A Simple Propositional $S5$ Tableau System*

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Abstract

We give a sound and complete propositional $S5$ tableau system of a particularly simple sort, having an easy completeness proof. It sheds light on why the satisfiability problem for $S5$ is less complex than that for most other propositional modal logics. We believe the system remains complete when quantifier rules are added. If so, it would allow us to get partway to an interpolation theorem for first-order $S5$, a theorem that is known to fail in general.

1 Introduction

The propositional modal logic $S5$ is peculiar in several respects. Most notably, propositional validity for it is `NP` complete, whereas for other standard propositional modal logics such as T or $S4$ it is `P-space`. But also, once quantifiers are added, other peculiarities show up: the interpolation theorem fails for first-order $S5$, but holds for first-order T and $S4$. Since it is generally possible to extract a proof of the interpolation theorem from a cut-free tableau or Gentzen calculus proof procedure, one cannot expect such a procedure to exist for first-order $S5$, and perhaps, by extension, not for the propositional part either. (There is a nice first-order $S5$ tableau system, using *prefixes*, but the cost of introducing the additional machinery of prefixes is that the tableau system is useless for proving interpolation.)

In this little paper we give a remarkably simple tableau system for propositional $S5$, and present its straightforward soundness and completeness proof. The rules help make it clear why we would expect propositional $S5$ to be computationally less complex than, say $S4$. In addition, they shed some light on interpolation, at least in the propositional setting. We leave open whether the first-order version of the $S5$ tableau system is complete. If it is, the status of interpolation is considerably clarified.

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2 Other Logics

"Nice" tableau systems are known for several modal logics. In this section we briefly sketch several, partly for comparison with the $S5$ system, and partly because it makes its presentation easier. The systems we give are for propositional K, $K4$, T, and $S4$. [1] can be consulted for a more detailed treatment.

A tableau is a tree, generally displayed with its root at the top. A tableau proof of a formula X begins with a tree containing just a root node, labeled $\neg X$. Then the tableau is "grown" using certain *branch extension rules* to be given below. A tableau branch is *closed* if it contains a syntactic contradiction, both Z and $\neg Z$ for some formula Z. If each branch is closed, the tableau itself is said to be closed. A closed tableau that is created by starting with $\neg X$ (a closed tableau *for* $\neg X$) is a tableau proof of X.

In order to state the branch extension rules most simply we use the *uniform notation* device of Smullyan, extended to the modal setting. For this purpose formulas, other than atoms and double negations, are classified into four categories: conjunctive or α formulas, disjunctive or β formulas, necessaries or ν formulas, and possibles or π formulas. For the α category two components are defined for each formula, denoted α_1 and α_2, and similarly for the β category. For the ν category only a single component is defined, denoted ν_0, and similarly for the π category. We begin with the classical connectives. Note that we have included iff, $\equiv$, somewhat artificially perhaps. Likewise we have omitted several of the less standard connectives such as joint denial. These, of course, could be added easily. The categories and components are presented in Table 1.

Conjunctive			Disjunctive		
α	α_1	α_2	β	β_1	β_2
$X \wedge Y$	X	Y	$\neg(X \wedge Y)$	$\neg X$	$\neg Y$
$\neg(X \vee Y)$	$\neg X$	$\neg Y$	$X \vee Y$	X	Y
$\neg(X \supset Y)$	X	$\neg Y$	$X \supset Y$	$\neg X$	Y
$X \equiv Y$	$X \supset Y$	$Y \supset X$	$\neg(X \equiv Y)$	$\neg(X \supset Y)$	$\neg(Y \supset X)$

Table 1: α- and β-Formulas and Components

Now, the branch extension rules for the propositional connectives are easily given, in Table 2.

$$\frac{\neg\neg X}{X} \qquad \frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 \mid \beta_2}$$

Table 2: Propositional Connective Rules

The intention here is, by now, well-known. If an α formula occurs on a tableau branch, that branch can be lengthened with two consecutive nodes, one labeled α_1, the other labeled α_2. The double negation case is similar, but simpler. If a β formula occurs on a branch, the end of the branch is split, with the left child labeled β_1 and the right child labeled β_2.

Next are the modal cases. We take both $\Box$ and $\Diamond$ as primitive. The categories and components are given in Table 3.

Necessary		Possible	
ν	ν_0	π	π_0
$\Box X$	X	$\Diamond X$	X
$\neg\Diamond X$	$\neg X$	$\neg\Box X$	$\neg X$

Table 3: ν- and π-Formulas and Components

For the modal logics T and $S4$, but not for K or $K4$, we have the branch extension rule given in Table 4.

$$\frac{\nu}{\nu_0}$$

Table 4: T and $S4$ Rule

Before stating the rest of the modal branch extension rules, we need one more piece of notation. For a set S of formulas, a corresponding (logic-dependent) set $S^\sharp$ is defined in Table 5.

Logic	$S^\sharp$
K, T	$\{\nu_0 \mid \nu \in S\}$
$K4$, $S4$	$\{\nu_0, \nu \mid \nu \in S\}$

Table 5: $S^\sharp$ Definition

Now we can give the remaining tableau rules. But these are of a different nature from the others. Previous branch extension rules actually *extended* branches. No information was lost. The next, and final, family of rules is what is sometimes called *destructive*—information disappears. The way they are used is this: if the set of formulas on a tableau branch matches the pattern shown above the line in the rule, that entire branch is *replaced* with the set shown below the line. (More correctly, in order not to loose the structure of the tree that has been created, we mark the formulas on the affected branch as "deleted," and lengthen the branch by adding the replacement formulas to the end.) The rules are in Table 6. They are logic-dependent, since the definition of $S^\sharp$ is.

This completes the definition of the tableau systems for propositional K, T, $K4$, and $S4$.

$$\frac{S, \pi}{S^\sharp, \pi_0}$$

Table 6: Destructive Rule

Example 2.1 Here is a proof, in the K system, of $\Box X \supset (\neg\Diamond\neg X \vee \neg\Diamond Y)$. Formulas are numbered for reference purposes.

1. $\neg\,[\Box X \supset (\neg\Diamond\neg X \vee \neg\Diamond Y)]$
2. $\Box X$
3. $\neg(\neg\Diamond\neg X \vee \neg\Diamond Y)$
4. $\neg\neg\Diamond\neg X$
5. $\neg\neg\Diamond Y$
6. $\Diamond\neg X$
7. $\Diamond Y$

Item 1 is the negation of the formula to be proved; 2 and 3 are from 1 by α; 4 and 5 are from 3 also by α; 6 and 7 are from 4 and 5 by double negation elimination.

At this point the Destructive Rule can be fired in two different ways. If we take π to be $\Diamond\neg X$ and S to be the set consisting of the remaining formulas on the branch, the rule has us replace the branch with the two formulas X (the only member of $S^\sharp$) and $\neg X$ (which is π_0), and we have closure. If we take π to be $\Diamond Y$ instead, the branch gets replaced with the formulas X and Y, and we do not have closure.

3 The $S5$ System

The tableau system for $S5$ follows the pattern above, but with certain modifications. We keep the rule stated in Table 4, and we have a rule of the form of that in Table 6, but with a new definition of $S^\sharp$. We give two versions for this since *either* gives us a sound and complete system for propositional $S5$. In order to state the two versions, we need some terminology.

Definition 3.1 A formula is *modalized* if every atomic subformula in it is within the scope of a modal operator. A formula is *trivially modalized* if it is a ν or a π formula.

Now the new definitions of $S^\sharp$ for $S5$ are given in Table 7.

We now have the $S5$ Branch Extension Rules fully specified (in two versions). There is one additional peculiarity to the tableau system however. In order to prove the formula X, we do not construct a tableau for $\neg X$, but for $\neg\Box X$!

$S^\sharp = \{X \in S \mid X \text{ is modalized}\}$
$S^\sharp = \{X \in S \mid X \text{ is trivially modalized}\}$

Table 7: $S^\sharp$ Definitions for $S5$

Example 3.2 Here is a proof of $X \supset \Box\Diamond X$ in the $S5$ system, using either version of the $S^\sharp$ definition. The tableau begins as follows.

1. $\neg\Box(X \supset \Box\Diamond X)$
2. $\neg(X \supset \Box\Diamond X)$
3. X
4. $\neg\Box\Diamond X$

Item 1 is the negated necessitation of the formula to be proved; 2 is from 1 by the Destructive Rule (take π to be $\neg\Box(X \supset \Box\Diamond X)$ and S to be $\{\pi\}$); 3 and 4 are from 2 by α.

Now use the Destructive Rule again, taking π to be formula 4 and S to be the entire set of formulas on the branch. In this case, $S^\sharp$ consists of formulas 1 and 4, and the tableau continues as follows.

1. $\neg\Box(X \supset \Box\Diamond X)$
4. $\neg\Box\Diamond X$
5. $\neg\Diamond X$
6. $\neg(X \supset \Box\Diamond X)$
7. X
8. $\neg\Box\Diamond X$
9. $\neg X$

Items 1, 4 and 5 are from the Destructive Rule application; 6 is from 1, again by the Destructive Rule; 7 and 8 are from 6 by α; 9 is from 5 by the rule in Table 4.

4 Soundness

There are two versions of semantics for $S5$, one in which accessibility is an equivalence relation, and one in which there is no accessibility relation at all. It is easy to move between the two versions—in effect a model of the second kind is one equivalence class from a model of the first kind. To keep things simple, we confine things to a semantics in which there is no accessibility relation at all. To be precise, here is the notion of $S5$ model we use.

Definition 4.1 An $S5$ *model* is a structure $\langle \mathcal{G}, \Vdash \rangle$, where $\mathcal{G}$ is a non-empty set (of possible worlds), and $\Vdash$ is a relation between members of $\mathcal{G}$ and atomic formulas.

Given an $S5$ model, the relation $\Vdash$ is extended to arbitrary formulas in the following standard way. (We take a small but sufficient number of connectives and modal operators as representative.) For $\Gamma \in \mathcal{G}$:

$$\begin{aligned}
\Gamma \Vdash (X \wedge Y) &\Leftrightarrow \Gamma \Vdash X \text{ and } \Gamma \Vdash Y \\
\Gamma \Vdash \neg X &\Leftrightarrow \text{not } \Gamma \Vdash X \\
\Gamma \Vdash \Box X &\Leftrightarrow \text{for all } \Delta \in \mathcal{G}, \Delta \Vdash X
\end{aligned}$$

A formula X is *valid in a model* $\langle \mathcal{G}, \Vdash \rangle$ if $\Gamma \Vdash X$ for all $\Gamma \in \mathcal{G}$. And X is simply *valid* if X is valid in all $S5$ models.

Now, soundness follows the standard pattern for tableaus. Call a set S of formulas *satisfiable* if there is some $S5$ model $\langle \mathcal{G}, \Vdash \rangle$, and some $\Gamma \in \mathcal{G}$, such that $\Gamma \Vdash X$ for all $X \in S$. Call a tableau branch satisfiable if the set of formulas on it is satisfiable. And call a tableau satisfiable if some branch is satisfiable. Satisfiability is a loop invariant for tableaus in the following sense.

Proposition 4.2 *If $\mathcal{T}$ is a satisfiable tableau, and $\mathcal{T}'$ is the result of applying a tableau rule on some branch of $\mathcal{T}$, then $\mathcal{T}'$ is also a satisfiable tableau.*

We leave the standard proof of this to you. It is applied in the usual way, with one small twist: a formula X is valid in all $S5$ models if and only if $\Box X$ is valid in all $S5$ models. This is easy to show. Now, suppose X were provable but not valid. Since X is not valid, neither is $\Box X$, and so $\{\neg\Box X\}$ must be satisfiable. Then the tableau proof of X starts off with a satisfiable tableau, since it starts off with $\neg\Box X$ and $\{\neg\Box X\}$ is satisfiable. But then only satisfiable tableaus can arise. Since we are assuming X is provable, we must be able to produce a closed tableau, and this is impossible since we would have a tableau that is both closed and satisfiable.

5 Completeness

As usual, completeness is less routine, and hence more interesting. Although completeness for tableau systems is often proved using a kind of "systematic" construction procedure, with a counter-example extracted from a failed tableau, we instead use a maximal consistent, Lindenbaum, style argument, which simplifies things considerably.

Definition 5.1 We say a set S of formulas is *consistent* if no tableau beginning with a finite subset of S closes. If $\mathcal{F}$ is a set of formulas, a subset S of $\mathcal{F}$ is *maximally consistent with respect to* $\mathcal{F}$ if S is consistent, and no subset of $\mathcal{F}$ that properly extends S is consistent.

Now the following holds with the usual proof, which we omit.

Lemma 5.2 (Lindenbaum) *If S is a consistent subset of $\mathcal{F}$, S extends to a set that is maximal consistent with respect to $\mathcal{F}$.*

Even if $\mathcal{F}$ is the set of all formulas, we cannot show the "usual" properties of maximal consistent sets that are associated with an axiomatic treatment, though we do have half of them, in the sense that what are usually equivalences become implications. (Once again we work with a minimal set of connectives and modal operators.)

Proposition 5.3 *If S is maximal consistent with respect to the set of all formulas, it is* downward saturated*, by which we mean the following:*

1. *Not both X and $\neg X$ are in S.*
2. *If $X \wedge Y$ is in S, so are both X and Y.*
3. *If $\neg(X \wedge Y)$ is in S, so is one of $\neg X$ or $\neg Y$.*
4. *If $\Box X$ is in S, so is X.*

The proof of this is straightforward, and once again we omit it. We also have the following, an analog of which is probably familiar in the context of axiom systems.

Proposition 5.4 *Suppose S is consistent and $\neg\Box X \in S$. Then $S^\sharp \cup \{\neg X\}$ is also consistent (using either version $S5$ of $S^\sharp$).*

Proof If $S^\sharp \cup \{\neg X\}$ were not consistent, there would be a closed tableau $\mathcal{T}$ starting with some finite subset of it. But then there would also be a closed tableau for a finite subset of S. Just make the first rule application the destructive modal rule, used on $\neg\Box X$, which is in S, and then copy the steps of $\mathcal{T}$. ■

Now for the completeness proof itself. We present the argument under the assumption that $S^\sharp$ is the set of modalized formulas in S—the argument is essentially the same if it is the set of trivially modalized formulas.

Assume C is a consistent set of *modalized* formulas. First, extend C to a set, C^*, that is maximally consistent with respect to the set of *all modalized formulas*. And, let $\mathcal{G}$ be the collection of all sets of formulas that extend C^* and are maximally consistent with respect to the set of *all formulas*. Not surprisingly, $\mathcal{G}$ will be the collection of possible worlds of a model.

First, a simple observation. If $\Gamma \in \mathcal{G}$, $\Gamma^\sharp = C^*$. In one direction, $C^* \subseteq \Gamma$ because Γ extends C^*. Since all members of C^* are modalized, it follows that $C^* \subseteq \Gamma^\sharp$. In the other direction, if C^* were a proper subset of $\Gamma^\sharp$, then, $\Gamma^\sharp$ would be a consistent, modalized, proper extension of C^*, which contradicts the maximality of C^*.

Now, for $\Gamma \in \mathcal{G}$, and atomic A, set $\Gamma \Vdash A$ iff $A \in \Gamma$. We thus have an $S5$ model $\langle \mathcal{G}, \Vdash \rangle$. We now need the following item, which should be familiar from axiomatic treatments. But note, in an axiomatic setting we have an equivalence in place of the implication below.

Theorem 5.5 (Truth Lemma) *For every formula X, and every $\Gamma \in \mathcal{G}$, if $X \in \Gamma$ then $\Gamma \Vdash X$.*

Proof By induction on formula degree. The positive atomic case is by definition.

Suppose $\neg A \in \mathcal{G}$, where A is atomic. By Proposition 5.3 $A \notin \Gamma$, and so by definition $\Gamma \nVdash A$, and hence $\Gamma \Vdash \neg A$.

The other propositional cases are also treated using Proposition 5.3.

Suppose $\Box X \in \Gamma$, and the result is known for X. Let Δ be an arbitrary member of $\mathcal{G}$. Then $\Delta^\sharp = \Gamma^\sharp$, as we observed above. But $\Box X$ is modalized, and so is in $\Gamma^\sharp$, hence in $\Delta^\sharp \subseteq \Delta$. Now, $X \in \Delta$, by Proposition 5.3 again. By the induction hypothesis, $\Delta \Vdash X$. Since $\Delta \in \mathcal{G}$ was arbitrary, it follows that $\Gamma \Vdash \Box X$.

Finally, suppose $\neg\Box X \in \Gamma$, and the result is known for $\neg X$. By Proposition 5.4, $\Gamma^\sharp \cup \{\neg X\}$ is consistent. Extend it to a maximal consistent subset of the set of all formulas, call it Δ. Since $\Gamma^\sharp = C^*$, it follows that $\Delta \in \mathcal{G}$. Since $\neg X \in \Delta$, by the induction hypothesis $\Delta \Vdash \neg X$. But then $\Gamma \Vdash \neg\Box X$. ■

Now, suppose X is not provable. This means there is no closed tableau beginning with $\neg\Box X$, hence the set $\{\neg\Box X\}$ is consistent; it is also modalized. Call it C, and use it to construct a model $\langle \mathcal{G}, \Vdash \rangle$ as above. In this model $\neg\Box X$ is a member of every world, hence $\Box X$ is false at every world. Then there must be some world at which X is false, and so X is not valid.

6 Concluding comments

Propositional $S5$ has a simpler satisfiability problem than propositional $S4$ (`NP` versus `P-space`). The tableau systems given above provide clear intuitions for why this is the case. (A formal proof can be based on this, but we do not do so.)

Suppose we are constructing an $S4$ tableau, and we have a branch whose set of formulas S contains two π-formulas, say $\Diamond X$ and $\Diamond Y$. If we apply the destructive modal rule to the first, the branch is replaced by $S^\sharp \cup \{X\}$, and the formula $\Diamond Y$ is eliminated—recall, this uses the $S4$ definition of $S^\sharp$. Likewise if we apply the rule on $\Diamond Y$ we get $S^\sharp \cup \{Y\}$ and $\Diamond X$ disappears. If we chose the "wrong" formula to work with, we may not produce a closed branch, though one might have been obtainable had we chosen differently. In short, in any systematic tableau construction procedure for $S4$, backtracking is essential.

The situation for $S5$ is quite different. Under the same circumstances as above, where a branch consists of a set S of formulas containing $\Diamond X$ and $\Diamond Y$, no matter which of these two we work with, the other remains—using either of the $S5$ definitions of $S^\sharp$. Backtracking is no longer necessary for completeness. More formally, for $S5$ there are more cases of rule permutability than for $S4$. Consequently one expects things to be simpler computationally.

We gave two tableau versions for $S5$, using two definitions of $S^\sharp$. Both have their advantages. Defining $S^\sharp$ to consist of the ν and π formulas of S means that to construct $S^\sharp$ we do not need to go beyond top-level connectives. A deeper analysis is unnecessary, and so if implemented, certain processing steps are simplified. On the other hand, defining $S^\sharp$ to consist of all modalized members of S can yield shorter tableau proofs, though the task of deciding on the membership of $S^\sharp$ becomes correspondingly more complex.

Finally something a little more speculative. The tableau system given above is for propositional $S5$. There is reason to believe it remains complete if quantifier rules are added, and constant-domain first-order $S5$ models are used. One of the pecularities of first-order $S5$ is that, unlike the situation with many other modal logics, an interpolation theorem is no longer provable. If we do, in fact, have first-order completeness, the situation would be somewhat clarified. It would be possible to extract a proof of the interpolation theorem for first-order $S5$ for implications *that are modalized*. This would get us halfway there, which is probably as close as is possible.

References

[1] Melvin C. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. D. Reidel Publishing Co., Dordrecht, 1983.