outcomes—we are seeing the *de re/de dicto* distinction at work. Since both readings are useful for different purposes, some mechanism must be introduced to disambiguate things.

Some time ago an abstraction device was proposed for modal logic, and this has proved quite useful [11, 12]. An extensive study of its virtues can be found in [7]. It will be defined properly below, but for now it should suffice to say that the two readings of $\Diamond P(c)$ just discussed correspond to two distinct syntactic expressions $\Diamond\langle\lambda x.P(x)\rangle(c)$ and $\langle\lambda x.\Diamond P(x)\rangle(c)$.

Throughout this paper, by a propositional modal logic **L**, I mean one characterized by a class of frames. If **L** is a propositional modal logic, I'll use $\mathbf{L}\lambda$ for the logic that syntactically allows relation symbols, constant symbols, abstraction, *but not quantification*. Semantically, I'll extend the possible world semantics for **L**, with a domain in which to interpret constant and relation symbols. I'll take this domain to be the same from world to world—constant domain semantics—but I'll allow constant symbols to be interpreted non-rigidly. I will assume one of the relation symbols is $=$, and I will use $\mathbf{L}\lambda_=$ for the logic determined by using **L** frames, but requiring $=$ to be interpreted by the equality relation on the domain. Since no quantifiers are present, we do not really have a first-order logic, but since a domain must be specified, there are some of the characteristics of one. Now, here are the results to be proved in this paper.

Theorem 1

1. If **L** is one of **K**, **D**, **T**, **B**, then $\mathbf{L}\lambda$ and $\mathbf{L}\lambda_=$ are decidable (this is not an exhaustive list).
2. $\mathbf{S5}\lambda$ is undecidable.
3. If **L** is between **K4** and **S5**, then $\mathbf{L}\lambda_=$ is undecidable.

To keep things simple, I have omitted function symbols. Allowing them keeps the decidable logics $\mathbf{L}\lambda$, listed in part 1 above, decidable. I don't know if the decidability of $\mathbf{L}\lambda_=$ is preserved if function symbols are allowed.

2 Syntax and Semantics

So far things have been described rather informally. Now it is time to get serious, beginning with syntax. Let **L** be a propositional modal logic. Recall that in this paper **L** will always be assumed to be determined by a class of frames.

I'll assume we have an alphabet of *variables* (typically x, y, x_1, ...), an alphabet of *constant symbols*, (typically c, d, c_1, ...), and for each n an alphabet of *n-ary relation symbols* (typically P, R, R_1, ...). One of the relation symbols is $=$, and I'll write $x = y$ rather than $=(x, y)$, in the usual way. A *term* is a constant symbol or a variable.

Definition 2 (Formula of Lλ) The set of formulas, and their free variables, is defined as follows.

1. If R is an n-ary relation symbol and $x_1, x_2, \ldots, x_n$ are variables, then $R(x_1, x_2, \ldots, x_n)$ is a formula, with $x_1, x_2, \ldots, x_n$ as its free variable occurrences.

2. if Φ is a formula, so are $\neg\Phi$, $\Box\Phi$, and $\Diamond\Phi$. Free variable occurrences are those of Φ.

3. If Φ and Ψ are formulas, so are $(\Phi \wedge \Psi)$, $(\Phi \vee \Psi)$, and $(\Phi \supset \Psi)$. Free variable occurrences are those of Φ together with those of Ψ.

4. If Φ is a formula, x is a variable, and t is a term, $\langle\lambda x.\Phi\rangle(t)$ is a formula. Free variable occurrences are those of Φ, except for occurrences of x, together with t if it is a variable.

I'll sometimes write $\Phi(x)$ to indicate x is a free variable that may have occurrences in Φ, and $\Phi(t)$ to denote the result of substituting t for free occurrences of x in Φ.

Next we turn to semantics. Besides the usual machinery of propositional modal logic, a domain must be provided to supply objects to serve as values for constant symbols and variables. Constant symbols will be interpreted non-rigidly, by functions mapping worlds to objects. Likewise a non-rigid interpretation must be supplied for relation symbols.

Definition 3 (Model of Lλ) A *model* is a structure $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$, where:

1. $\langle\mathcal{G}, \mathcal{R}\rangle$ is a frame for the propositional modal logic **L** ($\mathcal{G}$ is the set of possible worlds and $\mathcal{R}$ is the accessibility relation).

2. $\mathcal{D}$ is a non-empty set, the *domain*;

3. $\mathcal{I}$ is a mapping that assigns:

   (a) to each constant symbol some function from $\mathcal{G}$ to $\mathcal{D}$;

   (b) to each n-ary relation symbol some function from $\mathcal{G}$ to the power set of $\mathcal{D}^n$.

   If $\mathcal{I}(=)$ is the constant function assigning the equality relation on $\mathcal{D}$ to every member of $\mathcal{G}$, I'll say we have a model of $\mathbf{L}\lambda_=$.

While constant symbols are interpreted non-rigidly, variables are thought of as rigid, as in [7]. This is not the only possible way to do things—see [5, 4, 6]—but it is certainly the simplest approach.

Definition 4 (Valuation) A *valuation* v in a model $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ is a mapping that assigns to each variable some member of $\mathcal{D}$. A mapping $(v * \mathcal{I})$ from terms and worlds to $\mathcal{D}$ is defined as follows:

1. For a variable x, $(v * \mathcal{I})(x, \Gamma) = v(x)$.

2. For a constant symbol c, $(v * \mathcal{I})(c, \Gamma) = \mathcal{I}(c)(\Gamma)$.

Now the main semantic notion, which is symbolized by $\mathcal{M}, \Gamma \Vdash_v \Phi$, and is read: formula Φ is true in model $\mathcal{M}$, at possible world Γ, with respect to valuation v. For simplicity, take $\vee$, $\supset$, $\exists$, and $\Diamond$ as defined symbols, in the usual way.

Definition 5 (Truth in an Lλ Model) Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be an **L**λ model, and v be a valuation in it.

1. If $R(x_1, \ldots, x_n)$ is atomic, $\mathcal{M}, \Gamma \Vdash_v R(x_1, \ldots, x_n)$ iff $\langle v(x_1), \ldots, v(x_n) \rangle \in \mathcal{I}(R)(\Gamma)$.
2. $\mathcal{M}, \Gamma \Vdash_v \neg\Phi$ iff $\mathcal{M}, \Gamma \not\Vdash_v \Phi$.
3. $\mathcal{M}, \Gamma \Vdash_v \Phi \wedge \Psi$ iff $\mathcal{M}, \Gamma \Vdash_v \Phi$ and $\mathcal{M}, \Gamma \Vdash_v \Psi$.
4. $\mathcal{M}, \Gamma \Vdash_v \Box\Phi$ iff $\mathcal{M}, \Delta \Vdash_v \Phi$ for all $\Delta \in \mathcal{G}$.
5. $\mathcal{M}, \Gamma \Vdash_v \langle \lambda x.\Phi \rangle(t)$ if $\mathcal{M}, \Gamma \Vdash_{v'} \Phi$, where v' is like v except that $v'(x) = (v * \mathcal{I})(t, \Gamma)$.

As usual, a formula is called valid in a model if it is true at every world of it, with respect to every valuation, and valid in **L**λ if it is valid in all **L**λ models. Similarly for **L**$\lambda_=$ validity. Also, as usual, for closed formulas the specification of a valuation does not matter.

One can say some quite sophisticated things using this syntax and semantics. Here is an example from [7]. Define the following formula abbreviations.

$$\begin{array}{rcl} A_3 & = & \Box\langle \lambda y.\langle \lambda x.\Phi(y) \supset \Phi(c)\rangle(c)\rangle(c) \supset \\ & & \quad \langle \lambda y.\Box\langle \lambda x.\Phi(y) \supset \Phi(x)\rangle(c)\rangle(c) \\ A_4 & = & \langle \lambda x.\Box\Phi(x)\rangle(c) \supset \Box\langle \lambda x.\Phi(x)\rangle(c) \end{array}$$

Notice that A_4 is an arbitrary instance of *de re* implying *de dicto* for c, while A_3 is a more specialized instance of *de dicto* implying *de re*. It can be shown that $A_3 \supset A_4$ is valid in **K**λ. Thus if, for c, *de dicto* always implies *de re*, then in fact *de re* implies *de dicto* as well. (This works the other way around too.)

For an example involving equality, it is convenient to introduce the following abbreviation.

$$\mathbf{Rigid}(c) = \langle \lambda x.\Box\langle \lambda y.y = x\rangle(c)\rangle(c)$$

This characterizes what was called *local rigidity* in [7]. Now, $\mathbf{Rigid}(c) \supset A_4$ is valid in **K**$\lambda_=$.

What $\mathbf{Rigid}(c)$ says is that, at accessible worlds c keeps the same designation that it has in this one. I'll actually need a weaker notion saying that, from any accessible world one can always move to some world where c recovers the designation that it had in this world.

Definition 6 $\mathbf{Recur}(c) = \langle \lambda x.\Box\Diamond\langle \lambda y.y = x\rangle(c)\rangle(c)$.

3 Tableaus, In Brief

In [7] prefixed tableau systems are given for many standard first-order modal logics with non-rigid constant symbols (and function symbols) and the λ-abstraction mechanism. Rules for equality are also given. Here is a brief presentation of a tableau system for $\mathbf{K}\lambda$, which results when the quantifier rules from [7] are dropped.

A *prefix* is a finite sequence of positive integers. A *prefixed formula* is an expression of the form $\sigma\, X$, where σ is a prefix and X is a formula. I write prefixes using periods to separate integers, as in 1.2.3.2.1. If σ is a prefix and n is a positive integer, $\sigma.n$ is σ followed by a period followed by n. A prefix σ intuitively names a possible world in some modal model, and $\sigma\, X$ 'says' that X is true at the world σ names. The prefix $\sigma.n$ is intended to name a world that is accessible from the one that σ names. A tableau will contain prefixed formulas as node labels.

A tableau proof of a *closed formula* Z is a tree with $1\,\neg Z$ at its root, that is 'grown' according to certain *Branch Extension Rules*, given below, and that is *closed.* A tableau is closed if each branch is closed, and a branch is closed if it contains $\sigma\, X$ and $\sigma\,\neg X$, for some σ and some X. Finally, here are the Branch Extension Rules (taking $\wedge$ as a representative propositional connective). For any prefix σ,

$$\frac{\sigma\, X \wedge Y}{\begin{array}{c}\sigma\, X\\ \sigma\, Y\end{array}} \qquad \frac{\sigma\,\neg(X \wedge Y)}{\sigma\,\neg X \;\big|\; \sigma\,\neg Y} \qquad \frac{\sigma\,\neg\neg X}{\sigma\, X}$$

if the prefix $\sigma.n$ is new to the branch,

$$\frac{\sigma\,\Diamond X}{\sigma.n\, X} \qquad \frac{\sigma\,\neg\Box X}{\sigma.n\,\neg X}$$

if the prefix $\sigma.n$ already occurs on the branch,

$$\frac{\sigma\,\Box X}{\sigma.n\, X} \qquad \frac{\sigma\,\neg\Diamond X}{\sigma.n\,\neg X}$$

To take care of abstraction rules, a new family of constant symbols is introduced, which will only appear in proofs. If c is a (standard) constant symbol, and σ is a prefix, c_σ is an *extended constant symbol.* Think of it as what the non-rigid symbol c designates at the world named by σ.

$$\frac{\sigma\,\langle\lambda x.\Phi(x)\rangle(c)}{\sigma\,\Phi(c_\sigma)} \qquad \frac{\sigma\,\neg\langle\lambda x.\Phi(x)\rangle(c)}{\sigma\,\neg\Phi(c_\sigma)}$$

These rules correspond to $\mathbf{K}\lambda$—a proof of soundness and completeness can be extracted from the more general setting of [7]. A careful look at the completeness argument shows that the rules for $\Diamond X$ and $\neg\Box X$ never need to be applied to a formula more than once on a branch. Since each rule application reduces formula complexity, decidability of the system is immediate.

To extend the tableau system to one for $\mathbf{K}\lambda_=$, two more rules must be added.

First, reflexivity: If c_σ occurs on the tableau branch, and σ' is a prefix which also occurs on the tableau branch, then $\sigma'\,(c_\sigma = c_\sigma)$ can be added to the end of the branch.

Second, substitutivity: Let $\Phi(x)$ be a formula (allowing extended constant symbols) in which at most x occurs free, let t and u be extended constant symbols, and let $\Phi(t)$ be the result of substituting occurrences of t for all free occurrences of x in $\Phi(x)$, and similarly for $\Phi(u)$. If $\sigma_1(t = u)$ and $\sigma_2\Phi(t)$ both occur on a tableau branch, $\sigma_2\Phi(u)$ can be added to the end.

Once again a carefully formulated completeness argument shows that if a formula is unprovable, a *finite* tableau construction is sufficient to construct a counter-model. This ensures decidability.

Modifications to the specifically modal rules above yield systems for logics such as **K**, **D**, **T**, **B**, and these too can be used to establish decidability. See [3] for further discussion of tableau methods and decidability (without abstraction).

We now have the first part of Theorem 1.

4 Two Embeddings

There are well-known embeddings of classical first-order logic into first-order versions of **S5** and **S4**. They can be described easily: for **S5**, insert $\Box$ in front of every subformula; for **S4**, insert $\Box\Diamond$ in front of every subformula. The **S5** translation comes from [8] and [9]; the **S4** version comes from [2], where its connection with forcing was noted. In [10] the **S4** translation was a key step in the proof of the independence of the continuum hypothesis from the axioms of Zermelo-Fraenkel set theory. Now two variations of these translations are introduced, mapping first-order classical logic formulas into the language $\mathbf{L}\lambda$. The embeddings involve only a single non-rigid constant symbol, which we fix to be c. *Throughout I'll assume first-order classical formulas are defined as usual, and do not contain equality, constant, or function symbols.*

Definition 7 Let X be a formula of first-order classical logic. Modal formulas X° and X^*, in the language $\mathbf{L}\lambda$, are defined as follows.

1. For an atomic formula A, $A^\circ = \Box A$ and $A^* = \Box\Diamond A$.
2. $[\neg\Phi]^\circ = \Box\neg[\Phi]^\circ$ and $[\neg\Phi]^* = \Box\Diamond\neg[\Phi]^*$.
3. For $\odot$ one of $\wedge$, $\vee$, $\supset$, $[\Phi \odot \Psi]^\circ = \Box[\Phi \odot \Psi]^\circ$ and $[\Phi \odot \Psi]^* = \Box\Diamond[\Phi \odot \Psi]^*$.
4. $[(\forall x)\Phi]^\circ = \Box\langle\lambda x.\Phi^\circ\rangle(c)$ and $[(\forall x)\Phi]^* = \Box\Diamond\Box\langle\lambda x.\Phi^*\rangle(c)$.
5. $[(\exists x)\Phi]^\circ = \Diamond\langle\lambda x.\Phi^\circ\rangle(c)$ and $[(\exists x)\Phi]^* = \Box\Diamond\Diamond\langle\lambda x.\Phi^*\rangle(c)$.

I'll generally be interested in logics that are at least as strong as **KD4**, and for these part of the definition of the embedding for the existential quantifier will generally be simplified a bit, to $\Box\Diamond\langle\lambda x.\Phi^*\rangle(c)$.

Here is a statement of the central result. It provides undecidability for a range of logics, not quite the range that was promised, but the full version is then an easy consequence.

Theorem 8 Let Φ be a closed classical first-order formula. The following are equivalent.

1. Φ is classically valid.
2. $\Box\mathbf{Recur}(c) \supset \Phi^*$ is valid in $\mathbf{L}\lambda_=$, where **L** is between **KD4** and **S5**.
3. Φ^* is valid in $\mathbf{S5}\lambda$.
4. Φ° is valid in $\mathbf{S5}\lambda$.

The proof of this theorem will be spread over the next few sections.

5 Complete sequences

Complete sequences originated in Cohen's work on forcing, [1]. In [2] I modified them to prove a result concerning first-order **S4**. That work is further modified here to prove the following.

Proposition 9 Let Φ be a closed classical first-order formula. If Φ is classically valid then $\Box\mathbf{Recur}(c) \supset \Phi^*$ is valid in $\mathbf{KD4}\lambda_=$, and hence in $\mathbf{L}\lambda_=$ for any **L** stronger than **KD4**.

The proof of this Proposition occupies the rest of the section. Assume $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ is some $\mathbf{KD4}\lambda_=$ model. It will be convenient for this section to expand the language by adding a new family of free variables, called *parameters*, one for each member of $\mathcal{D}$. These will never be bound by λ abstracts or quantifiers, and it is understood that the only valuations that will be considered are such that $v(p) = p$ for every parameter. In short, we add names for members of the domain to the language. I'll extend the use of the word *closed* so that a formula whose only free variables are parameters is considered closed.

A few observations. Because of transitivity, $\Box X \supset \Box\Box X$ and $\Diamond\Diamond X \supset \Diamond X$ are valid in **KD4**. Because of seriality, $\Box X \supset \Diamond X$ is valid in **KD4**. It follows that $\Box\Diamond X \equiv \Box\Diamond\Box\Diamond X$ is also valid. One way, $\Box\Diamond X \supset \Box\Box\Diamond X \supset \Box\Box\Box\Diamond X \supset \Box\Diamond\Box\Diamond X$. The other way, $\Box\Diamond\Box\Diamond X \supset \Box\Diamond\Diamond\Diamond X \supset \Box\Diamond\Diamond X \supset \Box\Diamond X$. More generally, any sequence of modal operators beginning with $\Box$ and ending with $\Diamond$ can be replaced by $\Box\Diamond$, and conversely, in **KD4**. Also, since every formula of the form X^* begins with $\Box\Diamond$, it follows that $X^* \equiv \Box\Diamond X^*$ is valid in $\mathbf{KD4}\lambda$.

Lemma 10 Let Φ be any closed formula of classical first-order logic (allowing parameters), and let Γ be an arbitrary member of $\mathcal{G}$.

1. If $\mathcal{M}, \Gamma \Vdash_v \Phi^*$ and $\Gamma\mathcal{R}\Delta$ then $\mathcal{M}, \Delta \Vdash_v \Phi^*$.
2. If $\mathcal{M}, \Gamma \not\Vdash_v \Phi^*$ then $\mathcal{M}, \Delta \Vdash_v [\neg\Phi]^*$ for some $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$.
3. If $\mathcal{M}, \Gamma \Vdash_v [(\exists x)\Phi(x)]^*$ then $\mathcal{M}, \Delta \Vdash_v [\Phi(p)]^*$ for some $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$ and some parameter p.
4. If $\mathcal{M}, \Gamma \Vdash_v [\neg(\forall x)\Phi(x)]^*$ then $\mathcal{M}, \Delta \Vdash_v [\neg\Phi(p)]^*$ for some $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$ and some parameter p.
5. If $\mathcal{M}, \Gamma \Vdash_v [(\forall x)\Phi(x)]^*$ and $\mathcal{M}, \Gamma \Vdash_v \Box\Diamond\langle\lambda y.y = p\rangle(c)$ for a parameter p, then $\mathcal{M}, \Gamma \Vdash_v [\Phi(p)]^*$.

6. If $\mathcal{M}, \Gamma \Vdash_v [\neg(\exists x)\Phi(x)]^*$ and $\mathcal{M}, \Gamma \Vdash_v \Box\Diamond\langle\lambda y.y = p\rangle(c)$ for a parameter p, then $\mathcal{M}, \Gamma \Vdash_v [\neg\Phi(p)]^*$.

Proof Since $\Phi^* \equiv \Box\Diamond\Phi^*$, (1) is immediate.

Suppose $\mathcal{M}, \Gamma \not\Vdash_v \Phi^*$. Since $\Phi^* \equiv \Box\Diamond\Phi^*$, for some Δ with $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash_v \Box\neg\Phi^*$. But also $\Box\neg\Phi^* \supset \Box\Box\neg\Phi^* \supset \Box\Diamond\neg\Phi^*$, so $\mathcal{M}, \Delta \Vdash_v [\neg\Phi]^*$. We thus have (2).

If $\mathcal{M}, \Gamma \Vdash_v [(\exists x)\Phi(x)]^*$ then $\mathcal{M}, \Gamma \Vdash_v \Box\Diamond\langle\lambda x.\Phi^*(x)\rangle(c)$ so (making use of seriality) there is some accessible world Δ at which we have $\langle\lambda x.\Phi^*(x)\rangle(c)$. Let p be what c designates at Δ, that is, $p = \mathcal{I}(c)(\Delta)$. Then $\mathcal{M}, \Delta \Vdash_v \Phi^*(p)$, and we have (3).

(4) results by combining (2) and (3).

For (5), assume $\mathcal{M}, \Gamma \Vdash_v \Box\Diamond\Box\langle\lambda x.\Phi^*(x)\rangle(c)$ and $\mathcal{M}, \Gamma \Vdash_v \Box\Diamond\langle\lambda y.y = p\rangle(c)$. Let Δ be a world such that $\Gamma\mathcal{R}\Delta$. Then $\mathcal{M}, \Delta \Vdash_v \Diamond\Box\langle\lambda x.\Phi^*(x)\rangle(c)$, so for some Ω_1, $\Delta\mathcal{R}\Omega_1$ and $\mathcal{M}, \Omega_1 \Vdash_v \Box\langle\lambda x.\Phi^*(x)\rangle(c)$. Also $\mathcal{M}, \Omega_1 \Vdash_v \Diamond\langle\lambda y.y = p\rangle(c)$ so for some Ω_2 with $\Omega_1\mathcal{R}\Omega_2$, $\mathcal{M}, \Omega_2 \Vdash_v \langle\lambda y.y = p\rangle(c)$. Also $\mathcal{M}, \Omega_2 \Vdash_v \langle\lambda x.\Phi^*(x)\rangle(c)$ and it follows that $\mathcal{M}, \Omega_2 \Vdash_v \Phi^*(p)$. Thus $\mathcal{M}, \Delta \Vdash_v \Diamond\Phi^*(p)$ and since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash_v \Box\Diamond\Phi^*(p)$.

Finally, (6) is similar to (5).

Now, suppose Γ is a world of $\mathcal{M}$, and let $X_1, X_2, X_3, \ldots$ be an enumeration of all closed formulas of classical first-order logic (allowing parameters).. A *complete sequence* starting at Γ is any sequence $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$, of worlds constructed as follows.

$\Gamma_0 = \Gamma$.

Suppose Γ_n has been defined. There are several cases, depending on X_{n+1}. Parts of Lemma 10 play a role, in particular part 1.

1. If $\mathcal{M}, \Gamma_n \Vdash_v X^*_{n+1}$ and X_{n+1} is not of the form $(\exists x)\Phi(x)$, let $\Gamma_{n+1} = \Gamma_n$.

2. If $\mathcal{M}, \Gamma_n \Vdash_v X^*_{n+1}$ and X_{n+1} is $(\exists x)\Phi(x)$, there is a parameter p and a world Δ such that $\Gamma_n\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \Vdash_v [\Phi(p)]^*$. Choose one such Δ and set $\Gamma_{n+1} = \Delta$.

3. If $\mathcal{M}, \Gamma_n \not\Vdash_v X^*_{n+1}$ and X_{n+1} is not of the form $(\forall x)\Phi(x)$, there is a world Δ such that $\Gamma_n\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \Vdash_v [\neg X_{n+1}]^*$. Choose one such Δ and set $\Gamma_{n+1} = \Delta$.

4. Suppose $\mathcal{M}, \Gamma_n \not\Vdash_v X^*_{n+1}$ and X_{n+1} is $(\forall x)\Phi(x)$. By combining various parts of the Lemma, there is a parameter p and a world Δ such that $\Gamma_n\mathcal{R}\Delta$, and $\mathcal{M}, \Delta \Vdash_v [\neg\Phi(p)]^*$, and also $\mathcal{M}, \Delta \Vdash_v [\neg X_{n+1}]^*$. Choose such a Δ and set $\Gamma_{n+1} = \Delta$.

In a complete sequence, each world is accessible from its predecessor, and for each classical formula Φ, either Φ^* is true from some point in the sequence on, or $[\neg\Phi]^*$ is true from some point on. But much more than this can be said. A classical model is associated with a complete sequence as follows. Let $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ be a complete sequence. Call a parameter p *realized* in this sequence if $\mathcal{I}(c)(\Gamma_i) = p$ for some Γ_i in the complete sequence, where $i > 0$. Let D be the set of parameters that are realized. For an n-place relation symbol P, define $I(P)$ to be the n-place relation on D such that $\langle p_1, \ldots , p_n\rangle \in \mathcal{I}(P)$ if and only if $[P(p_1, \ldots , p_n)]^*$ is true from some point on in the complete sequence. We thus have a classical model $\langle D, I\rangle$ which is *associated with* the complete sequence.

Lemma 11 Let $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ be a complete sequence, and let $\langle D, I\rangle$ be the classical model associated with it. Also, assume $\mathcal{M}, \Gamma_0 \Vdash_v \Box\mathbf{Recur}(c)$. Then, for each closed classical first-order formula Φ (allowing parameters from D), Φ is true in $\langle D, I\rangle$ if and only if $\mathcal{M}, \Gamma_i \Vdash_v \Phi^*$ for some Γ_i (and hence for all Γ_i from some point on in the complete sequence).

Proof The proof is by induction on formula complexity. For simplicity, I'll assume $\neg$, $\wedge$, and $\forall$ are the only connectives and quantifiers.

Atomic Case: Here the definition of associated model gives us what we need.

Negation Case: Assume the result for Φ. If $\neg\Phi$ is true in $\langle D, I\rangle$ then Φ is false, so by the induction hypothesis, Φ^* is not true at any world of the complete sequence. Then by construction, $[\neg\Phi]^*$ must be true at some Γ_i. In the other direction, suppose $\mathcal{M}, \Gamma_i \Vdash_v [\neg\Phi]^*$ for some Γ_i. If Φ^* held at some member of the complete sequence, there would be a single world at which both Φ^* and $[\neg\Phi]^*$ held, but this is impossible because $\neg\{\Phi^* \wedge [\neg\Phi]^*\}$ is easily verified to be **KD4** valid. Thus Φ^* fails at every member of the complete sequence, so by the induction hypothesis Φ is false in $\langle D, I\rangle$, and hence $\neg\Phi$ is true.

Conjunction Case: This is straightforward, using the **KD4** validity of $(\Phi^* \wedge \Psi^*) \equiv (\Phi \wedge \Psi)^*$. (In verifying this one needs the validity of $X^* \equiv \Box\Diamond X^*$.)

Universal Case (one direction): Suppose $(\forall x)\Phi(x)$ is true in $\langle D, I\rangle$, and the result is known for simpler formulas. Then $[(\forall x)\Phi(x)]^*$ must be true at some member of the complete sequence because otherwise, by construction, $[\neg\Phi(p)]^*$ would be true at some member of the complete sequence, for some parameter p. Then, as we saw in the Negation Case above, $[\Phi(p)]^*$ would be false at every member of the complete sequence, and so $\Phi(p)$ would be false in $\langle D, I\rangle$, which contradicts the supposition.

Universal Case (other direction): Suppose $[(\forall x)\Phi(x)]^*$ is true at Γ_i of the complete sequence, and the result is known for simpler formulas. Let p be an arbitrary member of D. Say p is realized at Γ_j, so $\mathcal{I}(c)(\Gamma_j) = p$. Since $\Box\mathbf{Recur}(c)$ is true at Γ_0, then $\mathbf{Recur}(c)$ is true at Γ_j, that is, at Γ_j we have $\langle\lambda x.\Box\Diamond\langle\lambda y.y = x\rangle(c)\rangle(c)$, and hence we also have $\Box\Diamond\langle\lambda y.y = p\rangle(c)$. Let n be the larger of i and j. Then at Γ_n we have both $[(\forall x)\Phi(x)]^*$ and $\Box\Diamond\langle\lambda y.y = p\rangle(c)$, so by Lemma 10, we have $[\Phi(p)]^*$ at Γ_n. By the induction hypothesis, $\Phi(p)$ is true in $\langle D, I\rangle$ and since p was arbitrary, $(\forall x)\Phi(x)$ is true.

Proof of Proposition 9 Suppose that $\Box\mathbf{Recur}(c) \supset \Phi^*$ is not valid in $\mathbf{KD4}\lambda_=$; say that $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ is a $\mathbf{KD4}\lambda_=$ model, $\Gamma \in \mathcal{G}$, and $\mathcal{M}, \Gamma \Vdash_v \Box\mathbf{Recur}(c)$ but $\mathcal{M}, \Gamma \nVdash_v \Phi^*$. By Lemma 10 there is some Δ with $\Gamma\mathcal{R}\Delta$ such that $\mathcal{M}, \Delta \Vdash_v [\neg\Phi]^*$. Of course also $\mathcal{M}, \Delta \Vdash_v \Box\mathbf{Recur}(c)$. Now, construct a complete sequence starting at Δ, and consider the associated model $\langle D, I\rangle$. In it $\neg\Phi$ will be true, and so Φ is not classically valid.

6 Extensions and alternatives

Proposition 12 Let Φ be a closed first-order classical formula (without parameters or equality). If Φ is classically valid then Φ^* is valid in $\mathbf{S5}\lambda$.

Proof Suppose Φ is classically valid, and let $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ be an $\mathbf{S5}\lambda$ model. Modify $\mathcal{I}$ so that it interprets the equality symbol by the equality relation on $\mathcal{D}$. Since

Φ^* does not contain equality, this has no effect on its truth or falsity at members of $\mathcal{G}$. By Proposition 9, $\Box\mathbf{Recur}(c) \supset \Phi^*$ must be valid in $\mathcal{M}$. It is not hard to see that $\mathbf{Recur}(c)$ is also valid in any $\mathbf{S5}\lambda_=$ model. It follows that Φ^* is valid in $\mathcal{M}$.

Proposition 9 can be given an alternative proof. Choose some standard axiom system for classical first-order logic. One can show, by induction on proof length, that if X has a proof, then $\Box\mathbf{Recur}(c) \supset (\forall X)^*$ is valid in $\mathbf{KD4}\lambda_=$, where $\forall X$ is the universal closure of X. There are no special tricks, but I omit the details. A similar argument, by induction on proof length, can be used to establish the following—I do not know of a direct semantic argument.

Proposition 13 Let Φ be a closed formula in the language of classical first-order logic, without equality. If Φ is classically valid then Φ° is valid in $\mathbf{S5}\lambda$.

The following connection between the two translations will play an important role in the next section. The proof is again a straightforward induction on formula degree, and is omitted.

Proposition 14 Let $\mathcal{M}$ be an $\mathbf{S5}\lambda$ model in which $\Diamond A \supset \Box A$ is valid, for all atomic A. Then, for every classical formula Φ, without equality, $\Phi^\circ \equiv \Phi^*$ is valid in $\mathcal{M}$.

7 The converse direction

The following completes the argument for Theorem 8.

Proposition 15 Let Φ be a closed classical first-order formula. If Φ is not valid classically, then Φ° is not valid in $\mathbf{S5}\lambda$. Further, Φ° fails in an $\mathbf{S5}\lambda$ model in which $\Diamond A \supset \Box A$ is valid for all atomic A, and hence Φ^* is not valid in $\mathbf{S5}\lambda$.

Proof Suppose Φ is false in the classical model $M = \langle D, I\rangle$. An $\mathbf{S5}\lambda$ model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ is specified as follows. $\mathcal{G} = \mathcal{D} = D$, that is, the domain and the collection of worlds are the same, both D. $\mathcal{R}$ always holds. For relation symbols: $\mathcal{I}(P)(\Gamma) = I(P)$ for all worlds Γ. (Hence $\Diamond A \supset \Box A$ is valid for atomic A.) For the constant symbol c, $\mathcal{I}(c)(\Gamma) = \Gamma$.

Claim: for a classical formula X, allowing free variables, X is true in M with respect to valuation v if and only if $\mathcal{M}, \Gamma \Vdash_v X^\circ$ for every $\Gamma \in \mathcal{M}$.

The proof of the claim is by induction on formula degree. Most cases are straightforward. I'll give the negation and universal quantifier cases in some detail.

Negation: Suppose $\neg\varphi$ is true in M with respect to valuation v, and the claim is known for simpler formulas. Since φ is not true in M, by the induction hypothesis φ° must be false at some world of $\mathcal{M}$, with respect to v. But $\varphi^\circ \equiv \Box\varphi^\circ$, so it must be that φ° is false at *every* world of $\mathcal{M}$, and so $\Box\neg\varphi^\circ$ is true at every world, that is, $[\neg\varphi]^\circ$ holds at every world. The converse direction is similar.

Universal Quantifier: Suppose $(\forall x)\varphi$ is true in M with respect to v, and the claim is known for simpler formulas. Let Γ be an arbitrary member of D. Then φ is true in M with respect to v', where v' is the x-variant of v such that $v'(x) = \Gamma$. By the induction hypothesis, φ° is true at every world of $\mathcal{M}$ with respect to v', and since Γ is one of the worlds, $\mathcal{M}, \Gamma \Vdash_{v'} \varphi^\circ$. But Γ is also a member of $\mathcal{D}$, and $\mathcal{I}(c)(\Gamma) = \Gamma$. It follows that $\mathcal{M}, \Gamma \Vdash_v \langle \lambda x.\varphi^\circ \rangle(c)$. Since Γ was arbitrary, $\langle \lambda x.\varphi^\circ \rangle(c)$ is true at every world, hence so is $\Box\langle \lambda x.\varphi^\circ \rangle(c)$, and thus $[(\forall x)\varphi]^\circ$ is true at every world of $\mathcal{M}$ with respect to v.

In the other direction, suppose $[(\forall x)\varphi]^\circ$ is true at every world of $\mathcal{M}$ with respect to v, and the claim is known for simpler formulas. Let Γ be an arbitrary world. Then $\mathcal{M}, \Gamma \Vdash_v \langle \lambda x.\varphi^\circ \rangle(c)$, so $\mathcal{M}, \Gamma \Vdash_{v'} \varphi^\circ$ where v' is the x-variant of v such that $v'(x) = \mathcal{I}(c)(\Gamma) = \Gamma$. But $\varphi^\circ \equiv \Box\varphi^\circ$, and so φ° is true at every world with respect to v'. Then by the induction hypothesis, φ is true in M with respect to v'. But Γ was an arbitrary world, that is, an arbitrary member of D, so v' is an arbitrary x-variant of v. Then $(\forall x)\varphi$ is true in M with respect to v.

8 Conclusion

By combining Propositions 9, 12, 13, and 15, the proof of Theorem 8 is complete. This gives us most of Theorem 1. What is missing is that, while undecidability has been established for logics between $\mathbf{KD4}\lambda_=$ and $\mathbf{S5}\lambda_=$, the range between $\mathbf{K4}\lambda_=$ and $\mathbf{KD4}\lambda_=$ has not been included. But the extension is easy, using the observation that X is $\mathbf{KD4}\lambda_=$ valid iff $(\Diamond\top \wedge \Box\Diamond\top) \supset X$ is $\mathbf{K4}\lambda_=$ valid.

In a preliminary announcement of the results of this paper there was an error, and the undecidability of logics between $\mathbf{K4}\lambda$ and $\mathbf{S5}\lambda$ was asserted. While the assertion is correct for $\mathbf{S5}\lambda$, for other logics in the range, equality had to be brought in. This still leaves, as an open problem, the decidability status of $\mathbf{L}\lambda$ for $\mathbf{L}$ between $\mathbf{K4}$ and $\mathbf{S5}$, except for $\mathbf{S5}$ itself.

There are other problems connected with the introduction of the abstraction mechanism. As I noted in Section 1, the decidability of $\mathbf{L}\lambda_=$ with function symbols is not known if $\mathbf{L}$ is one of $\mathbf{K}$, $\mathbf{D}$, $\mathbf{T}$, $\mathbf{B}$. In addition, referees for this paper suggested the following questions. For propositional modal logics that are decidable, and that remain so if an abstraction mechanism is added, are there any for which the satisfiability problem becomes computationally more complex? And, what is the status of the one-variable fragment, or of other natural subsystems for which the corresponding classical fragment is decidable?

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