Logics With Several Modal Operators

by

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§1 *Introduction*: There seems to have been little discussion to date of Kripke models for modal logic capable of treating more than one modal necessity operator at a time, or modal and deontic operators simultaneously. The theory is a simple extension of the case for a single operator. In this paper as a specific example we discuss (both model-theoretically and axiomatically) a modal logic having both an S4 and an S5 necessity operator, and give an intuitive sense to the logic. We also discuss deontic logics superimposed on S4, S4.2, S4.3, and S5, all of which have different intuitive meanings, and a deontic logic with two obligation operators.

§2 *Kripke model theory—single operators*: In [5], [6], and [7] Kripke introduced a successful semantics for several modal logics. For various reasons, in [2] we altered his notation, and here we continue with that altered version. Before presenting the generalization to several modal operators we present the single case to establish notation.

By a (general modal) model is meant a triple $\langle \mathcal{G}, \mathcal{R}, \models \rangle$ where $\mathcal{G}$ is a non-empty set, $\mathcal{R}$ is a binary relation on $\mathcal{G}$, and $\models$ is a relation between elements of $\mathcal{G}$ and formulas (we use $\not\models$ to mean $\models$ does not hold), satisfying, for any $\Gamma \in \mathcal{G}$,

1) $\Gamma \models (X \wedge Y) \Leftrightarrow \Gamma \models X$ and $\Gamma \models Y$
2) $\Gamma \models (X \vee Y) \Leftrightarrow \Gamma \models X$ or $\Gamma \models Y$
3) $\Gamma \models \sim X \quad \Leftrightarrow \Gamma \not\models X$
4) $\Gamma \models (X \supset Y) \Leftrightarrow \Gamma \not\models X$ or $\Gamma \models Y$
5) $\Gamma \models \Box X \quad \Leftrightarrow$ for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta \models X$

A formula X is valid in the model $\langle \mathcal{G}, \mathcal{R}, \models \rangle$ if for all $\Gamma \in \mathcal{G}$, $\Gamma \models X$.

Intuitively (see [5], [6], or [7]) $\mathcal{G}$ is a collection of possible worlds; $\Gamma \mathcal{R} \Delta$ means Δ is a world possible relative to Γ (we will see in some cases this may be read better otherwise); $\Gamma \models X$ means X is true in the world Γ.

Now, by putting various special conditions on $\mathcal{R}$, models for various modal logics are produced. For example, call a model $\langle \mathcal{G}, \mathcal{R}, \models \rangle$ an S4 model if $\mathcal{R}$ is reflexive and transitive. Call X S4-valid if X is valid in all S4 models. In [7] Kripke showed X is a theorem of S4 if and only if X is S4-valid. Similarly, if $\mathcal{R}$ is an equivalence relation, call $\langle \mathcal{G}, \mathcal{R}, \models \rangle$ an S5 model, and define S5-validity similarly. In [5] and [7] Kripke showed X is a theorem of S5 if and only if X is S5 valid. Other cases are possible.

§3 Kripke model theory—several operators: Suppose we have several necessity operators, $\Box_1, \Box_2, \ldots, \Box_n$ (some of which may be deontic). We generalize the notion of model in § 2 to n + 2-tuples $\langle \mathcal{G}, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n, \models \rangle$ where $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$ are relations on $\mathcal{G}$, and $\mathcal{G}$ and $\models$ are as above except that condition 5) is replaced by the n conditions: (i = 1, 2, ..., n)

$$5_i)\quad \Gamma \models \Box_i X \Leftrightarrow \text{for all } \Delta \in \mathcal{G} \text{ such that } \Gamma \mathcal{R}_i \Delta,\ \Delta \models X.$$

Now conditions on the $\mathcal{R}_i$ individually are to reflect the axiomatic properties of the $\Box_i$ indivually, and relations among the $\mathcal{R}_i$ are to reflect axiomatic properties relating the $\Box_i$.

We remark that if we define dual operators $\Diamond_i$ to be $\sim \Box_i \sim$ it follows that

$$\Gamma \models \Diamond_i X \Leftrightarrow \text{for some } \Delta \in \mathcal{G} \text{ such that } \Gamma \mathcal{R}_i \Delta,\ \Delta \models X.$$

Rather than continue on such an abstract level, let us proceed immediately to special cases.

§4 S4 + S5: In this section, instead of $\Box_1$ and $\Box_2$ we will use L (for the S5 operator) and $\Box$ (for the S4 operator) and we will use M and $\Diamond$ for their respective duals. Also we use $\mathcal{R}$ and $\mathcal{S}$ instead of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Axiomatically the system is as follows.

Rules: $\dfrac{X, X \supset Y}{Y}$ $\qquad\qquad$ $\dfrac{X}{LX}$

Axiom schemata: all tautologies

$$
\begin{array}{ll}
LX \supset X & \Box X \supset X \\
L(X \supset Y) \supset (LX \supset LY) & \Box(X \supset Y) \supset (\Box X \supset \Box Y) \\
LX \supset LLX & \Box X \supset \Box\Box X \\
\quad X \supset LMX & \\
\quad LX \supset \Box X &
\end{array}
$$

Model theoretically, the system corresponds to the models $\langle \mathcal{G}, \mathcal{R}, \mathcal{S}, \models \rangle$ ($\mathcal{R}$ is for L, $\mathcal{S}$ for $\Box$) where $\mathcal{R}$ is an equivalence relation, $\mathcal{S}$ is reflexive and transitive, and $\mathcal{S} \subseteq \mathcal{R}$ (that is, $\Gamma \mathcal{S} \Delta \Rightarrow \Gamma \mathcal{R} \Delta$).

It is simple to show that all the above axioms are valid in such models, and that the rules preserve validity. Thus all theorems of the above system are valid in all such models. To establish the converse we may use the Lindenbaum-Henkin approach, due to Scott and Makinson in this context. See [8], and also [4] for an outline.

Let $\mathcal{G}$ be the collection of all maximal consistent sets of formulas of the above axiomatic system. Let $\Gamma \models X$ if $X \in \Gamma$. Let $\Gamma \mathcal{R} \Delta$ if $\{X | LX \in \Gamma\} \subseteq \Delta$ and let $\Gamma \mathcal{S} \Delta$ if $\{X | \Box X \in \Gamma\} \subseteq \Delta$. The resulting structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{S}, \models \rangle$ is a model of the above type. Most of the properties are straightforward. The only non-trivial fact to establish is that $LX \in \Gamma \Leftrightarrow$ for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $X \in \Delta$, (and similarly for $\Box$ and $\mathcal{S}$).

If $LX \in \Gamma$ and $\Gamma \mathcal{R} \Delta$, by definition of $\mathcal{R}$, $X \in \Delta$.

Conversely, suppose $LX \notin \Gamma$. Let Γ^* be $\{Z | LZ \in \Gamma\}$. $\Gamma^* \cup \{\sim X\}$ is consistent, for otherwise

$$
\begin{array}{l}
\Gamma^*, \sim X \vdash X \\
\quad \Gamma^* \vdash X \\
\quad \Gamma \vdash LX \\
\quad LX \in \Gamma.
\end{array}
$$

Now extend $\Gamma^* \cup \{\sim X\}$ to a maximal consistent set Δ. Then $\Gamma \mathcal{R} \Delta$ and $X \notin \Delta$.

Thus $\langle \mathcal{G}, \mathcal{R}, \mathcal{S}, \models \rangle$ is a model of the above type. Finally, if X is not a theorem, $\{\sim X\}$ is consistent. Extend it to a maximal consistent set Γ. Then $\Gamma \in \mathcal{G}$, $X \notin \Gamma$ so $\Gamma \not\models X$. Thus X is not valid.

This system may be given the following intuitive interpretation. As usual, let $\mathcal{G}$ be the collection of all possible worlds, and let $\Gamma \models X$ mean X is true in the world Γ. Let $\Gamma \mathcal{R} \Delta$ mean Δ is a world logically possible relative to Γ. Let $\Gamma \mathcal{S} \Delta$ mean if Γ is the case, Δ could become the case (or is the case). Thus $\mathcal{S}$ is a time relation. For two worlds Γ and Δ, to say $\Gamma \mathcal{R} \Delta$ is to say Δ is a possible alternative state of affairs to Γ, while to say $\Gamma \mathcal{S} \Delta$ means Δ is a possible later state of affairs than Γ. (See [3]).

Under this interpretation the modal operators may be read:

LX as X is logically necessary
MX as X is logically possible
$\Box X$ as X is and must remain true
$\Diamond X$ as X is or might become true.

Thus we have $LX \supset \Box X$: if X is logically necessary, X is and will remain true. The model property corresponding to this is $\Gamma \mathcal{S} \Delta \Rightarrow \Gamma \mathcal{R} \Delta$: if Δ may become the case after Γ, Δ is a logically alternative state of affairs to Γ. We do not have the converses, $\Box X \supset LX$ or $\mathcal{R} \subseteq \mathcal{S}$. Because something must remain true it need not be logically true; and a conceivable alternate to a world need not be a later world (or even comparable in a time sense). We do, however, have the weaker $\vdash \Box X \Rightarrow \vdash LX$.

We also have some interesting modal reductions (besides the usual separate S4 and S5 ones).

$$\Box LX \equiv \Diamond LX \equiv L\Box X \equiv LX$$
$$\Box MX \equiv \Diamond MX \equiv M\Diamond X \equiv MX$$

The reader may supply the straightforward proofs and intuitive interpretations.

§5 *S4 + deontic*: We use O for the deontic obligation operator, and retain $\Box$ and $\Diamond$ for the modal operators. The basic system is as follows.

Rules: $\dfrac{X, X \supset Y}{Y}$ $\dfrac{X}{\Box X}$ $\dfrac{X}{OX}$

Axiom schemata: all tautologies

$$\Box(X \supset Y) \supset (\Box X \supset \Box Y) \qquad O(X \supset Y) \supset (OX \supset OY)$$
$$\Box X \supset X \qquad OX \supset \sim O \sim X$$
$$\Box X \supset \Box\Box X$$

The axioms are unexceptional. We will discuss axioms relating $\Box$ and O later.

In the corresponding models $\langle \mathcal{G}, \mathcal{S}, \mathcal{J}, \models \rangle$ ($\mathcal{S}$ corresponds to $\Box$, $\mathcal{J}$ to O), $\mathcal{S}$ reflexive and transitive, and $\mathcal{J}$ has the property that for any $\Gamma \in \mathcal{G}$ there is at least one $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{J} \Delta$. That this is indeed the corresponding model theory may be established as in §4.

We interpret this model theory as follows. We keep the same time interpretation for $\mathcal{S}$ as above. $\mathcal{J}$ may be considered an idealization relation: $\Gamma \mathcal{J} \Delta$ means Δ is an idealized version of Γ, a possible way Γ ought to be.

Various additional purely deontic axioms may be considered; we will mention some in §8.

As an axiom connecting the modal and deontic operators we may take $OX \supset \Diamond X$, that is, if X ought to be the case, X might become the case (presumably a moral code is defeatist if there is no possibility of fulfilling its laws). The corresponding model condition is: for each $\Gamma \in \mathcal{G}$ there is some $\Delta \in \mathcal{G}$ such that both $\Gamma \mathcal{S} \Delta$ and $\Gamma \mathcal{J} \Delta$. That is, at any time, it is possible to attain some ideal state.

The axiom introduced above is equivalent to $\Box X \supset \sim O \sim X$. We do not assume the stronger axiom $\Box X \supset OX$ since under this interpretation it reads: if X is true and must remain true, X ought to be true, which is false. The situation for S5 in §7 is different however.

§6 *S4.2 + deontic, S4.3 + deontic*: Suppose we add the axiom schema

$\Diamond \Box X \supset \Box \Diamond X$ to the above system, producing an S4.2 + deontic system. (S4.2 is from [1]). The effect of this axiom on $\mathcal{S}$ is to impose compatibility conditions. That is, if $\Gamma \mathcal{S} \Delta$ and $\Gamma \mathcal{S} \Pi$, for some $\Omega \in \mathcal{G}$, $\Delta \mathcal{S} \Omega$ and $\Pi \mathcal{S} \Omega$. It is a restriction on possible futures; all possible futures must be compatible, which is probably true if by world now we mean state of knowledge or collection of known physical laws. This together with the model condition corresponding to $OX \supset \Diamond X$ above says that some ideal versions of the present are compatible with any future (since all futures are compatible). This seems acceptable.

If, however, we add the axiom $\Box(\Box X \supset \Box Y) \vee \Box(\Box Y \supset \Box X)$ to the above, producing an S4.3 + deontic system (S4.3 is from [1]) the result is different. The effect of this axiom on $\mathcal{S}$ is to impose comparability conditions. If $\Gamma \mathcal{S} \Delta$ and $\Gamma \mathcal{S} \Pi$, either $\Delta \mathcal{S} \Pi$ or $\Pi \mathcal{S} \Delta$. Under the time interpretation for $\mathcal{S}$ this imposes a deterministic view of the future, that there is only one path of development possible, which may or may not be true. But this together with $OX \supset \Diamond X$ produces a clearly unsatisfactory situation, that if X ought to be the case, at some point in the future X will be the case. It seems that if we are going to superimpose a deontic logic on S4.3 and use the time interpretation we should drop $OX \supset \Diamond X$. Indeed, no connection between the deontic and modal operators seems desirable.

§7 *S5 + deontic*: The system of this section is as follows.

Rules: $\dfrac{X,\ X \supset Y}{Y}$ $\quad \dfrac{X}{LX}$ $\quad \dfrac{X}{OX}$

Axiom schemata: all tautologies

$$L(X \supset Y) \supset (LX \supset LY) \qquad O(X \supset Y) \supset (OX \supset OY)$$
$$LX \supset X \qquad OX \supset \sim O \sim X$$
$$LX \supset LLX$$
$$X \supset LMX$$

Again we may produce a corresponding model $\langle \mathcal{G}, \mathcal{R}, \mathcal{T}, \models \rangle$ ($\mathcal{R}$ corresponds to L, $\mathcal{T}$ to O) where $\mathcal{R}$ is an equivalence relation and $\mathcal{J}$ is such that for any $\Gamma \in \mathcal{G}$ there is at least one $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{J} \Delta$.

We may interpret L and $\mathcal{R}$ as in §4 and O and $\mathcal{J}$ as in §5. It now seems desirable to take as a connecting axiom $LX \supset OX$ (whose counterpart was rejected above). This says that if X is logically necessary, X ought to be the case. The corresponding model property is $\mathcal{J} \subseteq \mathcal{R}$ (i.e. $\Gamma \mathcal{J} \Delta \Rightarrow \Gamma \mathcal{R} \Delta$), that is, ideal worlds are logically possible. This is hopefully acceptable.

§8 Mixed deontic systems: It is possible to have systems of rules for behavior of various strengths. For example, presumably a moral code should be more fundamental than a legal system. Thus it is of interest to consider deontic systems with more than one obligation operator. With the above example in mind, we will use O for the moral code operator and J (jurisprudence) for the legal system operator. We begin with the following basic system.

Rules: $\dfrac{X,\ X \supset Y}{Y}$ $\quad \dfrac{X}{OX}$ $\quad \dfrac{X}{JX}$

Axiom schemata: all tautologies

$$O(X \supset Y) \supset (OX \supset OY) \qquad J(X \supset Y) \supset (JX \supset JY)$$
$$OX \supset {\sim} O {\sim} X \qquad JX \supset {\sim} J {\sim} X$$

The corresponding models $\langle \mathcal{G}, \mathcal{J}, \mathcal{V}, \models \rangle$ ($\mathcal{J}$ corresponds to O, $\mathcal{V}$ to J) have the property that for any $\Gamma \in \mathcal{G}$ there is a $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{J} \Delta$, and similarly for $\mathcal{V}$. Again $\Gamma \mathcal{J} \Delta$ is to mean Δ is a morally ideal version of Γ, and $\Gamma \mathcal{V} \Delta$ is to mean Δ is a legally ideal version of Γ (a place where the laws of Γ are obeyed).

We next consider some additional axioms concerning O and J separately.

Suppose we add $O(OX \supset X)$. The corresponding model condition [8] is $\Gamma \mathcal{J} \Delta \Rightarrow \Delta \mathcal{J} \Delta$, that is, an ideal version of Γ is an ideal version of itself. This seems acceptable. Similarly $J(JX \supset X)$ seems appropriate. Rougly, it says it is legally required that laws be obeyed. We note that as consequences we have $OOX \supset OX$ and $JJX \supset JX$.

Suppose we add $OX \supset OOX$. The corresponding condition on $\mathcal{J}$

is that it be transitive. This is conceivable if, for example, we have a moral code capable of judging itself, or if ideal worlds permitted improvement, possibly on some different 'level'. The corresponding legal axiom, $JX \supset JJX$ is certainly not acceptable.

Finally, as a connecting axiom let us consider first $O(JX \supset X)$. The corresponding model condition is $\Gamma \mathcal{J} \Delta \Rightarrow \Delta \mathcal{V} \Delta$, roughly, in an ideal world laws are obeyed. This of course assumes there is something inherently good about being law abiding, which is not the case. Let us consider instead the weaker axiom $O(JX \supset \sim O \sim X)$. This imposes a moral condition on the laws themselves, not on the people who live under them. The corresponding model condition is: if $\Gamma \mathcal{J} \Delta$ there is a Π such that $\Delta \mathcal{J} \Pi$ and $\Delta \mathcal{V} \Pi$. This can be read: if Δ is an ideal world there is at least one world Π which is both a morally ideal version of Δ and in which the laws of Δ are obeyed.

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