What Are Justification Logics?

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Abstract

Justification logic began with Sergei Artemov's work providing an arithmetic semantics for intuitionistic logic. As part of that work, a small number of explicit modal logics were introduced—logics in which there was a structure of terms that kept track of not just what was a necessary truth, but why it was necessary. These explicit modal logics were connected with standard modal logics such as S4, T, K, and others using Realization Theorems, essentially saying that modal operators concealed an underlying informational structure. Since Artemov's work, the phenomenon of justification logic has turned out to be very broad. For instance, I have shown that infinitely many modal logics have justification counterparts. In this paper I will sketch the basics and try to give some of the ideas behind formal justification proofs, justification semantics, and realization theorems.

Keywords: justification logic, modal logic, BHK semantics, intuitionistic logic, logic of proofs, realization

1 Introduction

Justification logics have been around for about 20 years now, but they are perhaps not as well-known as they should be. While the story is still being written, much is now known. For those of us who were present from early on, the scope of the subject has proved surprising. In this expository paper I will sketch the beginnings, and say something about the current state of things—not everything, of course, but at least the basics.

2 Origins

The subject began with Sergei Artemov's work on an arithmetical semantics for intuitionistic logic. As we all know, intuitionistic logic is meant to be constructive. There is some question about what this actually means. The Brouwer, Heyting, Kolmogorov (BHK) semantics was intended to explicate this. It is based on an informal idea of *proof.* We may take $\neg X$ as abbreviating $X \supset \bot$, that is, X implies *falsum* or *absurdity.*

- $\bot$ has no proof.
- A proof of $X \wedge Y$ consists of a proof of X and a proof of Y.
- A proof of $X \vee Y$ consists of a proof of X or a proof of Y.
- A proof of $X \supset Y$ consists of an algorithm converting any proof of X into a proof of Y.

All are straightforward except for implication. Among the issues that could be raised are these. Do we just need an algorithm converting a proof of X into a proof of Y, or should we also have a verification that the algorithm is correct? If a proof of correctness is wanted, what is the status of such a proof? What kind requirements should it meet? Presumably constructive requirements, but then there is a certain circularity in what we are doing. And so on. Nonetheless, these conditions, however understood, are clearly fundamental and can serve us as guiding principles and perhaps more.

To make something precise out of the BHK approach, it was asked whether it could be used as a guide in giving an arithmetic interpretation of Intuitionistic logic (where formal arithmetic is understood classically). In 1933 Gödel made an important first step in this direction. He noted that one could characterize intuitionistic "truth" using classical validity plus a semi-formal notion of provability. Gödel proposed some precise conditions that provability should have. In fact, these conditions are those of the well-known modal logic S4. Writing $\Box X$ to symbolize that X is provable in Gödel's sense, his provability conditions are easily given. First, we build on classical logic, so we have the following, using a language in which $\Box X$ is a formula whenever X is.

- All classical tautologies.
- *modus ponens*, $X, X \supset Y \Rightarrow Y$.

Added to these are certain provability conditions.

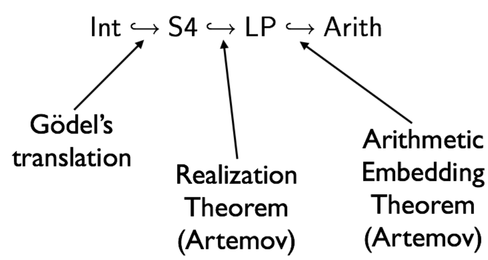
- $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$
- $\Box X \supset X$
- $\Box X \supset \Box\Box X$
- *necessitation* $X \Rightarrow \Box X$

The first of these axioms says that $\Box$ respects *modus ponens*; if X and $X \supset Y$ are provable, so is Y. The second, sometimes called *factivity*, says that anything that is provable is so. The third says that proofs can be checked; if X has a proof, that proof can be verified. These conditions have become the standard way of axiomatizing S4 and so may look very familiar.

Gödel apparently understood the BHK condition for implication as requiring a *verifiable algorithm* converting a proof of X into a proof of Y, and thus posited that proof of an implication should be represented as $\Box(\Box X \supset \Box Y)$. Following BHK, proof of a disjunction should be represented as $\Box X \vee \Box Y$, but given the S4 conditions one can prove this is equivalent to $\Box(\Box X \vee \Box Y)$, and this version is commonly used today since it parallels the structure used for implication. Similarly for conjunction. So, formally, we have the Gödel embedding, mapping propositional intuitionistic formulas into modal formulas, characterized very simply as: X^* is the result of putting $\Box$ in front of every subformula of X. For example, $[(A \wedge B) \supset A]^* = [\Box(\Box(\Box A \wedge \Box B) \supset \Box A)]$. Gödel noted, and McKinsey and Tarski proved that X is an intuitionistic theorem if and only if X^* is a theorem of S4.

Gödel thus reduced the problem of assigning an arithmetic interpretation of intuitionistic logic to that of doing something similar for S4, in effect modeling $\Box$ arithmetically. But things stopped here because, as Gödel himself noted, S4 does not embed into arithmetic, assuming one uses his provability predicate $(\exists y)(y$ is the Gödel number of a proof of $x)$ to interpret $\Box$. The problem is with the Factivity axiom. $\Box\bot \supset \bot$, or equivalently $\neg\Box\bot$, should embed as something provable but in fact it embeds as a consistency statement which, by Gödel's second incompleteness theorem, is not provable in Peano arithmetic.

In 1938 Gödel made another proposal: interpret $\Box$ as *explicit* provability. That is, introduce a representation for formal proofs, and allow different occurrences of $\Box$ to be translated using different such representations. This has the effect of moving the existential quantifier to the metalevel. Gödel's idea wasn't published until his collected papers appeared [11]. Meanwhile the idea was independently rediscovered by Sergei Artemov in the early 1990s. Artemov introduced a logic he called LP, standing for *logic of proofs*. See [1, 2] for his detailed presentations. LP is a kind of explicit modal logic. The following diagram (and commentary) shows why it is important.



Intuitionistic logic embeds into modal S4 via the mapping of Gödel, discussed earlier. On the other end, LP embeds into formal arithmetic (Peano will do) as Gödel thought would be the case. This is a result due to Artemov, and it will not concern us much here. In the middle, S4 embeds into LP, the first example of a Realization theorem. This result is also due to Artemov, and its generalizations will be a main topic in what follows. Putting all this together, intuitionistic logic has an arithmetic interpretation after all.

3 What Is LP?

In LP, and in other justification logics as well, $\Box X$ is not present; instead one finds $t{:}X$ where t is a *justification term* (originally called a *proof term* for LP). t is meant to represent a structured justification (which could be a proof) and $t{:}X$ is read: "X is so for reason t." So, what is a justification term? We begin specifically with LP, and generalize things later on.

- Variables, v_1, v_2, ... are proof terms.
- Constant symbols, c_1, c_2, ... are proof terms.
- If t and u are proof terms, so are $t + u$ and $t \cdot u$.
- If t is a proof term, so is $!t$.

Formulas are built up just as classical propositional formulas are, with one additional clause. If t is a proof term, and X is a formula, $t{:}X$ is a formula.

Here are the (informal) ideas.

- Variables stand for arbitrary justifications.
- Constants justify formulas that we do not further analyze; that is, axioms.

- $t \cdot u$ justifies X whenever u justifies some formula Y, and t justifies $Y \supset X$ (and thus $\cdot$ represents *modus ponens*).
- $t + u$ justifies X whenever t justifies X, or u justifies X (and thus $+$ represents a kind of *weakening*).
- If t justifies X, $!t$ justifies that fact (and thus ! is a *justification checker*).

An axiomatic presentation of LP will clearly embody the informal ideas just sketched. The only rule of inference is *modus ponens*. The following are axiom schemes, not particular axioms.

Classical Tautologies (one could also take an axiomatically complete subset; it makes little difference).

Application $t{:}(X \supset Y) \supset (s{:}X \supset (t \cdot s){:}Y)$

Weakening $s{:}X \supset (s + t){:}X$ and $t{:}X \supset (s + t){:}X$

Justification Checker $t{:}X \supset !t{:}(t{:}X)$

Factivity $t{:}X \supset X$

There is one more component to the axiomatization—a *constant specification*. A constant specification is a set of formulas of the form $c_1{:}c_2{:}\cdots c_n{:}X$ where X is an axiom from the list above, $n \geq 0$, and c_1, c_2, $\ldots$, c_n are justification constants. It is also required that if $c_1{:}c_2{:}\cdots c_n{:}X$ is in the constant specification, so is $c_2{:}\cdots c_n{:}X$. There are many kinds of constant specifications, but the only ones we will be interested in here are those that are *axiomatically appropriate*—for each axiom X and for each n there are constants c_1, c_2, $\ldots$, c_n so that $c_1{:}c_2{:}\cdots c_n{:}X$ is in the constant specification.

LP is axiomatized by the axioms above together with an (arbitrary) axiomatically appropriate constant specification, and with *modus ponens* as the only rule. Much more can be said about the role of constant specifications, but for what we discuss here finer details are not significant.

Here is a somewhat abbreviated example of a proof in LP.

1. $x{:}P \supset (x{:}P \vee y{:}Q)$ (tautology)
2. $a{:}(x{:}P \supset (x{:}P \vee y{:}Q))$ (given by constant specification)
3. $a{:}(x{:}P \supset (x{:}P \vee y{:}Q)) \supset (!x{:}x{:}P \supset [a{\cdot}!x]{:}(x{:}P \vee y{:}Q))$ (application)
4. $!x{:}x{:}P \supset [a{\cdot}!x]{:}(x{:}P \vee y{:}Q)$ (modus ponens)
5. $x{:}P \supset !x{:}x{:}P$ (justification checker)
6. $x{:}P \supset [a{\cdot}!x]{:}(x{:}P \vee y{:}Q)$ (classical logic, 4, 5)
7. $y{:}Q \supset [b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$ (similarly)
8. $x{:}P \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$ (weakening)
9. $y{:}Q \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$ (weakening)
10. $(x{:}P \vee y{:}Q) \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$

So we have $(x{:}P \vee y{:}Q) \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$ where a justifies the tautology $x{:}P \supset (x{:}P \vee y{:}Q)$ and b justifies the tautology $y{:}Q \supset (x{:}P \vee y{:}Q)$.

Before moving on, we note a fundamental result concerning LP. Its proof makes essential use of the presence of an axiomatically appropriate constant specification.

Theorem 3.1 (Internalization) *If X is a theorem of LP then $t{:}X$ is also a theorem for some justification term t.*

The proof constructs t so that it internalizes the steps of a proof of X. We omit details, which can be found in [1] where it appears as Corollary 5.5. There it is derived from the stronger looking Lifting Lemma (5.4), though a reversed derivation is also possible.

4 What Is Realization?

For any LP formula X, let X° be the result of replacing every justification term with $\Box$. This is called the *forgetful functor*. Here is an easy-to-prove result about it. It is easy to prove because it is easily checked to be true for axioms, and is easily seen to be preserved by modus ponens. The definitive statement can be found in [1] as Lemma 9.1, where the proof is simply said to be "straightforward."

Theorem 4.1 *If X is an LP theorem, X° is an S4 theorem.*

For LP and S4 the justification axioms were picked to make it easy to verify this. For instance, applying the forgetful functor to $s{:}(X \supset Y) \supset (t{:}X \supset [s \cdot t]{:}Y)$ gives the modal axiom $\Box(X^\circ \supset Y^\circ) \supset (\Box X^\circ \supset \Box Y^\circ)$. Much harder to prove is that there is a kind of converse. If X is a theorem of S4, there is a theorem Y of LP so that $Y^\circ = X$. Y is called a *realization* of X—essentially it replaces necessitations with structured terms that provide a kind of explanation for the correctness of the placement of the necessitation operators. Better yet, Y can always be taken to have distinct justification variables where X has negative occurrences of $\Box$. Positive $\Box$ occurrences become terms computed from these variables. Thus normal realizations show a kind of input/output structure for S4 theorems. Realizations meeting this condition are called normal realizations. For example, the S4 theorem $(\Box P \vee \Box Q) \supset \Box(\Box P \vee \Box Q)$ has the normal realization $(x{:}P \vee y{:}Q) \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$. (It is not unique.)

Realization is the essential connection between LP and S4. The result is stated in [1] as Theorem 9.4 where it appears with its original proof in a definitive form. There are several alternate proofs available now, as well.

Theorem 4.2 (Realization) *If Y is an S4 theorem, there is a normal realization X of Y such that X is an LP theorem.*

Note that this Theorem easily gives us that the forgetful functor is a mapping from S4 theorems *onto* LP theorems.

Often S4 is thought of as a logic of knowledge (with positive introspection). When it is, one often writes K (for "knows") instead of $\Box$. The S4 theorem $(KP \vee KQ) \supset K(KP \vee KQ)$ says something about the behavior of what we might call our implicit knowledge. Its realization, $(x{:}P \vee y{:}Q) \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$ makes reasoning about our knowledge explicit. If we have a reason for one of P or Q, this shows what we may do to verify that fact.

5 LP Was Just the Beginning

We've spent much time discussing modal S4 and justification LP. Of course K, T, D, K4, D4 are sublogics of S4, so if we just omit parts of the S4 and LP machinery, these are easily shown to be modal logics with justification counterparts that are connected via realization theorems.

Surprisingly, it has turned out that a broad range of canonical modal logics have justification counterparts. S5 was the first example that was not a sublogic of S4, [13, 14]. It needed new justification machinery, and realization for it introduced some new ideas that have subsequently been broadly applicable. My more recent work shows that the family of modal logics with justification counterparts is very big. Infinite, in fact, [7], though the proofs are for the most part not constructive.

Consider S4.2 as an example. Axiomatically, add to S4 the schema $\Diamond\Box X \supset \Box\Diamond X$, or equivalently, $\Box\neg\Box X \vee \Box\neg\Box\neg X$. Semantically, use S4 frames that are convergent: $u\mathcal{R}v_1$ and $u\mathcal{R}v_2$ implies there is some w with $v_1\mathcal{R}w$ and $v_2\mathcal{R}w$.

For a justification counterpart, add to LP two new two-place function symbols, f and g, and the axiom scheme $f(t,u){:}\neg t{:}X \vee g(t,u){:}\neg u{:}\neg X$. One must extend the notion of a constant specification to cover these new axioms too. We assume this in what follows, and Internalization is an immediate consequence. Call this logic J4.2. Here's an informal plausibility argument as to why the logic is interesting. LP cannot have contradictory justifications because of Factivity. In fact $\neg(t{:}X \wedge u{:}\neg X)$ is provable, and thus $\neg t{:}X \vee \neg u{:}\neg X$ is provable in LP. Informally, in any context one of the disjuncts must hold. $f(t,u){:}\neg t{:}X \vee g(t,u){:}\neg u{:}\neg X$ says we can "compute" a justification for whichever does hold.

J4.2 realizes S4.2. For example, here is an S4.2 theorem: $(\Diamond\Box A \wedge \Diamond\Box B) \supset \Diamond\Box(A \wedge B)$, or equivalently, $(\neg\Box\neg\Box A \wedge \neg\Box\neg\Box B) \supset \neg\Box\neg\Box(A \wedge B)$. Omitting the verification,

$$\vdash_{\mathsf{J4.2}} \{\neg[j_4 \cdot j_3{\cdot}!v_5 \cdot g(!v_3, j_2 \cdot v_5{\cdot}!v_9)]{:}\neg v_9{:}X \wedge \\ \neg[j_5 \cdot f(!v_3, j_2 \cdot v_5{\cdot}!v_9)]{:}\neg v_3{:}Y\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(X \wedge Y)$$

is a provable realization, where the various justification terms j_i come from Internalization, as follows.

j_1 justifies the theorem $X \supset (Y \supset (X \wedge Y))$
j_2 justifies the theorem $\neg[j_1 \cdot v_1 \cdot v_3]{:}(X \wedge Y) \supset (v_1{:}X \supset \neg v_3{:}Y)$
j_3 justifies the theorem $v_5{:}\neg[j_1 \cdot v_1 \cdot v_3](X \wedge Y) \supset \{\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}Y \supset \neg[v_2 \cdot v_1]{:}X\}$
j_4 justifies the theorem $\neg!v_9{:}v_9{:}X \supset \neg v_9{:}X$
j_5 justifies the theorem $\neg!v_3{:}v_3{:}Y \supset \neg v_3{:}Y$

The simplest justification logic, known as J, has just $\cdot$ and $+$ as operation symbols, and axiomatically has the LP axioms except for Justification Checker and Factivity. All other justification logics are built on J by the addition of new operation symbols and axioms covering them. S4.2 above is an example, but there are many, many others, as we will see.

6 How Is Realization Proved?

Broadly speaking, there are two families of realization proofs. Some realization proofs are algorithmic and hence constructive. They need, as input, not a modal validity, but a cut free modal proof. The first realization proof used a sequent calculus system for S4, [1, 2]. But it seems that cut freeness can be in sequent calculi, tableaus, nested sequents, prefixed tableaus, hypersequents;

all have been successfully used for this purpose. However, most modal logics don't have cut free proof systems of these types. The best known ones do, of course, but there are lots of others.

I introduced a non-constructive way of proving realization for S4 and LP, [5, 7]. It makes use of a semantics for justification logics that will be discussed shortly. This non-constructive approach has turned out to apply to a very wide range of *canonical* modal logics, [6]. It has a two-part structure, and that structure adapts well to the algorithmic approaches too. I won't go into details here, but stage 1 produces a *quasi-realizer*, then stage 2 converts that to a realizer, [8]. Stage 1 does not involve $+$, or substitution for justification variables, and this makes things simpler. Stage 2 is constructive, independently of the modal logic involved. There is a quasi-realizer-to-realizer algorithm based on formula structure, and not proof structure. Stage 1 may or may not be constructive. It is here that cut-free proofs come in, if available. This two-part division simplifies the overall proof structure, and makes implementation more feasible. An implementation even exists for S4 and LP, [9].

7 Models

Modal possible world models are completely standard. To fix notation, a model is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{G}$ is a set of *possible worlds*, $\mathcal{R}$ is the *accessibility relation* on worlds, and $\mathcal{V}$ assigns truth values to propositional atoms at each world. We write $\mathcal{M}, \Gamma \Vdash X$ to mean fomula X is true at possible world Γ of modal model $\mathcal{M}$. The truth conditions are boolean at each world, together with the fundamemtal: $\mathcal{M}, \Gamma \Vdash \Box X$ provided $\mathcal{M}, \Delta \Vdash X$ for every possible world Δ of $\mathcal{M}$ that is accessible from Γ.

There are several semantics that have been introduced for justification logics. The one we need here has come to be known as *Fitting semantics.* Models are now structures $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ where $\langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ is a modal model and $\mathcal{E}$ is what is called an *evidence function.* An evidence function maps terms and formulas to sets of possible worlds. The intuition is that to say possible world Γ is in $\mathcal{E}(t, X)$ is to say that at state Γ, justification term t is *relevant* evidence for X. Relevant evidence is not assumed to be conclusive. For instance, in a jury trial a judge may decide that the testimony of a witness would be relevant, but it is then up to the jury to decide on its truth.

Conditions on an evidence function depend on the justification logic in question. Since $\cdot$ and $+$ are included in every justification logic, we always assume the following conditions.

- $\mathcal{E}(t, X) \cap \mathcal{E}(s, X \supset Y) \subseteq \mathcal{E}(s \cdot t, Y)$
- $\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$

The first of these says that any state at which t is relevant to X and s is relevant to $X \supset Y$ is a state at which $s \cdot t$ is relevant to Y. The other condition is analogous. Evaluation of truth at possible worlds of a Fitting model is similar to that in modal models in that it is boolean at each world. The key new item concerns formulas of the form $t{:}X$, analogous to $\Box X$. It is as follows.

$$\mathcal{M}, \Gamma \Vdash t{:}X \text{ if and only if } \begin{cases} \mathcal{M}, \Delta \Vdash X \text{ for every } \Delta \text{ accessible from } \Gamma \\ \text{and} \\ \Gamma \in \mathcal{E}(t, X) \end{cases}$$

Loosely this says that X should be necessary at Γ in the familiar modal sense, and t should be relevant to X at Γ.

Operators on justification terms besides $\cdot$ and $+$ will depend on the particular justification logic in question, and will have their own special conditions. For instance, the ! operator of LP adds the following requirements. Corresponding to S4, frames are transitive and reflexive. In addition, $\mathcal{E}$ is monotonic, that is, $\Gamma\mathcal{R}\Delta$ implies if $\Gamma \in \mathcal{E}(t, X)$ then $\Delta \in \mathcal{E}(t, X)$. And finally, $\mathcal{E}(t, X) \subseteq \mathcal{E}(!t, t{:}X)$.

Other operators, frame conditions, and evidence function conditions are possible. The potential range is very broad.

8 How Is Completeness Proved?

All completeness theorems that have been proved so far are for justification counterparts of canonical modal logics and is shown by a justification version of a canonical model construction. The following sketches the general construction of a canonical model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ for axiomatic LP, though it applies more generally.

- $\mathcal{G}$ is the set of all LP maximally consistent sets. We use Γ, Δ for arbitrary members of $\mathcal{G}$.
- Let $\Gamma^\sharp = \{X \mid t{:}X \in \Gamma \text{ for some } t\}$. Set $\Gamma\mathcal{R}\Delta$ if $\Gamma^\sharp \subseteq \Delta$.
- $\Gamma \in \mathcal{E}(t, X)$ if $t{:}X \in \Gamma$ for some t.
- $\Gamma \in \mathcal{V}(X)$ if $X \in \Gamma$.

Now one proves the usual Truth Lemma. (In fact the proof is easier than in the modal case.) One also proves that $\langle \mathcal{G}, \mathcal{R} \rangle$ is a reflexive and transitive frame. This is a key step, and it is where justification terms must 'fit together' correctly. Also the appropriate conditions on the evidence function must be shown, but for LP this is easy. See [10] for the details, which are omitted here.

9 The Current State of Things

As we have noted several times, what works for LP and S4 works for a wide range of modal and justification logics. Here's the most general result so far. The proof is non-constructive, and the Fitting semantics just presented is essential. A full presentation can be found in [7] where the result appears as Theorem 12.1.

Theorem 9.1 *Suppose KL is a canonical modal logic and JL is an axiomatic candidate for a justification counterpart. Suppose the canonical justification model for JL is built on a frame for KL. Then a Realization Theorem connects KL and JL.*

Here is a corollary which shows the wide extent of the phenomenon of modal logics having justification counterparts. *Geach logics* are those axiomatized over modal K by axiom schemes of the form $\Diamond^k\Box^l X \supset \Box^m\Diamond^n X$, where $k, l, m, n \geq 0$. Equivalently, the schemes have the form $\Box^k\neg\Box^l X \vee \Box^m\neg\Box^n\neg X$. This restatement is because possibility is not as natural as necessity in a justification setting. Geach logics were introduced in [12] where a very general completeness argument was given. Among Geach logics are D, T, B, K4, S4, S4.2, and S5. The Lemmon-Scott methods adapt to the justification setting, and Theorem 9.1 can be used to show the following.

Corollary 9.2 *Any modal logic axiomatized by Geach formulas has a justification counterpart with a connecting Realization theorem.*

Note that this shows the family of modal logics with justification counterparts is infinite. It does not exhaust Theorem 9.1 however. The modal logic S4.3 can be shown to have a justification counterpart, but it is not a Geach logic. A plausible guess is that justification counterparts can be found for all Sahlquist logics, but that is still a guess.

To make things more complicated, there are now two modal logics, Gödel-Löb and Grzegorczyk, that are not canonical but have had justification counterparts created, and realization theorems proved. The first of these is from [15], and the second is an unpublished result of mine. For these cases, the realization proof is constructive, but no semantics is known.

It is clear that justification logics are quite general. How general is gradually being revealed, but much is known. And that's where things are now.

A general survey of justification logics can be found in [3]. Information about non-constructive proofs of realization is in [7] and [8]. Since the talk was given on which this paper was based, a book presenting the subject in considerable detail has been completed. It will appear in 2019 from Cambridge University Press, with the title "Justification Logic, Reasoning About Reasons," authored by Sergei Artemov and Melvin Fitting.

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