The Strict/Tolerant Idea and Bilattices

Melvin Fitting

September 27, 2019

Abstract

Strict/tolerant logic is a formally defined logic that has the same consequence relation as classical logic, though it differs from classical logic at the metaconsequence level. Specifically, it does not satisfy a cut rule. It has been proposed for use in work on theories of truth because it avoids some objectionable features arising from the use of classical logic. Here we are not interested in applications, but in the formal details themselves. We show that a wide range of logics have strict/tolerant counterparts, with the same consequence relations but differing at the metaconsequence level. Among these logics are Kleene's K_3, Priest's LP, and first degree entailment, FDE. The primary tool we use is the *bilattice*. But it is more than a tool, it seems to be the natural home for this kind of investigation.

Keywords: strict/tolerant, bilattice, many valued logic, Kleene logic, logic of paradox, first degree entailment

1 Introduction

A natural companion to the question "What is a logic?" (which won't be asked here) is the question "When are logics the same?" It is common to say that sameness for logics means they have the same consequence relations. But then there is the curious example of ST, which stands for *strict/tolerant* for reasons that will become clear later. The idea of holding premises and conclusions of a consequence relation to different standards comes from [22, 23, 24], where the standards for premises were weaker. Today it corresponds to what is called TS, for *tolerant/strict*. Holding premises to stronger standards was introduced in [14, 15] and today is called ST. It turns out that the ST consequence relation coincides with that of classical logic, but a good case has been made that ST is not identical with classical logic because the two differ at the metaconsequence level. In fact, [5] shows there is a hierarchy of logical pairs, with ST and classical logic at the bottom, where each pair agrees at the consequence level, the metaconsequence level, the metametaconsequence level, and so on up to some arbitrary finite level, and then they differ at the next level. Very curious indeed, and very interesting.

In this paper we examine a different sort of generalization of the ST phenomenon: wide instead of high. We show there is a family of logic pairs consisting of an ST-like logic and a corresponding classical-like logic, where each pair agrees on consequences but differs on metaconsequences. We do not examine working our way up the meta,

meta^2, meta^3, ... hierarchy as in [5]. Instead we complicate the structure of the truth value space itself, of course going to three values and beyond. We set up the basics for study, but we leave the meta levels to another time or to other people.

The machinery we use comes from bilattice theory, with the original ST/classical example as the simplest case. We sketch the necessary bilattice background, to keep this paper relatively self-contained.

2 ST, Classical Logic, and one new example

Logics can be specified proof theoretically, or semantically. In this paper we make no use of proof theoretic methods. The work is entirely semantic.

Many valued logics are specified by giving a set of truth values, an interpretation for propositional connectives, and a specification of what counts as "true." A bit more precisely, let $\mathcal{T}$ be a non-empty set of truth values, and for each logical connective (in this paper conjunction, disjunction, and negation) assume we have a corresponding operation on $\mathcal{T}$. We will overload the use of the symbols $\wedge$, $\vee$ and $\neg$ to serve as logical connectives and also as operations on $\mathcal{T}$, with context determining which is intended. And finally a non-empty proper subset $\mathcal{D}$ of the truth value space is specified as the *designated truth values*, often with some structural properties imposed.

With respect to a many valued logic, a valuation v is a mapping from propositional variables to truth values, that is, to $\mathcal{T}$. A valuation extends to all formulas in the usual way, for instance setting $v(X \wedge Y) = v(X) \wedge v(Y)$, where on the left $\wedge$ is an operation symbol, and on the right $\wedge$ is the corresponding operation on $\mathcal{T}$. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas. For a valuation v we write $v \Vdash \Gamma \Rightarrow \Delta$ provided, if $v(X) \in \mathcal{D}$ for every $X \in \Gamma$ then $v(Y) \in \mathcal{D}$ for some $Y \in \Delta$. More informally, $v \Vdash \Gamma \Rightarrow \Delta$ provided that if every member of Γ is designated under v then some member of Δ also is. A sequent $\Gamma \Rightarrow \Delta$ is *valid* in a many valued logic provided, for every valuation v in that logic, $v \models \Gamma \Rightarrow \Delta$. We take this notion of validity as determining the consequence relation of the many valued logic.

Among the best-known three-valued logics are Kleene's strong, K_3, from [19, 20], and Priest's logic of paradox, LP, from [25] but with truth tables originating in [3]. We can take the truth values of both to be 0, $\frac{1}{2}$ and 1. (Other names for these values will also be used from time to time in this paper.) The intended intuition is that in K_3 the value $\frac{1}{2}$ represents a truth value *gap* while in LP it represents a *glut*. Either way, the truth tables for propositional operators turn out to be the same. Assume we have an ordering so that $0 \le \frac{1}{2} \le 1$. Conjunction, $\wedge$, is greatest lower bound (equivalently minimum in this case); disjunction, $\vee$, is least upper bound (or maximum); and negation, $\neg$, is an order reversal so that $\neg 0 = 1$, $\neg 1 = 0$, and $\neg\frac{1}{2} = \frac{1}{2}$. The two logics differ in their choice of designated values. For K_3 the designated value set is $\{1\}$ while for LP it is $\{\frac{1}{2}, 1\}$. We do not go into the motivation for these choices; discussions are available in many places—see [26] for instance.

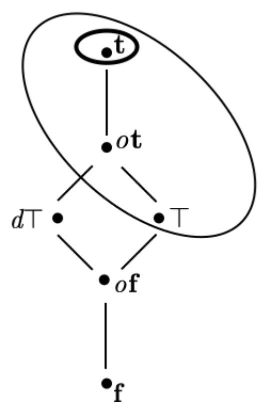
The logic known as ST combines aspects of both K_3 and LP through a mixed definition of consequence. Note that since the space of truth values is the same for K_3 and for LP, and the behavior of logical connectives is the same, these two standard logics have the same valuation behavior. $\Gamma \Rightarrow \Delta$ is taken to be valid in ST provided, for every valuation, if every member of Γ is designated in the K_3 sense, then some member of

Δ is designated in the LP sense. Now the reason for the name *strict/tolerant* becomes a bit clearer: members of the antecedent Γ are held to stricter standards, only 1 is acceptable, while we are more tolerant with members of Δ accepting both 1 and $\frac{1}{2}$.

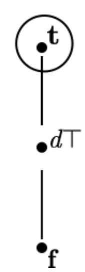
Of course classical logic also fits the many valued paradigm. Truth values are 0 and 1, with $\wedge$ and $\vee$ defined as greatest lower bound and least upper bound respectively, and negation as order reversal. $\{1\}$ is the set of designated truth values. And $\Gamma \Rightarrow \Delta$ is defined in the expected way: every valuation mapping all members of Γ to a designated value must map some member of Δ to a designated value.

The important connection between ST and classical logic is very simply stated: they have the same consequence relation, see [7, 5] among other places.

But it has been argued that they still are not the same logics because they differ at the metainference level. In particular, classical logic validates the cut rule but ST does not, and there are other metainferences on which they differ as well. Current work in [5] generalizes this result upward, as we discussed in section 1. We will generalize it laterally. We will show there is an abundance of pairs of many valued logics where one logic is analogous to ST, the other to classical logic, such that both agree on consequence but differ on metaconsequence. Indeed, there are strict/tolerant analogs for strong Kleene logic itself, for the logic of paradox of Priest, and for first degree entailment. We present one example now, to give an idea of things. It will, perhaps, seem a bit mysterious, but motivations and proofs for our assertions will come later on.



(a) $\mathsf{ST}\langle\mathcal{NINE}, \{\mathbf{t}, o\mathbf{t}, \top\}\rangle$



(b) $\mathsf{C}\langle\mathcal{NINE}, \{\mathbf{t}, o\mathbf{t}, \top\}\rangle$, or Kleene's K_3

Figure 1: A Strict/Tolerant Pair

In Figure 1 two lattices are shown. The lattice names will be explained in Section 7. The truth value names trace back to Ginsberg, d is supposed to represent default, for instance. Here the truth value names play no role other than letting us specify what node we are talking about. Think of both lattices as having an ordering relation, $\leq$, represented graphically as upwards (with reflexivity tacitly assumed). For both lattices, $\wedge$ and $\vee$ are interpreted as greatest lower and least upper bound respectively. Negation is order reversal for both, so $\neg d\top = d\top$ in each, for instance. For the lattice in Figure 1b the only designated truth value is $\mathbf{t}$, shown circled, thus this is a presentation of strong Kleene logic, K_3. For the lattice in Figure 1a we introduce both a strict and a tolerant designated set, analogous to what is done with the logic ST. In the present example, the strict set of truth values is $\{\mathbf{t}\}$, shown heavily circled, and the tolerant set is $\{\mathbf{t}, o\mathbf{t}, \top\}$, shown lightly circled. We say $\Gamma \Rightarrow \Delta$ is valid in the resulting strict/tolerant logic provided that for every valuation, if every member of Γ is strictly designated then some member of Δ is tolerantly designated. It will be shown later on that the logics corresponding to these two lattices have the same consequence relation, but differ at the metaconsequence level, and are thus connected in the same way that ST and classical logic are. As we said earlier, we do not analyze higher level differences in this paper.

3 ST and $\mathcal{FOUR}$

Our unifying machinery will be bilattices. Before discussing the general machinery we begin with the paradigm example, the Belnap-Dunn system, called $\mathcal{FOUR}$. This was presented in a very influential paper, [6]. Its truth values were intended to represent sets of ordinary truth values, only true ($\mathbf{t}$), only false ($\mathbf{f}$), neither ($\bot$), both ($\top$). It has two partial orderings, one on degree of truth, one on degree of information. All this is shown in Figure 2, in which the information ordering is vertical, and is denoted $\leq_k$. This has become customary in bilattice literature, with k standing for *knowledge*, though i for *information* would probably be better. The truth ordering is denoted $\leq_t$ and is shown horizontally.

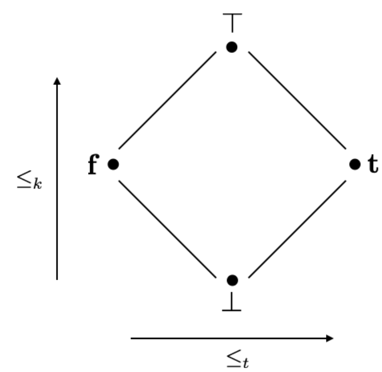


Figure 2: The Bilattice $\mathcal{FOUR}$

Each of the two orderings gives us the structure of a bounded, distributive lattice. For the truth ordering, greatest lower bound is symbolized using $\wedge$ and least upper bound by $\vee$. A negation operation, denoted $\neg$, is a horizontal symmetry, $\neg\mathbf{t} = \mathbf{f}$, $\neg\mathbf{f} = \mathbf{t}$, $\neg\top = \top$ and $\neg\bot = \bot$. The De Morgan laws hold, so with respect to $\leq_t$ we have a De Morgan algebra. The $\leq_k$ ordering plays an important role, but we postpone discussion until we have introduced the full notion of bilattice, for which $\mathcal{FOUR}$ is the simplest non-trivial example.

In order to turn $\mathcal{FOUR}$ into a many valued logic, a set of designated truth values must be specified. This is taken to be $\{\mathbf{t}, \top\}$, which one can think of as *at least true*. The values we would naturally think of as consistent are $\mathbf{f}$, $\bot$, $\mathbf{t}$, and the $\leq_t$ ordering, restricted to them, gives us the operations of the strong Kleene logic, $\mathsf{K_3}$. Likewise the set of designated truth values of $\mathcal{FOUR}$, restricted to $\{\mathbf{f}, \bot, \mathbf{t}\}$, gives us

$\{\mathbf{t}\}$, appropriate for $\mathsf{K_3}$. Similarly $\leq_t$ restricted to $\mathbf{f}$, $\top$, $\mathbf{t}$ gives us the operations of LP, and the set of designated truth values of $\mathcal{FOUR}$, similarly restricted, gives us $\{\mathbf{t}, \top\}$, appropriate for LP.

When working with ST we need both three valued logics $\mathsf{K_3}$ and LP and, although their representation in $\mathcal{FOUR}$ as described above is quite natural, it has the consequence of giving us different carrier sets for the two logics, with one containing $\perp$ and the other $\top$. To avoid this, we do not work with the representation of $\mathsf{K_3}$ just described. Instead we work with the set $\{\mathbf{f}, \top, \mathbf{t}\}$, and we refer to $\{\mathbf{t}, \top\}$ as *tolerantly* designated, and $\{\mathbf{t}\}$ as *strictly* designated. That is, we have one space of truth values, and two versions of designated value. We will do something similar for other bilattices, when we come to them.

Much more can be said about $\mathcal{FOUR}$, but this is enough for the time being. It is better to continue our discussion after the general family of bilattices has been introduced.

4 Bilattices

A *bilattice* is an algebraic structure with two lattice orderings. Various conditions can be imposed, connecting the orderings. We start at the simplest level.

A *pre-bilattice* is a structure $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ where each of $\leq_t$ and $\leq_k$ are bounded partial orderings on $\mathcal{B}$. (Notice that we overload $\mathcal{B}$ to stand for both the structure with its orderings, and for its domain. This should cause no confusion since context can sort things out. We do similar things with other structures as well.) Think of the members of domain $\mathcal{B}$ as generalized truth values. The relation $\leq_t$ is intended to order degree of truth in some sense (though it was noted in [27] that the ordering is really about truth-and-falsity, and that to separate the two something more complex than a bilattice is needed, namely a *trilattice*. We do not persue this point here). Meet and join operations with respect to this ordering are denoted $\wedge$ and $\vee$, and the least and greatest elements are denoted $\mathbf{f}$ and $\mathbf{t}$. The other relation, $\leq_k$, is intended to order degree of information, again in some sense. The meet operation with respect to this ordering is denoted $\otimes$ and is called *consensus*; the join operation is denoted $\oplus$ and is called *gullability*, or sometimes *accept all*. The least and greatest elements with respect to this ordering are denoted $\perp$ and $\top$. The Belnap-Dunn structure $\mathcal{FOUR}$ from Figure 2 is the simplest pre-bilattice.

If a pre-bilattice has an operation $\neg$ that reverses $\leq_t$, preserves $\leq_k$, and is an involution, such an operation is simply called *negation*. Formally, the conditions are as follows.

($\mathsf{Neg\text{-}1}$) $a \leq_t b$ implies $\neg b \leq_t \neg a$

($\mathsf{Neg\text{-}2}$) $a \leq_k b$ implies $\neg a \leq_k \neg b$

($\mathsf{Neg\text{-}3}$) $\neg\neg a = a$

Bilattices were introduced by Ginsberg in [17, 18], who defined a bilattice to be a pre-bilattice with negation (though without using the terminology 'pre-bilattice'). $\mathcal{FOUR}$ is the simplest bilattice in Ginsberg's sense. In any such bilattice, $\neg\mathbf{t} = \mathbf{f}$, $\neg\mathbf{f} = \mathbf{t}$, $\neg\top = \top$, $\neg\perp = \perp$. It is not hard to show that we also have De Morgan's Laws for the t operations and something akin to them for the k operations.

(NDeM-1) $\neg(a \wedge b) = (\neg a \vee \neg b)$

(NDeM-2) $\neg(a \vee b) = (\neg a \wedge \neg b)$

(NDeM-3) $\neg(a \otimes b) = (\neg a \otimes \neg b)$

(NDeM-4) $\neg(a \oplus b) = (\neg a \oplus \neg b)$

A pre-bilattice may have a negation-like operation with respect to $\leq_k$ as well. If one exists it is denoted $-$ and is called *conflation*, with the following conditions.

(Con-1) $a \leq_k b$ implies $-b \leq_k -a$

(Con-2) $a \leq_t b$ implies $-a \leq_t -b$

(Con-3) $- - a = a$

(Con-4) $-\neg a = \neg - a$

The last condition, that negation and conflation commute, is occasionally not assumed, but will be here. $\mathcal{FOUR}$ is an example of a bilattice with conflation, where $-\top = \bot$, $-\bot = \top$, $-\mathbf{t} = \mathbf{t}$, $-\mathbf{f} = \mathbf{f}$. When conflation is present, we have dual versions of the De Morgan laws given earlier.

(CDeM-1) $-(a \wedge b) = (-a \wedge -b)$

(CDeM-2) $-(a \vee b) = (-a \vee -b)$

(CDeM-3) $-(a \otimes b) = (-a \oplus -b)$

(CDeM-4) $-(a \oplus b) = (-a \otimes -b)$

Monotonicity conditions for the operations with respect to the ordering defining it are standard, because we have lattice structures. Thus, for instance, $a \leq_t b$ implies $a \wedge c \leq_t b \wedge c$. A bilattice is called *interlaced* if such conditions hold across the two orderings. More precisely, we have interlacing if the following hold.

(Int-1) $a \leq_t b$ implies $a \otimes c \leq_t b \otimes c$

(Int-2) $a \leq_t b$ implies $a \oplus c \leq_t b \oplus c$

(Int-3) $a \leq_k b$ implies $a \wedge c \leq_k b \wedge c$

(Int-4) $a \leq_k b$ implies $a \vee c \leq_k b \vee c$

In any interlaced bilattice $\mathbf{f} \wedge \mathbf{t} = \bot$, $\mathbf{f} \vee \mathbf{t} = \top$, $\bot \otimes \top = \mathbf{f}$, and $\bot \oplus \top = \mathbf{t}$. Once again $\mathcal{FOUR}$ is an example, this time of an interlaced bilattice. There are bilattices that are not interlaced. $\mathcal{DEFAULT}$, shown in Figure 3, is an example of one. In it $\mathbf{f} \leq_t d\mathbf{f}$ but $\mathbf{f} \otimes d\top = d\top \not\leq_t d\mathbf{f} = d\mathbf{f} \otimes d\top$. $\mathcal{DEFAULT}$ goes back to [17], but will play no further role here.

The following plays an important role in [4] when establishing representation theorems for interlaced bilattices. Representation theorems are discussed in Section 8.

Proposition 4.1 *In an interlaced bilattice:*

(1) *if* $a \leq_k b$ *then* $a \leq_k x \leq_k b$ *if and only if* $a \wedge b \leq_t x \leq_t a \vee b$;

(2) *if* $a \leq_t b$ *then* $a \leq_t x \leq_t b$ *if and only if* $a \otimes b \leq_k x \leq_k a \oplus b$.

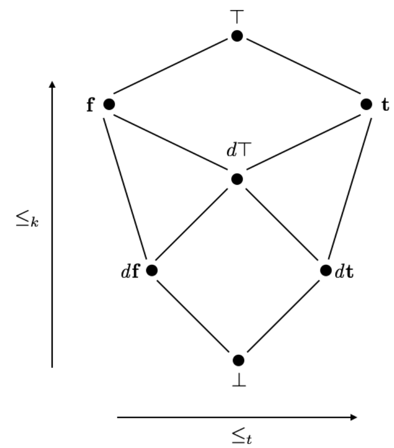


Figure 3: The Bilattice $\mathcal{DEFAULT}$

Proof We give the proof of the first, taken from [4], to give an idea of the uses of interlacing. The second part is similar. Throughout, assume $a \leq_k b$ (it is actually needed in only one part).

Suppose $a \leq_k x \leq_k b$. Using interlacing, $a \vee a \vee b \leq_k x \vee a \vee b \leq_k b \vee a \vee b$, and hence $a \vee b \leq_k x \vee a \vee b \leq_k a \vee b$. Then $x \vee a \vee b = a \vee b$ and so $x \leq_t a \vee b$. By a dual argument, $a \wedge b \leq_t x$, and so $a \wedge b \leq_t x \leq_t a \vee b$.

Now suppose $a \wedge b \leq_t x \leq_t a \vee b$. Using interlacing, $a \otimes (a \wedge b) \leq_t a \otimes x \leq_t a \otimes (a \vee b)$. We have $a \leq_k b$ so by interlacing again, $a = a \wedge a \leq_k a \wedge b$ and hence $a \otimes (a \wedge b) = a$. Similarly $a \otimes (a \vee b) = a$. Then $a \leq_t a \otimes x \leq_t a$, so $a \otimes x = a$, and so $a \leq_k x$. By a dual argument, $x \leq_k b$. ■

A bilattice is *distributive* if all possible distributive laws hold. For instance, not only should $\wedge$ and $\vee$ distribute over each other, as in $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, but over $\otimes$ and $\oplus$ as well, for example $a \wedge (b \otimes c) = (a \wedge b) \otimes (a \wedge c)$. Altogether there are 12 such distributive laws combining $\wedge$, $\vee$, $\otimes$, and $\oplus$.

$\mathcal{FOUR}$ is a distributive bilattice. Figure 4 shows a distributive bilattice, $\mathcal{NINE}$, a bit more complex than $\mathcal{FOUR}$. This time the node names come from [2]. It is rather easy to show that every distributive bilattice is interlaced. The converse is not true.

In Section 8 we will discuss bilattice representation theorems, which will help account for where our examples are coming from.

5 Consistent, AntiConsistent, Exact

The bilattice $\mathcal{FOUR}$ from Figure 2 is already complex enough to contain a subset consisting of classical truth values, a subset of consistent truth values appropriate for Kleene's strong three valued logic K_3, and a subset of what we might call anticonsistent truth values, appropriate for Priest's logic of paradox, LP. We next give structural conditions that single these sets out, and we suggest that the analogous sets in other bilattices should play analogous roles. For the rest of this section, $\mathcal{B}$ is an interlaced bilattice with a negation and a conflation.

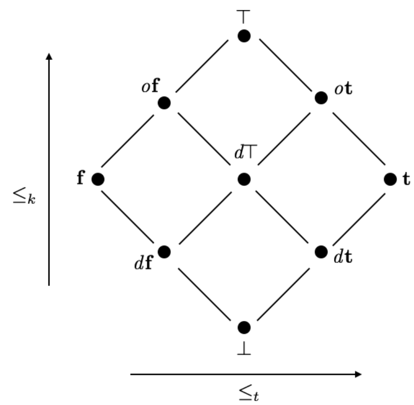


Figure 4: The Bilattice $\mathcal{NINE}$

Definition 5.1 *$a \in \mathcal{B}$ is consistent if $a \leq_k -a$, anticonsistent if $-a \leq_k a$, and exact if $a = -a$.*

In $\mathcal{FOUR}$, as desired, the consistent values are $\{\mathbf{f}, \bot, \mathbf{t}\}$, those of Kleene's logic, the anticonsistent values are $\{\mathbf{f}, \top, \mathbf{t}\}$, those of Priest's logic, and the exact values are the familiar classical $\{\mathbf{f}, \mathbf{t}\}$. In $\mathcal{NINE}$ the exact values are $\{\mathbf{f}, d\top, \mathbf{t}\}$, the consistent values are the exact ones together with $\{d\mathbf{f}, \bot, d\mathbf{t}\}$, and the anticonsistent values are the exact ones plus $\{o\mathbf{f}, \top, o\mathbf{t}\}$. The following says certain features of $\mathcal{FOUR}$ extend quite generally to interlaced bilattices with negation and conflation.

Proposition 5.2 *In $\mathcal{B}$, the sets of exact values, consistent values, and anticonsistent values each contain $\mathbf{f}$ and $\mathbf{t}$, and are closed under $\wedge$, $\vee$, and $\neg$, while $\bot$ is consistent and $\top$ is anticonsistent.*

Proof Suppose a, b are both consistent. Then $a \leq_k -a$ and $b \leq_k -b$. Using (Int-3) and (CDeM-1), $a \wedge b \leq_k -a \wedge -b = -(a \wedge b)$. Hence $a \wedge b$ is consistent. All the other claims have similar proofs. ■

The following says that every consistent value is below an exact value, and every anticonsistent value is above an exact value.

Proposition 5.3 *For $a \in \mathcal{B}$:*

1. *if a is consistent then $a \leq_k b$ for some exact b,*
2. *if a is anticonsistent then $b \leq_k a$ for some exact b.*

Proof We show part 2; part 1 is similar. Suppose a is anticonsistent, so that $-a \leq_k a$. Using interlacing, $a \wedge -a \leq_k a \wedge a = a$. Let $b = a \wedge -a$. Then $b \leq_k a$, and b is exact because $-b = -(a \wedge -a) = -a \wedge - - a = -a \wedge a = a \wedge -a = b$. ■

Proposition 5.4 *For $a, b \in \mathcal{B}$, if $a \leq_k b$ and both a and b are exact, then $a = b$.*

Proof If $a \leq_k b$ then $-b \leq_k -a$, and if also a and b are both exact, $b \leq_k a$. ■

It is not the case that exact, consistent, anticonsistent is always an exhaustive classification. Figure 6, discussed in Section 9, shows a bilattice that is distributive, hence is interlaced, and has a conflation. But in it neither $\langle \bot, \bot \rangle$ nor $\langle \top, \top \rangle$ is exact, consistent, or anticonsistent.

6 Logical Bilattices

For this section, as in the previous one, $\mathcal{B}$ is an interlaced bilattice with negation and conflation.

Definition 6.1 The set of *logical formulas* is built up from a set of propositional letters, typically P, Q, ..., using the binary symbols $\wedge$, $\vee$ and $\neg$.

Note that there is no implication. A discussion of implication in the bilattice context can be found in [2], also see [28].

Definition 6.2 A *valuation* in bilattice $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ is a mapping v from the set of propositional letters to members of $\mathcal{B}$. Valuations extend uniquely to the set of all logical formulas in the familiar way

$$\begin{aligned} v(X \wedge Y) &= v(X) \wedge v(Y) \\ v(X \vee Y) &= v(X) \vee v(Y) \\ v(\neg X) &= \neg v(X) \end{aligned}$$

and we will use the same symbol v for this extension too.

Proposition 6.3 *If a valuation v in a bilattice maps every propositional letter to a consistent truth value, it maps every formula to a consistent truth value. Similarly for the exact truth values, and for the anticonsistent truth values.*

Proof Immediate, by Proposition 5.2 ■

Valuations have an important monotonicity property that is fundamental to Kripke-style theories of truth, [9, 12, 13]. Though formal work on self-reference and truth does not concern us here, monotonicity retains its importance.

Proposition 6.4 *Let v and w be valuations in bilattice $\mathcal{B}$. If $v(P) \leq_k w(P)$ for every propositional letter P then $v(X) \leq_k w(X)$ for every logical formula X.*

Proof This is an immediate consequence of (Neg-2) for negation and the interlacing conditions (Int-3) and (Int-4). ■

As noted earlier, in $\mathcal{FOUR}$ a particular set of *designated* truth values is standard, $\{\mathbf{t}, \top\}$. Its properties were nicely generalized in [2].

Definition 6.5 A *prime bifilter* on $\mathcal{B}$ is a non-empty subset $\mathcal{F}$ of $\mathcal{B}$ that is not the entire of $\mathcal{B}$ and that meets the following conditions.

(PBif-1) $(a \wedge b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ and $b \in \mathcal{F}$

(PBif-2) $(a \otimes b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ and $b \in \mathcal{F}$

(PBif-3) $(a \vee b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ or $b \in \mathcal{F}$

(PBif-4) $(a \oplus b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ or $b \in \mathcal{F}$

A *logical bilattice* is a pair $\langle \mathcal{B}, \mathcal{F} \rangle$ where $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

$\mathcal{FOUR}$ has exactly one prime bifilter, $\{\mathbf{t}, \top\}$. $\mathcal{NINE}$ has two prime bifilters, $\{\mathbf{t}, o\mathbf{t}, \top\}$ and $\{\mathbf{t}, o\mathbf{t}, \top, d\mathbf{t}, d\top, o\mathbf{f}\}$.

Proposition 6.6 *A prime bifilter is upward closed in both bilattice orderings.*

Proof Suppose $\mathcal{F}$ is a prime bifilter, $a \in \mathcal{F}$, and $a \leq_k b$. Then $b = a \oplus b$ so $b \in \mathcal{F}$ by (PBif-4) of Definition 6.5. The case of the t ordering is similar. ∎

In Section 2 a definition of validity for a sequent in a many valued logic was given. That definition includes the case of a logical bilattice $\langle \mathcal{B}, \mathcal{F} \rangle$ once we specify that the prime bifilter $\mathcal{F}$ is the set of designated values.

In [1, 2] a very nice result is shown: the valid sequents of any logical bilattice are the same as they are for $\mathcal{FOUR}$ using the prime bifilter $\{\mathbf{t}, \top\}$.

7 A Family of Strict/Tolerant Logics

In Section 3 we reformulated ST so that it was incorporated into the structure of $\mathcal{FOUR}$. The role of the three-member space of truth values common to LP and to K_3 was played by the "upper" part of $\mathcal{FOUR}$, which amounts to the anticonsistent part. The set of what we called the tolerantly designated truth values was $\{\mathbf{t}, \top\}$, the only prime bifilter for $\mathcal{FOUR}$. The set of strictly designated truth values was $\{\mathbf{t}\}$, the subset of the prime bifilter consisting of the exact values. This now is the paradigm for our generalization. We begin by setting up the machinery we need, and then prove our general theorems on the existence of a family of ST-like logics. Given all the work that has gone into the development of bilattices over the years, this theorem is quite easy to establish. It is the family of logics, and the bilattice setting in which they appear that is significant.

Definition 7.1 Let $\mathcal{B}$ be an interlaced bilattice with negation and conflation, and let $\mathcal{F}$ be a prime bifilter on $\mathcal{B}$, so that $\langle \mathcal{B}, \mathcal{F} \rangle$ is a logical bilattice. Throughout this definition we write $\mathcal{A}$ for the set of anticonsistent members of $\mathcal{B}$, and $\mathcal{E}$ for the set of exact members.

(1) $\mathcal{D}_t\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{A}$, the subset of $\mathcal{F}$ consisting of anticonsistent members of $\mathcal{B}$. This is our *tolerant* set of designated values.

(2) $\mathcal{D}_s\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{E}$, the subset of $\mathcal{F}$ consisting of exact members of $\mathcal{B}$. This is our *strict* set of designated values.

(3) $\mathsf{ST}\langle \mathcal{B}, \mathcal{F} \rangle$ is the analog of strict/tolerant logic associated with the logical bilattice $\langle \mathcal{B}, \mathcal{F} \rangle$. Its set of truth values is $\mathcal{A}$. A sequent $\Gamma \Rightarrow \Delta$ is valid in this logic provided, for every valuation v mapping propositional letters to $\mathcal{A}$, if v maps every formula in Γ to $\mathcal{D}_s\langle \mathcal{B}, \mathcal{F} \rangle$ then v maps some formula in Δ to $\mathcal{D}_t\langle \mathcal{B}, \mathcal{F} \rangle$.

(4) $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$ is the analog of classical logic associated with the logical bilattice $\langle \mathcal{B}, \mathcal{F} \rangle$. Its set of truth values is $\mathcal{E}$, with $\mathcal{D}_s\langle \mathcal{B}, \mathcal{F} \rangle$ as the set of designated truth values. A sequent $\Gamma \Rightarrow \Delta$ is valid in this logic provided, for every valuation v mapping propositional letters to $\mathcal{E}$, if v maps every formula in Γ to $\mathcal{D}_s\langle \mathcal{B}, \mathcal{F} \rangle$ then v maps some formula in Δ to $\mathcal{D}_s\langle \mathcal{B}, \mathcal{F} \rangle$.

A few remarks before moving to a central result. Since our logical formulas only contain $\wedge$, $\vee$, and $\neg$, in evaluating formulas in the various structures above only the $\leq_t$ ordering comes into play. As we noted in Proposition 5.2, both $\mathcal{A}$ and $\mathcal{E}$ are closed under $\wedge$, $\vee$, and $\neg$.

In $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$, the set $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle = \mathcal{F}\cap\mathcal{E}$ is a prime filter. For instance, suppose $a, b \in \mathcal{E}$, the set of truth values of $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$, and $a \vee b \in \mathcal{F} \cap \mathcal{E}$. Then $a \vee b \in \mathcal{F}$ and so one of a or b is in $\mathcal{F}$ since it is a prime bifilter in $\mathcal{B}$. So one of of a or b is in $\mathcal{F} \cap \mathcal{E}$. Similarly for the other prime filter conditions.

Similar remarks apply partially to $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$. Here the set $\mathcal{D}_t\langle\mathcal{B},\mathcal{F}\rangle = \mathcal{F} \cap \mathcal{A}$ of tolerant truth values will constitute a prime filter within the set of anticonsistent truth values, which is the domain used for $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$. This does not extend to the set $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle = \mathcal{F} \cap \mathcal{E}$ of strict truth values. For instance, in the strict/tolerant FDE example shown much later in Figure7, $\langle \mathbf{t}, \bot\rangle \vee \langle \mathbf{t}, \top\rangle = \langle \mathbf{t}, \mathbf{f}\rangle$, which is in the set $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$, but neither $\langle \mathbf{t}, \bot\rangle$ nor $\langle \mathbf{t}, \top\rangle$ is in this set.

Proposition 7.2 *Let $\langle\mathcal{B},\mathcal{F}\rangle$ be a logical bilattice, where $\mathcal{B}$ is an interlaced bilattice with negation and conflation. The logics $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$ and $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$ validate the same sequents.*

Proof

Left to Right: Assume $\Gamma \Rightarrow \Delta$ is valid in $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$; we show $\Gamma \Rightarrow \Delta$ is valid in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$.

Let v be a valuation mapping propositional letters to $\mathcal{E}$, and suppose v maps every formula in Γ to $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$; we show v maps some formula in Δ to $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$. Since $\Gamma \Rightarrow \Delta$ is valid in $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$ and v maps all of Γ to $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$, then for some $Y \in \Delta$, $v(Y) \in \mathcal{D}_t\langle\mathcal{B},\mathcal{F}\rangle$. But by Proposition 6.3, $v(Y)$ must be exact, and so in $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$.

Right to Left: Assume $\Gamma \Rightarrow \Delta$ is *not* valid in $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$. We show $\Gamma \Rightarrow \Delta$ is *not* valid in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$.

By our assumption there is a valuation v mapping propositional letters to $\mathcal{A}$, the anticonsistent members of $\mathcal{B}$, mapping every formula in Γ to $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$, but for some $Y \in \Delta$, $v(Y)$ is not in $\mathcal{D}_t\langle\mathcal{B},\mathcal{F}\rangle$.

Define a new valuation v' as follows. For each propositional letter P, if $v(P)$ is exact, let $v'(P) = v(P)$. If $v(P)$ is anticonsistent but not exact, by Proposition 5.3, there is some exact $a \leq_k v(P)$; choose one such a and set $v'(P) = a$. By its definition v' maps all propositional letters to exact members of $\mathcal{B}$, and hence by Proposition 6.3, v' maps every logical formula to an exact member of $\mathcal{B}$. We show $v' \not\models \Gamma \Rightarrow \Delta$ in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$.

Since $v'(P) \leq_k v(P)$ for each propositional letter then for every logical formula X, $v'(X) \leq_k v(X)$ by Proposition 6.4. Since both v and v' map all members of Γ to exact members of $\mathcal{B}$ then, by Proposition 5.4. v and v' agree on members of Γ. So v' maps every member of Γ to $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$.

We have a logical formula $Y \in \Delta$ such that $v(Y) \notin \mathcal{D}_t\langle\mathcal{B},\mathcal{F}\rangle$. We show $v'(Y) \notin \mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$, which will finish the proof. Well, otherwise $v'(Y)$ would be exact (which it is) and in the prime bifilter $\mathcal{F}$. But $v'(Y) \leq_k v(Y)$ and prime bifilters are upward closed in both bilattice orderings, Proposition 6.6, so $v(Y)$ would be in $\mathcal{F}$ (which it is not).

■

It has been vehemently argued whether or not, despite validating the same sequents, classical logic and strict/tolerant logic are the same logic. See [5] for a good summary of this, as well as further references to the issue. Their difference is that they do not agree at the metaconsequence level, something that has been generalized upward as we noted at the beginning of this paper. A similar phenomenon applies to the bilattice based generalizations considered in this paper, and with the same examples.

A metaconsequence is represented by the following general form.

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma_0 \Rightarrow \Delta_0}$$

Here the members of Γ_i and Δ_i are taken to be schemata. Validity is understood to mean each instance of such a scheme is valid. Validity for an instance, with respect to a logic, actually has two versions, local and global. The global version is: if each sequent above the line is valid in the logic, so is the sequent below. The local version is: for each valuation, if that valuation validates each sequent above the line then that valuation validates the sequent below. Local is easily seen to imply global. It is the local version that is appropriate here. The particular metaconsequence scheme of interest is the familiar one of cut.

Proposition 7.3 *Let $\langle\mathcal{B},\mathcal{F}\rangle$ be a logical bilattice, where $\mathcal{B}$ is an interlaced bilattice with negation and conflation. The metaconsequence scheme*

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}$$

is locally valid in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$ but not in $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$.

Proof

Local Validity in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$: Assume we have a specific instance of the Cut scheme, and let v be a mapping from propositional letters to exact members of $\mathcal{B}$. Reasoning in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$ we show that if $v \not\models \Gamma \Rightarrow \Delta$ then either $v \not\models \Gamma, A \Rightarrow \Delta$ or $v \not\models \Gamma \Rightarrow \Delta, A$.

Assume $v \not\models \Gamma \Rightarrow \Delta$. Then $v(X) \in \mathcal{F}$ for every $X \in \Gamma$ and $v(Y) \notin \mathcal{F}$ for every $Y \in \Delta$. Either $v(A) \in \mathcal{F}$ or $v(A) \notin \mathcal{F}$. If we have the first, then $v(X) \in \mathcal{F}$ for every X in Γ, A, so $v \not\models \Gamma, A \Rightarrow \Delta$. If we have the second, then $v(Y) \notin \mathcal{F}$ for every $Y \in \Delta, A$, so $v \not\models \Gamma \Rightarrow \Delta, A$.

Local Non-Validity in $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$: Let $\Gamma \Rightarrow \Delta$ be any specific sequent that is not valid in $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$, and let P be a propositional letter that does not occur in Γ or in Δ. Let us say v is a valuation such that $v \not\models \Gamma \Rightarrow \Delta$ in $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle$. That is, v maps propositional letters to anticonsistent members of $\mathcal{B}$, maps every member of Γ to $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$, and maps no member of Δ to $\mathcal{D}_t\langle\mathcal{B},\mathcal{F}\rangle$.

Since P does not occur in Γ or Δ we are free to reassign a value to P without affecting the behavior of v on Γ or Δ. The bilattice value $\top$ must be in $\mathcal{F}$ because $\mathcal{F}$ is non-empty and we have Proposition 6.6. Set $v(P) = \top$. Then $v \models \Gamma, P \Rightarrow \Delta$ because v does not map every member of Γ, P to $\mathcal{D}_s\langle\mathcal{B},\mathcal{F}\rangle$, since $\top$ is anticonsistent but not exact. But also $v \models \Gamma \Rightarrow \Delta, P$ because v maps some member of Δ, P to $\mathcal{D}_t\langle\mathcal{B},\mathcal{F}\rangle$ since $v(P) = \top$ is anticonsistent and is in $\mathcal{F}$.

Thus v is a counterexample to the local validity, in $\mathsf{ST}\langle\mathcal{B}, \mathcal{F}\rangle$, of the following metainference

$$\frac{\Gamma, P \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, P}{\Gamma \Rightarrow \Delta}.$$

■

We conclude this section with a few examples. Starting in Section 10 we discuss where such examples 'really' come from.

Example 7.4 In Figure 2 we gave the ur-bilattice, $\mathcal{FOUR}$. For it the exact values are just $\mathbf{f}$ and $\mathbf{t}$, the classical ones, and the anticonsistent ones are these together with $\top$. The only prime bifilter is $\{\mathbf{t}, \top\}$ which, if taken as designated in the set of anticonsistent values, gives us LP. Then $\mathsf{ST}\langle\mathcal{FOUR}, \{\mathbf{t}, \top\}\rangle$ is the usual strict/tolerant logic ST while $\mathsf{C}\langle\mathcal{FOUR}, \{\mathbf{t}, \top\}\rangle$ is just classical logic, and the two theorems above specialize to what are, in effect, the beginnings of the subject.

Example 7.5 Figure 4 shows a bilattice, $\mathcal{NINE}$, having two prime bifilters. We, quite arbitrarily, choose to work with the smaller one, $\{\mathbf{t}, o\mathbf{t}, \top\}$. The exact members of $\mathcal{NINE}$ are $\{\mathbf{f}, d\top, \mathbf{t}\}$ and the overlap with the prime bilfilter contains just $\mathbf{t}$. Thus the analog of classical logic from the original ST example turns out to be K_3. There are six anticonsistent values, and these, displayed a bit differently, are shown in Figure 1.

8 Bilattice Representation Theorems

Where do bilattices come from? There is an intuitively appealing way of constructing them that is completely representative, in the sense that every bilattice with 'reasonable' properties is isomorphic to a bilattice constructed in this way. In bilattice history this construction dates from [17], with subsequent extensions by others. In fact, many of the ideas predate bilattices as such, though that was not generally known until later. See [16] for an interesting prehistory. In this section we sketch the ideas, without the proofs, and then add an extension that will be applied to the present investigation in Section 9.

A central intuition for truth values in a bilattice is that they encode evidence for and evidence against an assertion, treating positive and negative evidence independently. An interesting family of examples is based on groups of experts. Suppose we have one group whose members announce their opinions for something, or don't, and another group similarly announcing opinions against, or keeping silent. The two groups could be distinct, overlap, or be identical. We can identify the opinions in favor with the set of experts declaring for, and similarly for the set of experts against. In this way a generalized truth value becomes a pair of sets of experts, the set of those for, and the set of those against. We have an increase in knowledge, or more properly information, if additional experts declare their opinions. We have an increase in degree of truth (understood loosely) if additional experts declare in favor while some withdraw from declaring against. This is a good model to have in mind while reading the following, but it is not fully general. The collection of all sets of experts, drawn from some fixed group, is a lattice under the subset ordering, but not all lattices are of this kind, hence the move to general lattice structures, L_1 intuitively representing evidence for, and L_2 intuitively representing evidence against.

Definition 8.1 (Bilattice Product) Let $L_1 = \langle L_1, \leq_1 \rangle$ and $L_2 = \langle L_2, \leq_2 \rangle$ be bounded lattices. Their *bilattice product* is defined as follows.

$$\begin{gathered} L_1 \odot L_2 = \langle L_1 \times L_2, \leq_t, \leq_k \rangle \\ \langle a, b \rangle \leq_k \langle c, d \rangle \text{ iff } a \leq_1 c \text{ and } b \leq_2 d \\ \langle a, b \rangle \leq_t \langle c, d \rangle \text{ iff } a \leq_1 c \text{ and } d \leq_2 b \end{gathered}$$

Note the reversal of the $\leq_2$ ordering in the definition of $\leq_t$. The following items are now rather straightforward to check. In stating the results we assume that 0_1 and 0_2 are the least members of L_1 and L_2, and 1_1 and 1_2 are the greatest. We write $\sqcup_1$ and $\sqcup_2$ for the respective joins, and $\sqcap_1$ and $\sqcap_2$ for the meets. If the two lattices are identical, we omit subscripts. Also recall that a De Morgan algebra is a bounded distributive lattice with a De Morgan involution, written here as an overbar, such that $\overline{a \sqcap b} = \overline{a} \sqcup \overline{b}$ and $\overline{\overline{a}} = a$. (The other De Morgan law follows.) It is often the case in what follows that the distributivity laws of De Morgan algebras are not needed. By a *non-distributive De Morgan algebra* we mean something meeting the conditions for a De Morgan algebra except, possibly, satisfaction of the distributive laws.

(BP-1) $L_1 \odot L_2$ is always a pre-bilattice that is interlaced. In $L_1 \odot L_2$ the extreme elements are $\perp = \langle 0_1, 0_2 \rangle$, $\top = \langle 1_1, 1_2 \rangle$, $\mathbf{f} = \langle 0_1, 1_2 \rangle$, and $\mathbf{t} = \langle 1_1, 0_2 \rangle$. The bilattice operations evaluate to the following.

$$\begin{aligned} \langle a, b \rangle \wedge \langle c, d \rangle &= \langle a \sqcap_1 c, b \sqcup_2 d \rangle \\ \langle a, b \rangle \vee \langle c, d \rangle &= \langle a \sqcup_1 c, b \sqcap_2 d \rangle \\ \langle a, b \rangle \otimes \langle c, d \rangle &= \langle a \sqcap_1 c, b \sqcap_2 d \rangle \\ \langle a, b \rangle \oplus \langle c, d \rangle &= \langle a \sqcup_1 c, b \sqcup_2 d \rangle \end{aligned}$$

(BP-2) If L_1 and L_2 are distributive lattices then $L_1 \odot L_2$ is a distributive bilattice.

(BP-3) If $L_1 = L_2 = L$ then $L \odot L$ is an bilattice with negation, where $\neg \langle a, b \rangle = \langle b, a \rangle$.

(BP-4) If $L_1 = L_2 = L$ is a non-distributive De Morgan algebra then $L \odot L$ is a bilattice with a conflation that commutes with negation, where $-\langle a, b \rangle = \langle \overline{b}, \overline{a} \rangle$.

Combining several of the items above, if L is a De Morgan algebra (which assumes distributivity), then $L \odot L$ is a distributive bilattice with a negation and a conflation that commute.

What is more difficult to establish is that these conditions reverse. For instance, if we have an interlaced bilattice, it is isomorphic to $L_1 \odot L_2$, where L_1 and L_2 are bounded lattices, and L_1 and L_2 are unique up to isomorphism. And so on. Thus we have very general *representation theorems*. These results were proved over time, and various parts can be found in [17, 10, 11, 4].

We will not need a detailed proof of these representation theorems but a few basic items from the proof will be of importance to us, since we will be adding one more piece. For an interlaced bilattice $\mathcal{B}$, L_1 can be taken to be $\{x \vee \perp \mid x \in \mathcal{B}\}$ and L_2 to be $\{x \wedge \perp \mid x \in \mathcal{B}\}$, each with the ordering resulting when $\leq_t$ is restricted to L_1 or L_2 respectively. If we have a bilattice with negation the lattices L_1 and L_2 just described are isomorphic and we can simply use L consisting of $\{x \vee \perp \mid x \in \mathcal{B}\}$ with the ordering induced by $\leq_t$. In Proposition 8.3 we make use of these pieces of the proof to add one new part to the representation theorem collection.

Suppose $\mathcal{B}$ is a bilattice with negation, $L = \{x \vee \bot \mid x \in \mathcal{B}\}$, and $f : \mathcal{B} \to L$ is defined by $f(x) = x \vee \bot$. This mapping is always many-one. For instance if $\mathcal{B} = \mathcal{NINE}$ from Figure 4, $f(\top) = f(o\mathbf{t}) = f(\mathbf{t}) = \mathbf{t}$. Even in the paradigm case of $\mathcal{FOUR}$ from Figure 2, $f(\top) = f(\mathbf{t}) = \mathbf{t}$. Thus each member of the lattice L, generated by the proof of the representation theorem, always has multiple pre-images in the bilattice $\mathcal{B}$ that we are representing. We will show that there are special and unique pre-images, of particular significance in our current strict/tolerant investigation. These are simply the *exact* members (provided we have the conflation machinery to define them).

The following Lemma provides everything we need for our proof of the central role of exact bilattice members. Using the bilattice representation results above, much of it could be left as an exercise in computation. Instead we give direct proofs, which provide some insights of their own.

Lemma 8.2 *Assume $\mathcal{B}$ is an interlaced bilattice with a negation and a conflation. For every $x, y \in \mathcal{B}$:*

(1) $(x \vee \bot) \wedge -(x \vee \bot)$ *is exact;*

(2) $x = (x \vee \bot) \wedge (x \vee \top)$*;*

(3) $(x \wedge y) \vee \bot = (x \vee \bot) \wedge (y \vee \bot)$*;*

(4) $[(x \vee \bot) \wedge -(x \vee \bot)] \vee \bot = x \vee \bot$*;*

(5) *if x and y are exact then $x \vee \bot \leq_t y \vee \bot$ if and only if $x \leq_t y$;*

(6) *if x and y are exact and $x \vee \bot = y \vee \bot$ then $x = y$.*

Proof

(1) Exactness is simple.

$$\begin{aligned} -[(x \vee \bot) \wedge -(x \vee \bot)] &= [-(x \vee \bot) \wedge --(x \vee \bot)] \\ &= [-(x \vee \bot) \wedge (x \vee \bot)]. \end{aligned}$$

(2) (This is Corollary 2.8 part 4 in [4].) Since $\bot \leq_k x \leq_k \top$, using interlacing, $x \vee \bot \leq_k x \vee x \leq_k x \vee \top$, and so $x \vee \bot \leq_k x \leq_k x \vee \top$. Then by Proposition 4.1, $(x \vee \bot) \wedge (x \vee \top) \leq_t x \leq_t (x \vee \bot) \vee (x \vee \top)$ so in particular, $(x \vee \bot) \wedge (x \vee \top) \leq_t x$. Also $x \leq_t x \vee \bot$ and $x \leq_t x \vee \top$, so $x \leq_t (x \vee \bot) \wedge (x \vee \top)$.

(3) From $\bot \leq_k x$ by interlacing, $x \vee \bot \leq_k x \vee x = x$. Similarly $y \vee \bot \leq_k y$. Then $\bot \leq_k (x \vee \bot) \wedge (y \vee \bot) \leq_k x \wedge y$. Then by Proposition 4.1, $(x \wedge y) \wedge \bot \leq_t (x \vee \bot) \wedge (y \vee \bot) \leq_t (x \wedge y) \vee \bot$, so in particular $(x \vee \bot) \wedge (y \vee \bot) \leq_t (x \wedge y) \vee \bot$. Also $x \wedge y \leq_t x$ so $(x \wedge y) \vee \bot \leq_t (x \vee \bot)$, and similarly $(x \wedge y) \vee \bot \leq_t (y \vee \bot)$. Then $(x \wedge y) \vee \bot \leq_t (x \vee \bot) \wedge (y \vee \bot)$.

(4) Using item (3),

$$\begin{aligned} [(x \vee \bot) \wedge -(x \vee \bot)] \vee \bot &= [(x \vee \bot \vee \bot) \wedge (-(x \vee \bot) \vee \bot)] \\ &= [(x \vee \bot) \wedge (-x \vee \top \vee \bot)] \\ &= [(x \vee \bot) \wedge (-x \vee \mathbf{t})] \\ &= [(x \vee \bot) \wedge \mathbf{t}] \\ &= x \vee \bot \end{aligned}$$

(5) If $x \le_t y$ then $x \vee \bot \le_t y \vee \bot$, using the interlacing conditions. In the other direction, suppose $x \vee \bot \le_t y \vee \bot$ and both x and y are exact. Then

$$
\begin{aligned}
x &= (x \vee \bot) \wedge (x \vee \top) \quad \text{part (2)} \\
&= (x \vee \bot) \wedge (-x \vee \top) \quad \text{exactness} \\
&= (x \vee \bot) \wedge -(x \vee \bot) \\
&\le_t (y \vee \bot) \wedge -(y \vee \bot) \quad \text{interlacing} \\
&= (y \vee \bot) \wedge (-y \vee \top) \\
&= (y \vee \bot) \wedge (y \vee \top) \quad \text{exactness} \\
&= y \quad \text{part (2)}
\end{aligned}
$$

(6) This follows from part (5).

■

Proposition 8.3 *Suppose L is a non-distributive De Morgan algebra, and $\mathcal{B} = L \odot L$. The set of exact members of $\mathcal{B}$, under the ordering $\le_t$, is isomorphic to L.*

Proof The proofs of the usual bilattice representation theorems discussed earlier say that $\mathcal{B}$ is isomorphic to $L' \odot L'$ where $L' = \{x \vee \bot \mid x \in \mathcal{B}\}$ with ordering $\le_t$ restricted to L'. They also say this is unique up to isomorphism, so L and L' are isomorphic. It is enough, then, to show that L' and $\mathcal{E}$ are isomorphic, where $\mathcal{E}$ is the set of exact members of $\mathcal{B}$.

Let $f : \mathcal{E} \to L'$ be defined by $f(x) = x \vee \bot$. We show that f is $1-1$, onto, and an order isomorphism. We begin with onto. An arbitrary member of L' must be $x \vee \bot$ for some $x \in \mathcal{B}$. Let y be $(x \vee \bot) \wedge -(x \vee \bot)$. By Lemma 8.2 part (1), y is exact and by part (4), $f(y) = x \vee \bot$. Hence f is onto. It is $1-1$ by Lemma 8.2 part (6). Finally, we have an order isomorphism by Lemma 8.2 part (5). ■

9 Logical De Morgan Algebras

Quite a few common many valued logics validate De Morgan's laws. An extensive investigation of these can be found in [21], where applications to the theory of truth were examined. In that paper, being a prime filter was one of the conditions considered for the set of designated truth values. We will take it as central here, and we investigate the resulting family with respect to its relation to bilattices. Actually, since the distributive laws assumed in De Morgan algebras play little role here, we use the more general family of *non-distributive* De Morgan algebras. Everything we say applies, of course, if we also have distributivity.

Definition 9.1 (Non-Distributive Logical De Morgan Algebras) Let L be a non-distributive De Morgan algebra (writing $\sqcap$ and $\sqcup$ for meet and join, and overbar for De Morgan complement). A subset D of L is a *prime filter* in L if it meets the following two conditions:

1. $a \sqcap b \in D$ if and only if $a \in D$ and $b \in D$
2. $a \sqcup b \in D$ if and only if $a \in D$ or $b \in D$.

We call the pair $\langle L, D\rangle$ a non-distributive *logical De Morgan algebra*, thinking of it as a many valued logic with D as the set of designated truth values.

We will show that each member of the family of logics determined by non-distributive logical De Morgan algebras has a strict/tolerant version. Classical logic is determined by the best known De Morgan example, and so is part of a large family with strict/tolerant logics.

Using the bilattice construction sketched in Section 8, if L is a non-distributive De Morgan algebra then $L \odot L$ is an interlaced bilattice with negation and conflation. This can be extended from algebras to logics, as we will now show.

Lemma 9.2 *Let $\langle L, D\rangle$ be a non-distributive De Morgan logic. Then $\langle L \odot L, D \times L\rangle$ is a logical bilattice (interlaced, with negation and conflation).*

Proof Given earlier items, all that needs to be shown is that $D \times L$ is a prime bifilter in $L \odot L$, Definition 6.5. In the following, $\langle x, y\rangle$ and $\langle z, w\rangle$ are any two members of $L \odot L$. Since we are working with L throughout, membership in L is automatic and can be mentioned or dropped whenever useful. We show one prime bifilter case as sufficiently representative:

$$\begin{aligned}
\langle x, y\rangle \vee \langle z, w\rangle \in D \times L \text{ iff } & \langle x \sqcup z, y \sqcap w\rangle \in D \times L \\
\text{iff } & x \sqcup z \in D \\
\text{iff } & x \in D \text{ or } z \in D \\
\text{iff } & (x \in D \text{ and } y \in L) \text{ or } (z \in D \text{ and } w \in L) \\
\text{iff } & \langle x, y\rangle \in D \times L \text{ or } \langle z, w\rangle \in D \times L.
\end{aligned}$$

■

Proposition 9.3 *Let $\langle L, D\rangle$ be a non-distributive logical De Morgan algebra. $\langle L, D\rangle$ is isomorphic to the bilattice based logic structure $\mathsf{C}(\langle L\odot L, D\times L\rangle)$ from Definition 7.1. To state this more precisely, first recall that $\mathsf{C}(\langle L \odot L, D \times L\rangle)$ is the many valued logic $\langle \mathcal{E}, (D \times L) \cap \mathcal{E}\rangle$, where $\mathcal{E}$ is the set of exact members of $L \odot L$. Then, there is an isomorphism between $\mathcal{E}$ and L that pairs the members of $(D \times L) \cap \mathcal{E}$ with those of D.*

Proof Begin with a non-distributive logical De Morgan algebra $\langle L, D\rangle$, and construct the interlaced bilattice $L \odot L$. Proposition 8.3 says the set of exact members of $L \odot L$ is isomorphic to L. We can extract more information from the proof of that proposition. We know the mapping $f(x) = x \vee \bot$ maps the exact members of the bilattice isomorphicallly to a non-distributive De Morgan algebra $L' = \{x \vee \bot \mid x \in L \odot L\} = \{x \vee \bot \mid x \in L \odot L \text{ and } x \text{ exact}\}$. And further, L' must be isomorphic to L. We now examine the details of the mapping f.

Let $\langle a, b\rangle$ be an arbitrary member of $L \odot L$. Then $f(\langle a, b\rangle) = \langle a, b\rangle \vee \bot = \langle a, b\rangle \vee \langle 0, 0\rangle = \langle a \sqcup 0, b \sqcap 0\rangle = \langle a, 0\rangle$. Now the isomorphism from L' to L is obvious: $\langle a, 0\rangle \mapsto a$. Then further, the mapping from $L \odot L$ to L is in fact just $\langle a, b\rangle \mapsto a$, and so it was this that was shown in the proof of Proposition 8.3 to be an order preserving isomorphism when restricted to the exact members of $L \odot L$. So to finish the present proof, we must show this mapping is $1 - 1$ and onto between $(D \times L) \cap \mathcal{E}$ and D.

We first show the mapping $\langle a, b\rangle \mapsto a$, restricted to $(D \times L) \cap \mathcal{E}$, is onto D. Suppose $a \in D$. Of course $\langle a, \overline{a}\rangle \in D \times L$ and $-\langle a, \overline{a}\rangle = \langle \overline{\overline{a}}, \overline{a}\rangle = \langle a, \overline{a}\rangle$, so $\langle a, \overline{a}\rangle$ is exact. Thus $\langle a, \overline{a}\rangle \in (D \times L) \cap \mathcal{E}$, and of course $\langle a, \overline{a}\rangle \mapsto a$.

Finally we show the mapping $\langle a, b\rangle \mapsto a$, restricted to the exact members of $L \odot L$, is $1-1$. To show this it is enough to show that if $\langle a, b\rangle$ and $\langle a, c\rangle$ are both exact, then $\langle a, b\rangle = \langle a, c\rangle$. If $\langle a, b\rangle$ is exact, $\langle a, b\rangle = -\langle a, b\rangle = \langle \overline{b}, \overline{a}\rangle$, so $a = \overline{b}$. Similarly $a = \overline{c}$, and it follows that $\overline{b} = \overline{c}$ and hence $b = c$. ■

10 Generating Strict/Tolerant Examples

We now have everything we need for the central result of this paper.

Proposition 10.1 *For each non-distributive logical De Morgan algebra there is a strict/tolerant logic having the same consequence relation but differing from it at the metaconsequence level. There is an algorithm for constructing the strict/tolerant logic from the logical De Morgan algebra.*

Proof We present the algorithm and cite the various earlier results proven earlier that establish what we need.

Gen-1 Start with a (non-distributive) logical De Morgan algebra, $\langle L, D\rangle$.

Gen-2 $\langle L \odot L, D \times L\rangle$ is an interlaced logical bilattice with negation and conflation.

Gen-3 Using notation from Definition 7.1, $\mathsf{ST}\langle L \odot L, D \times L\rangle$ is a strict/tolerant logic analog and $\mathsf{C}\langle L \odot L, D \times L\rangle$ is a classical logic analog.

Gen-4 By Proposition 7.2, $\mathsf{ST}\langle L \odot L, D \times L\rangle$ and $\mathsf{C}\langle L \odot L, D \times L\rangle$ validate the same sequents.

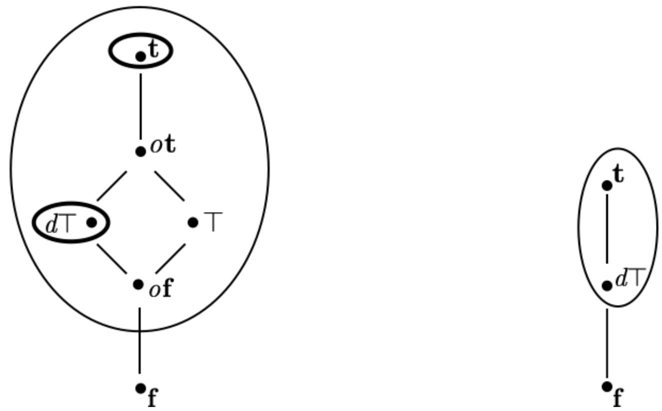
Gen-5 By Proposition 7.3, $\mathsf{ST}\langle L \odot L, D \times L\rangle$ and $\mathsf{C}\langle L \odot L, D \times L\rangle$ differ at the metaconsequence level.

Gen-6 Finally, the structure $\mathsf{C}\langle L \odot L, D \times L\rangle$ is isomorphic to the logical De Morgan algebra $\langle L, D\rangle$ with which we began, by Proposition 9.3.

■

Example 10.2 Continuing Example 7.4. Let L be the lattice $\{0, 1\}$ with the ordering $0 \leq 1$, and let D be $\{1\}$. $\langle L, D\rangle$ is not just some logical De Morgan algebra but is that of classical logic, the most basic of all. The bilattice product $L \odot L$ is isomorphic to $\mathcal{FOUR}$ from Figure 2, with $\perp$ corresponding to $\langle 0, 0\rangle$, $\mathbf{f}$ to $\langle 0, 1\rangle$, $\mathbf{t}$ to $\langle 1, 0\rangle$, and $\top$ to $\langle 1, 1\rangle$. The logical bilattice $\langle L \odot L, D \times L\rangle$ is then isomorphic to $\mathcal{FOUR}$ with $\{\mathbf{t}, \top\}$ as designated values. It follows that $\mathsf{C}(\langle L \odot L, D \times L\rangle)$ is classical logic and $\mathsf{ST}(\langle L \odot L, D \times L\rangle)$ is the usual version of strict/tolerant logic, ST.

Example 10.3 Continuing Example 7.5. We start with the Kleene strong three valued logic, $\{0, \frac{1}{2}, 1\}$, with $\{1\}$ as designated truth value. These give us a (distributive) logical De Morgan algebra, $\mathsf{K}_3 = \langle \{0, \frac{1}{2}, 1\}, \{1\}\rangle$, Kleene's strong three-valued logic. We use this to create the logical bilatice, $\langle \mathsf{K}_3 \odot \mathsf{K}_3, \{1\} \times \mathsf{K}_3\rangle$, which is isomorphic to $\mathcal{NINE}$, from Figure 4. Then, as discussed in Example 7.5, this generates the strict/tolerant logic pair from Figure 1.



(a) Strict/Tolerant LP (b) Standard LP

Figure 5: A Strict/Tolerant Counterpart of LP

Example 10.4 This time we do something like Example 10.3 but modify the work so that we produce a strict/tolerant counterpart of LP, the logic of paradox, instead of K_3. Formally, the only difference between LP and K_3 is the choice of designated truth values. For LP, from $\{0, \frac{1}{2}, 1\}$ we take $\{\frac{1}{2}, 1\}$ as designated, so we have the (again distributive) logical De Morgan algebra $\langle\{0, \frac{1}{2}, 1\}, \{\frac{1}{2}, 1\}\rangle$. The bilattice $\mathcal{NINE}$, from Figure 4 is still the bilattice we must work with (isomorphically). The prime bifilter we now want from $\mathcal{NINE}$ is $\{d\mathbf{t}, \mathbf{t}, d\top, o\mathbf{t}, o\mathbf{f}, \top\}$ (though note that $d\mathbf{t}$ is not anticonsistent) and the intersection of this with the exact members is $\{d\top, \mathbf{t}\}$. The details are much like those of Example 10.3 and we wind up with the diagrams shown in Figure 5, which can be compared with the earlier ones. The strictly designated values are $\{d\top, \mathbf{t}\}$ and the tolerantly designated values are $\{\mathbf{t}, d\top, o\mathbf{t}, o\mathbf{f}, \top\}$.

As a simple instance, it is well-known that $P, \neg P \Rightarrow Q$ is not valid in LP, as the valuation $v(P) = d\top$, $v(Q) = \mathbf{f}$ shows. The same valuation, in the strict/tolerant version also works as a counterexample.

Example 10.5 We start with the bilattice $\mathcal{FOUR}$, shown in Figure 2. As is standard, we take $\{\mathbf{t}, \top\}$ as designated truth values. Using the $\leq_t$ ordering, the resulting logic is the well-known *first degree entailment*, FDE. Thus we have a logical De Morgan algebra, $\langle\langle\mathcal{FOUR}, \leq_t\rangle, \{\mathbf{t}, \top\}\rangle$, completing Gen-1 of the construction outlined earlier.

For Gen-2 of the construction we form the bilattice product $\langle\langle\mathcal{FOUR}, \leq_t\rangle \odot \langle\mathcal{FOUR}, \leq_t\rangle$, which is shown in Figure 6 and given the name $\mathcal{SIXTEEN}$. It may be best to think of the construction simply as formal, without trying to attach intuitive significance to possible meanings for node labels. $\mathcal{SIXTEEN}$ becomes a logical bilattice when we take as designated values $\{\mathbf{t}, \top\} \times \{\mathbf{t}, \top, \mathbf{f}, \bot\}$. That is,

Designated Values: $\langle\mathbf{t}, \mathbf{t}\rangle, \langle\mathbf{t}, \top\rangle, \langle\mathbf{t}, \mathbf{f}\rangle, \langle\mathbf{t}, \bot\rangle, \langle\top, \mathbf{t}\rangle, \langle\top, \top\rangle, \langle\top, \mathbf{f}\rangle, \langle\top, \bot\rangle$.

We began with $\mathcal{FOUR}$, using the ordering $\leq_t$, and so our De Morgan operation is the negation of $\mathcal{FOUR}$. Then conflation in $\mathcal{SIXTEEN}$ is: $-\langle a, b\rangle = \langle\neg b, \neg a\rangle$. It is now easy to check that the truth values of $\mathcal{SIXTEEN}$ divide up as shown below. What might be a bit surprising, after the previous examples, is that not everything falls into the exact, consistent, anticonsistent categories.

Exact Values: $\langle\mathbf{f}, \mathbf{t}\rangle$, $\langle\bot, \bot\rangle$, $\langle\top, \top\rangle$, $\langle\mathbf{t}, \mathbf{f}\rangle$.

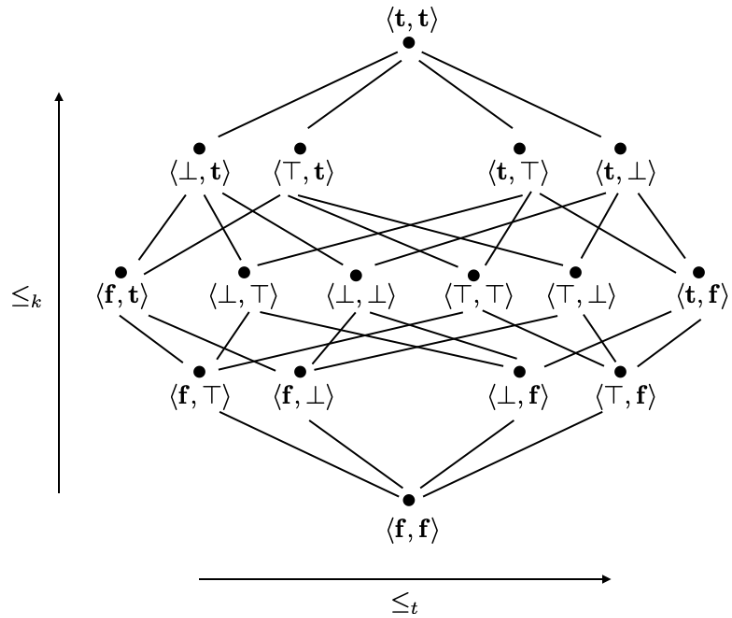


Figure 6: The Bilattice $\mathcal{SIXTEEN}$

Consistent Values: Exact together with $\langle \mathbf{f}, \mathbf{f} \rangle$, $\langle \mathbf{f}, \bot \rangle$, $\langle \mathbf{f}, \top \rangle$, $\langle \bot, \mathbf{f} \rangle$, $\langle \top, \mathbf{f} \rangle$.

AntiConsistent Values: Exact together with $\langle \bot, \mathbf{t} \rangle$, $\langle \top, \mathbf{t} \rangle$, $\langle \mathbf{t}, \bot \rangle$, $\langle \mathbf{t}, \top \rangle$, $\langle \mathbf{t}, \mathbf{t} \rangle$.

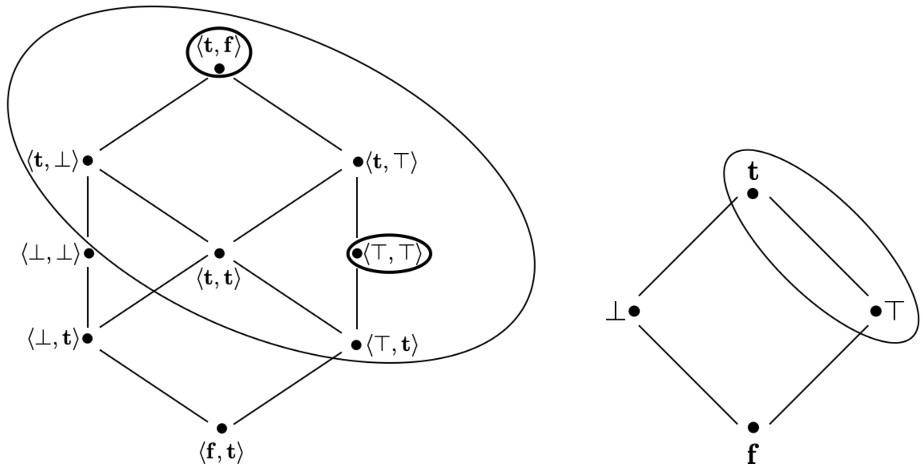
None of the above: $\langle \bot, \top \rangle$, $\langle \top, \bot \rangle$.

We now move on to Gen-3 of the construction. By Proposition 9.3 the classical logic analog, $\mathsf{C}\langle \mathcal{SIXTEEN}, \{\mathbf{t}, \top\} \times \{\mathbf{t}, \top, \mathbf{f}, \bot\}\rangle$, is isomorphic to the bilattice $\mathcal{FOUR}$ under the $\leq_t$ ordering, with $\{\mathbf{t}, \top\}$ designated. As to $\mathsf{ST}\langle \mathcal{SIXTEEN}, \{\mathbf{t}, \top\} \times \{\mathbf{t}, \top, \mathbf{f}, \bot\}\rangle$, it has as members the anticonsistent values from $\mathcal{SIXTEEN}$, with the ordering induced by $\leq_t$. The set of strictly designated values is the intersection of the set of Designated Values for $\mathcal{SIXTEEN}$ with the set of Exact Values, and this is $\{\langle \top, \top \rangle, \langle \mathbf{t}, \mathbf{f} \rangle\}$. Finally, the set of tolerantly designated values is the intersection of the set of Designated Values with the set of Anticonsistent values, and this is $\{\langle \top, \top \rangle, \langle \mathbf{t}, \mathbf{f} \rangle, \langle \top, \mathbf{t} \rangle, \langle \mathbf{t}, \bot \rangle, \langle \mathbf{t}, \top \rangle, \langle \mathbf{t}, \mathbf{t} \rangle\}$. All this is shown schematically in Figure 7a. The standard formulation of FDE is shown as part 7b. Our general results show that these validate the same consequence relation, but differ on the metaconsequence level.

11 And More?

The family of what we called logical De Morgan algebras (distrubutive or not) is mostly made up of examples of purely technical interest. But the fact that K_3, LP, and FDE have strict/tolerant counterparts may have useful consequences, or at least consequences that someone might argue are useful. I leave this to others. But there are some more technical items that I plan to develop further in subsequent work.

Here we looked at K_3, strong Kleene logic. There is also weak Kleene logic. This was generalized to the bilattice context, in [13], using what I called “cut down operations.”



(a) Strict/Tolerant FDE (b) Standard FDE

Figure 7: A Strict/Tolerant Counterpart of FDE

Such operations have been further investigated in [8], and dualized in [29]. It is likely that strict/tolerant analogs based on cut down (or track down) operations can be developed, similar to what has been done here.

Analogous to strict/tolerant logic, but with things reversed, there is also tolerant/strict logic; see [14, 15] for background. This is a more complicated family than that of strict/tolerant logic, and will be investigated in a separate paper.

In a private communication Eduardo Alejandro Barrio raised the question of what is the minimum size of a strict/tolerant counterpart. It may be the case that the algorithm given as proof of Proposition 10.1 produces minimal sized counterparts, but perhaps not. This is open.

Finally, [5] generalizes the original strict/tolerant phenomenon in an 'upward' direction, as we discussed in Section 1. Their work examines the structure of consequence, metaconsequence, metametaconsequence, and so on. It is likely that this work also generalizes to the present setting, but it is deferred to a later paper.

References

[1] Ofer Arieli and Arnon Avron. "Reasoning with logical bilattices". In: *Journal of Logic, Language, and Information* 5.1 (1996), pp. 25–63.

[2] Ofer Arieli and Arnon Avron. "The Value of Four Values". In: *Artificial Intelligence* 102 (1998), pp. 97–141.

[3] F. G. Asenjo. "A Calculus of Antinomies". In: *Notre Dame Journal of Formal Logic* 7 (1966), pp. 103–105.

[4] Arnon Avron. "The structure of interlaced bilattices". In: *Mathematical Structures in Computer Science* 6.3 (1996), pp. 287–299.

[5] Eduardo Alejandro Barrio, Federico Pailos, and Damian Szmuc. "Hierarchies of Classical and Paraconsistent Logics". submitted for publication.

[6] Nuel D. Belnap Jr. "A useful four-valued logic". In: *Modern Uses of Multiple-Valued Logic*. Ed. by Jon Michael Dunn and George Epstein. D. Reidel, 1977.

[7] P. Cobreros et al. "Tolerant, classical, strict". In: *Journal of Philosophical Logic* 41.2 (2012), pp. 347–385.

[8] Thomas Macaulay Ferguson. "Cut-Down Operations on Bilattices". In: *2015 IEEE International Symposium on Multiple-Valued Logic*. 2015, pp. 24–29.

[9] Melvin C. Fitting. "Bilattices and the theory of truth". In: *Journal of Philosophical Logic* 18 (1989), pp. 225–256.

[10] Melvin C. Fitting. "Bilattices in logic programming". In: *The Twentieth International Symposium on Multiple-Valued Logic*. Ed. by George Epstein. IEEE. 1990, pp. 238–246.

[11] Melvin C. Fitting. "Bilattices and the semantics of logic programming". In: *Journal of Logic Programming* 11 (1991), pp. 91–116.

[12] Melvin C. Fitting. "A Theory of Truth that prefers falsehood". In: *Journal of Philosophical Logic* 26 (1997), pp. 477–500.

[13] Melvin C. Fitting. "Bilattices are nice things". In: *Self-Reference*. Ed. by Thomas Bolander, Vincent Hendricks, and Stig Andur Pederrsen. Center for the Study of Language and Information, 2006. Chap. 3, pp. 53–77.

[14] Szymon Frankowski. "Formalization of a plausible inference". In: *Bulletin of the Section of Logic* 33 (2004), pp. 41–52.

[15] Szymon Frankowski. "P-consequence versus q-consequence". In: *Bulletin of the Section of Logic* 33 (2004), pp. 197–207.

[16] George Gargov. "Knowledge, Uncertaity and Ignorance in Logic: Bilattices and beyond". In: *Journal of Applied Non-Classical Logics* 9.2-3 (1999), pp. 195–283.

[17] Matthew L. Ginsberg. "Multivalued logics: a uniform approach to reasoning in artificial intelligence". In: *Computational Intelligence* 4 (1988), pp. 265–316.

[18] Matthew L. Ginsberg. "Bilattices and modal operators". In: *Journal of Logic and Computation* 1.1 (1990), pp. 41–69.

[19] Stephen C. Kleene. "On notation for ordinal numbers". In: *Journal of Symbolic Logic* 3 (1938), pp. 150–155.

[20] Stephen C. Kleene. *Introduction to Metamathematics*. Princeton, NJ: D. Van Nostrand, 1950.

[21] Hannes Leitgeb. "Truth and the Liar in De Morgan-Valued Models". In: *Notre Dame Journal of Formal Logic* 40.4 (1999), pp. 496–514.

[22] Grzegorz Malinowski. "Q-Consequence Operation". In: *Reports on Mathematical Logic* 24 (1990), pp. 49–59.

[23] Grzegorz Malinowski. "Referential and Inferential Many-Valuedness". In: *Paraconsistency, the Logical Way to the Inconsistent*. Ed. by W. A. Carnielli, M. E. Coniglio, and I. M. L. DOttaviano. Vol. 228. Lecture Notes in Pure and Applied Mathematics 228. Marcel Dekker, 2002, pp. 341–352.

[24] Grzegorz Malinowski. "That $p + q = c$(onsequence)". In: *Bulletin of the Section of Logic* 36.1/2 (2007), pp. 7–19.

[25] Graham Priest. "The Logic of Paradox". In: *Journal of Philosophical Logic* 8.1 (1979), pp. 219–241.

[26] Graham Priest. *An Introduction to Non-Classical Logic; From If to Is*. Second Edition. Cambridge University Press, 2008.

[27] Y. Shramko and H. Wansing. "Some useful 16-valued logics: how a computer network should think". In: *Journal of Philosophical Logic* 34.121-153 (2005).

[28] Y. Shramko and H. Wansing. "Truth and Falsehood: An Inquiry into Generalized Logical Values". In: *Trends in Logic*. Vol. 36. Springer, 2011.

[29] Damian Szmuc. "Track-down operations on bilattices". In: *Proceedings of the 48th IEEE International Symposium on Multiple-Valued Logic, Los Alamitos, California*. Ed. by Robert Wille and Martin Lukac. 2018, pp. 74–79.