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Subformula Results in Some Propositional Modal Logics

1. Introduction. Say the modal logic K has been formulated axiomatically, with a rule of necessitation, but without a rule of subsitution. To turn K into T one may add to the axioms of K all formulas of the form $\Box A \supset A$. But, in demonstrating that some particular formula X is a theorem of T, not all of these new axioms will be used. We show one needs only those in which $\Box A$ is a subformula of X, the formula being demostrated in T. We also establish a similar relationship between T and $S4$, and between $S4$ and $S5$.

Our proof methods make use of Kripke's model theory [2]. Unfortunately, our methods are not general. Each result seems to require an argument with its own peculiarities, and so the techniques apparently do not even cover the relationship between B and $S5$. It would be interesting to know if our theorems could be established by a more uniform approach.

2. Preliminaries. Formulas are built up as usual, with $\wedge$, $\sim$ and $\Box$ as primitive, and $\supset$ defined.

Let L be a set of formulas. By a *derivation from* L we mean a sequence $X_1, X_2, \ldots, X_n$, of formulas, such that, for each X_i, one of:

1) X_i is a classical tautology
2) X_i is a member of L
3) For some $j, k < i$, $X_j = (X_k \supset X_i)$
4) For some $j < i$, $X_i = \Box X_j$.

We say X is *derivable from* L if X is the last term of a derivation from L. (Note that there is no rule of substitution; the members of L themselves are the axioms. This differs from, say [3].)

We write $\vdash_L X$ to mean X is derivable from L. Let S also be a set of formulas. We write $S \vdash_L X$ to mean X is derivable from $S \cup L$. Thus $S \vdash_L X$ and $\vdash_{S \cup L} X$ mean the same, but the different notation is useful for emphasis.

We adopt the usual abbreviations, and write

$$X \vdash_L Y \quad \text{for} \quad \{X\} \vdash_L Y$$
$$S, X_1, \ldots, X_n \vdash_L Y \quad \text{for} \quad S \cup \{X_1, \ldots, X_n\} \vdash_L Y$$

Let $\rightarrow$ be a new symbol. Its use is given by the following.

$S \vdash_L \emptyset \rightarrow X$ means $S \vdash_L X$

$S \vdash_L \{A_1, \ldots, A_n\} \rightarrow X$ means $S \vdash_L ((A_1 \wedge \ldots \wedge A_n) \supset X)$

And, if Γ is an infinite set of formulas, $S \vdash_L \Gamma \rightarrow X$ means, for some finite subset Δ of Γ, $S \vdash_L \Delta \rightarrow X$. (Note that $\vdash_L \Gamma \rightarrow X$ in our notation would be written $\Gamma \vdash_L X$ in [3], except for the omission of the substitution rule.)

Let f be some fixed false statement, say $(A \wedge \sim A)$. Call a set Γ of formulas *S-inconsistent in L*, if $S \vdash_L \Gamma \rightarrow f$. If Γ is not S-inconsistent in L, call *Γ S-consistent in L*. The following results hold for this notion.

1) If Δ is S-consistent in L it can be extended to a set Γ S-consistent in L-having no proper S-consistent extension, that is, to a maximal S-consistent set.

2) If $\Gamma \cup \{X\}$ and $\Gamma \cup \{\sim X\}$ are both S-inconsistent in L, so is Γ.

3) If Γ is maximal S-consistent in L, $(X \wedge Y) \in \Gamma$ iff $X \in \Gamma$ and $Y \in \Gamma$.

4) If Γ is maximal S-consistent in L, $S \subseteq \Gamma$. More generally, $S \vdash_L X$ implies $X \in \Gamma$.

5) Suppose all formulas of the form $\Box(A \supset B) \supset (\Box A \supset \Box B)$ are in L. Then if $\{\Box X_1, \ldots, \Box X_n, \sim\Box Y\}$ is S-consistent in L, so is $\{X_1, \ldots, X_n, \sim Y\}$.

We will be using Kripke models. We assume the basic results about them are known [1], [2]. We use the following notation. A *frame* is a pair $\langle G, R \rangle$ where G is a non-empty set and R is a binary relation on G. A *model* is a triple $\langle G, R, \models \rangle$ where $\langle G, R \rangle$ is a frame and $\models$ is a relation between members of G and formulas such that, for $\Gamma \in G$,

$$\Gamma \models (X \wedge Y) \quad \text{iff} \quad \Gamma \models X \quad \text{and} \quad \Gamma \models Y$$
$$\Gamma \models \sim X \quad \text{iff} \quad \text{not-}\Gamma \models X$$
$$\Gamma \models \Box X \quad \text{iff} \quad \Delta \models X \quad \text{for all} \quad \Delta \in G \text{ for which } \Gamma R \Delta.$$

A formula X is *valid* in a model $\langle G, R, \models \rangle$ if, for each $\Gamma \in G$, $\Gamma \models X$.

3. Results. Let K be the set of all formulas of the form $\Box(A \supset B) \supset (\Box A \supset \Box B)$. Let T be the set consisting of the members of K together with all formulas of the form $\Box A \supset A$.

THEOREM 1. *For each formula X, let $T(X)$ be the set of formulas of the form $\Box A \supset A$ where $\Box A$ is a subformula of X. Then $\vdash_T X$ iff $T(X) \vdash_K X$.*

PROOF. Let X be a formula, fixed for the proof. Trivially, if $T(X) \vdash_K X$ then $\vdash_T X$. Now suppose not-$T(X) \vdash_K X$. Let G consist of all maximal $T(X)$-consistent sets in K. If $\Gamma, \Delta \in G$, let $\Gamma R \Delta$ mean, for each subformula $\Box A$ of X, if $\Box A \in \Gamma$ then $A \in \Delta$. Then $\langle G, R \rangle$ is a frame.

Suppose $\Box A$ is a subformula of X, and $\Box A \in \Gamma \in G$. By item 4 of 2, $(\Box A \supset A) \in \Gamma$ and it follows that $A \in \Gamma$. Hence $\Gamma R \Gamma$, so R is reflexive.

If A is any atomic formula, set $\Gamma \models A$ if $A \in \Gamma$. Then $\models$ extends uniquely to all formulas to make $\langle G, R, \models \rangle$ a model. Suppose this done. Let A be any subformula of X, and $\Gamma \in G$. We claim $\Gamma \models A$ iff $A \in \Gamma$. This is shown by induction on the degree of A. If A is atomic, the result is true by definition. If A is $(B \wedge C)$ or $\sim B$ the result is true by item 3) in 2. If A is $\Box B$ and A is a subformula of X, so is B. Suppose the result known for B; we show it holds for $\Box B$, that is, for A.

Suppose $\Box B \in \Gamma$. Let $\Gamma R \Delta$. By definition of R, $B \in \Delta$ so $\Delta \models B$. It follows that $\Gamma \models \Box B$.

Suppose $\Box B \notin \Gamma$. Then $\sim\Box B \in \Gamma$. Let $\Gamma^0 = \{Z \mid \Box Z \in \Gamma\}$. By item 5 in 2, $\Gamma^0 \cup \{\sim B\}$ is $T(X)$-consistent in K. Extend it to a maximal $T(X)$-consistent set in K, call it Δ. Clearly $\Gamma R \Delta$; $\sim B \in \Delta$, so $B \notin \Delta$. Then not-$\Delta \models B$ by induction hypothesis, so not-$\Gamma \models \Box B$.

Thus for subformulas A of X, $\Gamma \models A$ iff $A \in \Gamma$. Now, we are supposing not-$T(X) \vdash_K X$. Then $\{\sim X\}$ is $T(X)$-consistent in K. Extend it to a maximal $T(X)$-consistent set Δ. Then $\Delta \in G$, and $X \notin \Delta$, so not-$\Delta \models X$. Thus is not valid in the model $\langle G, R, \models\rangle$. But R is reflexive, so by now-standard results [1], [2] not-$\vdash_T X$. This completes the proof.

Let *S4* be the set of formulas consisting of the members of T together with all formulas of the form $\Box A \supset \Box\Box A$.

THEOREM 2. *For each formula X, let $S4(X)$ be the set of formulas of the form $\Box A \supset \Box\Box A$ where $\Box A$ is a subformula of X. Then $\vdash_{S4} X$ iff $S4(X) \vdash_T X$.*

PROOF. Let X be some fixed formula. Suppose not-$S4(X) \vdash_T X$. Let G consist of all maximal $S4(X)$-consistent sets in T. If $\Gamma, \Delta \in G$, let $\Gamma R \Delta$ mean, for each subformula $\Box A$ of X, if $\Box^n A \in \Gamma$ then $\Box^{n-1} A \in \Delta$. We claim R is transitive (as well as reflexive).

Suppose $\Gamma R \Delta$ and $\Delta R \Omega$. Let $\Box A$ be a subformula of X, and suppose $\Box^n A \in \Gamma$. We show $\Box^{n-1} A \in \Omega$. Well, $(\Box A \supset \Box\Box A) \in S4(X)$, so $S4(X) \vdash_T (\Box A \supset \Box\Box A)$. Using the necessitation rule, and $\Box(Z \supset W) \supset (\Box Z \supset \Box W)$, $S4(X) \vdash_T (\Box^n A \supset \Box^{n+1} A)$ and so, by item 4 of 2, $(\Box^n A \supset \Box^{n+1} A) \in \Gamma$. Since $\Box^n A \in \Gamma$ then $\Box^{n+1} A \in \Gamma$. Since $\Gamma R \Delta$, $\Box^n A \in \Delta$, and since $\Delta R \Omega$, $\Box^{n-1} A \in \Omega$. Thus $\Gamma R \Omega$, and R is transitive.

The rest of the proof is a simple modification of that of Theorem 1, which we omit.

Let *S5* be the set of formulas consisting of the members of *S4* together with all formulas of the form $\sim\Box A \supset \Box\sim\Box A$.

LEMMA 3. *For each formula X, let $S5^*(X)$ be the set of all formulas of the form $\sim\Box MA \supset \Box\sim\Box MA$ where M is any string (possibly empty) of $\Box$ and $\sim$ symbols, and $\Box A$ is a subformula of X. Then $\vdash_{S5} X$ iff $S5^*(X) \vdash_{S4} X$.*

PROOF. Let X be some fixed formula. Suppose not-$S5^*(X) \vdash_{S4} X$. Let G consist of all maximal $S5^*(X)$-consistent sets in *S4*. If $\Gamma, \Delta \in G$, let $\Gamma R \Delta$ mean, for each subformula $\Box A$ of X, if $\Box MA \in \Gamma$ then $MA \in \Delta$, where M is any string of $\Box$ and $\sim$ symbols. We claim R is symmetric (as well as reflexive and transitive).

Suppose $\Gamma R \Delta$. Let $\Box A$ be a subformula of X, and suppose $\Box MA \in \Delta$. We claim $MA \in \Gamma$. For if not, $\sim MA \in \Gamma$. Then $\sim\Box MA \in \Gamma$, so $\Box\sim\Box MA \in \Gamma$. Then $\sim\Box MA \in \Delta$ since $\Gamma R \Delta$, and this is a contradiction. Thus $MA \in \Gamma$; $\Delta R \Gamma$, so R is symmetric.

The rest of the proof is similar to that of Theorem 1, so we omit it.

LEMMA 4. *Let A^* be the formula $(\sim\Box A \supset \Box\sim\Box A)\wedge(\sim\Box\sim A \supset \Box\sim\Box\sim A)$. Then each of the following is derivable from S4:*

$$A^* \supset (\sim\Box MA \supset \Box\sim\Box MA) \tag{1}$$

$$A^* \supset (\sim\Box\sim MA \supset \Box\sim\Box\sim MA) \tag{2}$$

where M is any string of $\Box$ and $\sim$ symbols.

PROOF. By induction on the length of M. If M is of length 0, the result is immediate. Now suppose the result is known for M of length m, and suppose M' is of length $m+1$. Then either $M' = \sim M$ or $M' = \Box M$.

case a) $M' = \sim M$. Then (1) for M' is the same as (2) for M. And (2) for M' follows from (1) for M on insertion of two double-negations.

case b) $M' = \Box M$. Then (1) for M' follows from (1) for M on replacing two $\Box$ symbols by $\Box\Box$. And (2) for M' follows from (1) for M using the following $S4$ theorem:

$$(\sim\Box Z \supset \Box\sim\Box Z) \supset (\sim\Box\sim\Box Z \supset \Box\sim\Box\sim\Box Z).$$

THEOREM 5. *For each formula X, let $S5(X)$ consist of all formulas of the forms $\sim\Box A \supset \Box\sim\Box A$ and $\sim\Box\sim A \supset \Box\sim\Box\sim A$, where $\Box A$ is a subformula of X. Then $\vdash_{S5} X$ iff $S5(X) \vdash_{S4} X$.*

PROOF. Immediate from the above two lemmas.

Unfortunately, the above techniques do not seem to extend very far. We give the following as an example of the difficulties.

Let B be the set of formulas consisting of the members of T together with all formulas of the form $\sim A \supset \Box\sim\Box A$.

THEOREM 6. *For each formula X, let $B^*(X)$ consist of all formulas of the form $\sim MA \supset \Box\sim\Box MA$ where M is any string of $\Box$ and $\sim$ symbols, and $\Box A$ is a subformula of X. Then $\vdash_B X$ iff $B^*(X) \vdash_T X$.*

PROOF. As usual. This time let $\Gamma R \Delta$ mean, whenever $\Box MA \in \Gamma$ then $MA \in \Delta$ where M is any string of $\Box$ and $\sim$ symbols, and $\Box A$ is a subformula of X. We claim R is symmetric. For, let $\Gamma R \Delta$, and suppose $\Box MA \in \Delta$. If $MA \notin \Gamma$, then $\sim MA \in \Gamma$, so $\Box\sim\Box MA \in \Gamma$, so $\sim\Box MA \in \Delta$, a contradicion. Now finish as in earlier proofs.

This is the analog of Lemma 3. The trouble is, there seems to be no analog to Lemma 4. The function of Lemma 4 was to replace the generally infinite set $S5^*(X)$ by the finite set $S5(X)$. Some comparable way of replacing $B^*(X)$ by a finite set would be nice. Even better would be a uniform approach to the above. Each result was obtained by a suitable complication of a standard completeness proof, but each result used a different complication. A uniform approach, whatever it may be like, should allow the extension of the above to other logics as well.

References

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