1. Introduction. Say the modal logic K has been formulated axiomatically, with a rule of necessitation, but without a rule of substitution. To turn K into T one may add to the axioms of K all formulas of the form $\Box A \supset A$. But, in demonstrating that some particular formula K is a theorem of T, not all of these new axioms will be used. We show one needs only those in which $\Box A$ is a subformula of K, the formula being demostrated in K. We also establish a similar relationship between K and K and between K and K and K between K and K and K between K between K and K between K and K between K between K and K between K between K and K between K

Our proof methods make use of Kripke's model theory [2]. Unfortunately, our methods are not general. Each result seems to require an argument with its own peculiarities, and so the techniques apparently do not even cover the relationship between B and S5. It would be interesting to know if our theorems could be established by a more uniform approach.

2. Preliminaries. Formulas are built up as usual, with \wedge , \sim and \square as primitive, and \supset defined.

Let L be a set of formulas. By a derivation from L we mean a sequence X_1, X_2, \ldots, X_n , of formulas, such that, for each X_i , one of:

- 1) X_i is a classical tautology
- 2) X_i is a member of L
- 3) For some $j, k < i, X_j = (X_k \supset X_i)$
- 4) For some $j < i, X_i = \square X_j$.

We say X is derivable from L if X is the last term of a derivation from L. (Note that there is no rule of substitution; the members of L themselves are the axioms. This differs from, say [3].)

We write $\vdash_L X$ to mean X is derivable from L. Let S also be a set of formulas. We write $S \vdash_L X$ to mean X is derivable from $S \cup L$. Thus $S \vdash_L X$ and $\vdash_{S \cup L} X$ mean the same, but the different notation is useful for emphasis.

We adopt the usual abbreviations, and write

$$\begin{matrix} X \vdash_L Y & \text{for} & \{X\} \vdash_L Y \\ S, X_1, \dots, X_n \vdash_L Y & \text{for} & S \cup \{X_1, \dots, X_n\} \vdash_L Y \end{matrix}$$

Let \rightarrow be a new symbol. Its use is given by the following.

$$\begin{array}{lll} \mathcal{S} \vdash_{L} \mathcal{O} \rightarrow X & \text{means} & \mathcal{S} \vdash_{L} X \\ \mathcal{S} \vdash_{L} \{A_{1}, \ldots, A_{n}\} \rightarrow X & \text{means} & \mathcal{S} \vdash_{L} \left((A_{1} \land \ldots \land A_{n}) \supset X \right) \end{array}$$

And, if Γ is an infinite set of formulas, $S \vdash_L \Gamma \to X$ means, for some finite subset Δ of Γ , $S \vdash_L \Delta \to X$. (Note that $\vdash_L \Gamma \to X$ in our notation would be written $\Gamma \vdash_L X$ in [3], except for the omission of the substitution rule.)

Let f be some fixed false statement, say $(A \land \sim A)$. Call a set Γ of formulas S-inconsistent in L, if $S \vdash_L \Gamma \rightarrow f$. If Γ is not S-inconsistent in L, call ΓS -consistent in L. The following results hold for this notion.

- 1) If Δ is S-consistent in L it can be extended to a set Γ S-consistent in L-having no proper S-consistent extension, that is, to a maximal S-consistent set.
 - 2) If $\Gamma \cup \{X\}$ and $\Gamma \cup \{\sim X\}$ are both S-inconsistent in L, so is Γ .
 - 3) If Γ is maximal S-consistent in L, $(X \land Y) \in \Gamma$ iff $X \in \Gamma$ and $Y \in \Gamma$.
- 4) If Γ is maximal S-consistent in L, $S \subseteq \Gamma$. More generally, $S \vdash_L X$ implies $X \in \Gamma$.
- 5) Suppose all formulas of the form $\Box(A\supset B)\supset (\Box A\supset \Box B)$ are in L. Then if $\{\Box X_1,\ldots,\Box X_n,\, \sim \Box Y\}$ is S-consistent in L, so is $\{X_1,\ldots,X_n,\, \sim Y\}$.

We will be using Kripke models. We assume the basic results about them are known [1], [2]. We use the following notation. A frame is a pair $\langle G, R \rangle$ where G is a non-empty set and R is a binary relation on G. A model is a triple $\langle G, R, \models \rangle$ where $\langle G, R \rangle$ is a frame and \models is a relation between members of G and formulas such that, for $\Gamma \in G$,

 $\Gamma \models (X \land Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$ $\Gamma \models \sim X$ iff not $\Gamma \models X$

 $\Gamma \models \Box X$ iff $\Delta \models X$ for all $\Delta \in G$ for which $\Gamma R \Delta$. A formula X is *valid* in a model $\langle G, R, \models \rangle$ if, for each $\Gamma \in G$, $\Gamma \models X$.

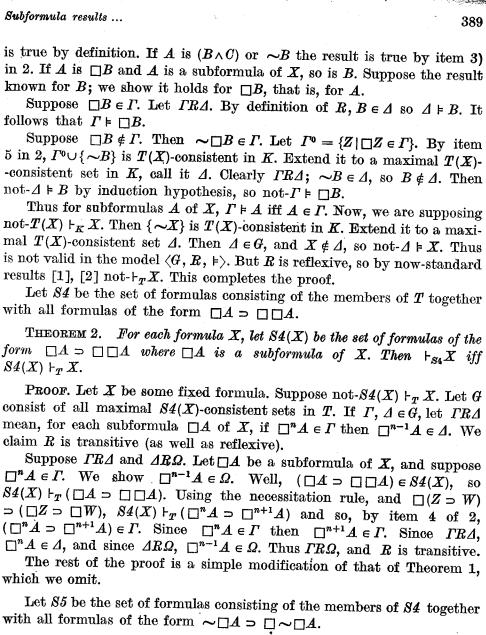
3. Results. Let K be the set of all formulas of the form $\square(A \supset B) \supset (\square A \supset \square B)$. Let T be the set consisting of the members of K together with all formulas of the form $\square A \supset A$.

THEOREM 1. For each formula X, let T(X) be the set of formulas of the form $\Box A \supset A$ where $\Box A$ is a subformula of X. Then $\vdash_T X$ iff $T(X) \vdash_K X$.

PROOF. Let X be a formula, fixed for the proof. Trivially, if $T(X) \vdash_K X$ then $\vdash_T X$. Now suppose not- $T(X) \dotplus_K X$. Let G consist of all maximal T(X)-consistent sets in K. If Γ , $\Delta \in G$, let $\Gamma R \Delta$ mean, for each subformula $\Box A$ of X, if $\Box A \in \Gamma$ then $A \in \Delta$. Then $\langle G, R \rangle$ is a frame.

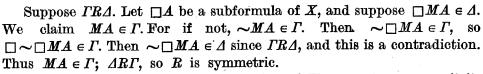
Suppose $\Box A$ is a subformula of X, and $\Box A \in \Gamma \in G$. By item 4 of 2, $(\Box A \supset A) \in \Gamma$ and it follows that $A \in \Gamma$. Hence $\Gamma R \Gamma$, so R is reflexive.

If A is any atomic formula, set $\Gamma \models A$ if $A \in \Gamma$. Then \models extends uniquely to all formulas to make $\langle G, R, \models \rangle$ a model. Suppose this done. Let A be any subformula of X, and $\Gamma \in G$. We claim $\Gamma \models A$ iff $A \in \Gamma$. This is shown by induction on the degree of A. If A is atomic, the result



LEMMA 3. For each formula X, let S5*(X) be the set of all formulas of the form $\sim \square MA \supset \square \sim \square MA$ where M is any string (possibly empty) of \square and \sim symbols, and $\square A$ is a subformula of X. Then $\vdash_{SS} X$ iff $S5*(X) \vdash_{S4} X$.

PROOF. Let X be some fixed formula. Suppose not-S5*(X) $\vdash_{S4}X$. Let G consist of all maximal S5*(X)-consistent sets in S4. If Γ , $\Delta \in G$, let $\Gamma R \Delta$ mean, for each subformula $\Box A$ of X, if $\Box M A \in \Gamma$ then $M A \in \Delta$, where M- is any string of \square and \sim symbols. We claim R is symmetric (as well as reflexive and transitive).



The rest of the proof is similar to that of Theorem 1, so we omit it.

LEMMA 4. Let A^* be the formula $(\sim \Box A \supset \Box \sim \Box A) \land (\sim \Box \sim A)$ $\supset \Box \sim \Box \sim A$. Then each of the following is derivable from S4:

$$A^* \supset (\sim \square MA \supset \square \sim \square MA) \tag{1}$$

$$A^* \supset (\sim \square \sim MA \supset \square \sim \square \sim MA) \tag{2}$$

where M is any string of \square and \sim symbols.

PROOF. By induction on the length of M. If M is of length 0, the result is immediate. Now suppose the result is known for M of length m, and suppose M' is of length m+1. Then either $M' = \sim M$ or $M' = \square M$.

case a) $M' = \sim M$. Then (1) for M' is the same as (2) for M. And (2) for M' follows from (1) for M on insertion of two double-negations.

case b) $M' = \square M$. Then (1) for M' follows from (1) for M on replacing two \square symbols by $\square \square$. And (2) for M' follows from (1) for M using the following S4 theorem:

$$(\sim \square Z \supset \square \sim \square Z) \supset (\sim \square \sim \square Z \supset \square \sim \square \sim \square Z).$$

THEOREM 5. For each formula X, let S5(X) consist of all formulas of the forms $\sim \Box A \supset \Box \sim \Box A$ and $\sim \Box \sim A \supset \Box \sim \Box \sim A$, where $\Box A$ is a subformula of X. Then $\vdash_{S5}X$ iff $S5(X) \vdash_{S4}X$.

PROOF. Immediate from the above two lemmas.

Unfortunately, the above techniques do not seem to extend very far. We give the following as an example of the difficulties.

Let B be the set of formulas consisting of the members of T together with all formulas of the form $\sim A \supset \square \sim \square A$.

THEOREM 6. For each formula X, let $B^*(X)$ consist of all formulas of the form $\sim MA \supset \square \sim \square MA$ where M is any string of \square and \sim symbols, and $\square A$ is a subformula of X. Then $\vdash_B X$ iff $B^*(X) \vdash_T X$.

PROOF. As usual. This time let $\Gamma R \Delta$ mean, whenever $\square M A \in \Gamma$ then $MA \in \Delta$ where M is any string of \square and \sim symbols, and $\square A$ is a subformula of X. We claim R is symmetric. For, let $\Gamma R \Delta$, and suppose $\square M A \in \Delta$. If $MA \notin \Gamma$, then $\sim MA \in \Gamma$, so $\square \sim \square MA \in \Gamma$, so $\sim \square MA \in \Delta$, a contradiction. Now finish as in earlier proofs.

This is the analog of Lemma 3. The trouble is, there seems to be no analog to Lemma 4. The function of Lemma 4 was to replace the generally infinite set $S5^*(X)$ by the finite set S5(X). Some comparable way of

replacing $B^*(X)$ by a finite set would be nice. Even better would be a uniform approach to the above. Each result was obtained by a suitable complication of a standard completeness proof, but each result used a different complication. A uniform approach, whatever it may be like, should allow the extension of the above to other logics as well.

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