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INTERPOLATION FOR FIRST ORDER S5

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Abstract. An interpolation theorem holds for many standard modal logics, but first order **S5** is a prominent example of a logic for which it fails. In this paper it is shown that a first order **S5** interpolation theorem can be proved provided the logic is extended to contain propositional quantifiers. A proper statement of the result involves some subtleties, but this is the essence of it.

§1. Introduction. While an interpolation theorem can be proved for many first order modal logics, **S5** is a notable exception, [5]. One might naturally suspect that if more machinery were available to build interpolant candidates, an interpolation theorem could be obtained. This is the route taken in [2], for instance, where world-designating propositional variables are added. Propositional quantifiers can be seen as a limited, but natural, second order construct, and they also provide what is needed. This will be shown by a method based on a tableau proof procedure.

In a modal setting, propositional quantification is not as straightforward as one might think. I'll begin by reviewing a little of what is known about it in a purely propositional setting—there seems to be no study of it in a first order context. I give a tableau system for proving first order modal formulas involving propositional quantifiers. Showing completeness of this system brings in machinery originally developed for higher type classical logic. Then I properly state the interpolation theorem, and give its constructive proof.

§2. Propositional quantification background. Propositional quantifiers have never become mainstream, but they have a history that threads its way through the entire of modern logic. Bertrand Russell [13], for instance, used them to define disjunction and negation from implication. Classically, adding propositional quantifiers to propositional logic obviously leaves things decidable, since one can think of them as ranging over $\{true, false\}$, and apply a truth table analysis. They do, however, raise the complexity of satisfiability from *NP* complete to *P-space* complete [15].

Adding propositional quantifiers to propositional modal logics is much trickier. There are different versions, because there are two notions of frames.

1. Most commonly, a frame is a structure $\langle \mathscr{G}, \mathscr{R} \rangle$, with $\mathscr{G}$ a set of possible worlds, and $\mathscr{R}$ an accessibility relation. Then *propositions* are arbitrary subsets of $\mathscr{G}$, and propositional quantifiers range over these subsets. Such frames are called

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second order frames, by analogy with "true" second order classical models. They also supply the *primary interpretation of propositional quantifiers*. These propositionally quantified modal logics are denoted **S4**π+, **S5**π+, and so on.

2. Defining frames as above leads to problems since the fit between Kripke and algebraic semantics is not satisfactory [10, Chapter 1]. To get around this, *generalized frames* were introduced in [17]. These are structures $\langle \mathcal{G}, \mathcal{R}, \mathcal{P} \rangle$, where $\mathcal{G}$ and $\mathcal{R}$ are as before, and $\mathcal{P}$ is a *designated collection* of subsets of $\mathcal{G}$ called *propositions*, required to be closed under natural operations corresponding to formula syntax. Frames of this kind are called *first order frames*, and are analogous to *Henkin models* for classical second order logic. They also supply the *secondary interpretation of propositional quantifiers*. If propositional quantifiers are restricted to range over members of $\mathcal{P}$, the logics are denoted **S4**π, **S5**π, and so on.

If one adds straightforward axioms for propositional quantifiers to the customary modal proof machinery, one can prove completeness results with respect to first order frames [3, 4]. Reasonable tableau rules work well too. That is, **S4**π, **S5**π, and so on, have natural proof procedures, as one would expect.

Second order frames are more problematic. Fine and Kripke showed **S4**π+, and several other propositionally quantified modal logics, are recursively isomorphic to second order classical logic (unpublished, but a weaker version is in [4]). But, **S5**π+ is decidable [4, 11]! (**S5**π+ embeds in monadic second order classical logic, and this is decidable, [1].) Also, **S5**π+ can be axiomatized [4, 11] by adding the following to **S5**π: $(\exists X)\big[X \wedge (\forall Y)\big(Y \supset \Box(X \supset Y)\big)\big]$. There is also a logic, **S5**$\pi-$, corresponding to a semantics in which no closure conditions are imposed on the collection of propositions in a generalized frame. It too has been axiomatized, in [4].

§3. **Syntax and semantics.** The modal language, L, has first-order free variables, $x, y, \ldots$, and propositional variables, $X, Y, \ldots$. I'll use $\alpha, \beta, \ldots$ to denote variables of either kind. To keep things simple, I do not include constant or function symbols in the language. Atomic formulas are $\bot$ and $\top$, propositional variables, and expressions of the form $R(x_1, \ldots, x_n)$, where R is a relation symbol and $x_1, \ldots, x_n$ are first-order variables. Formulas are built up with $\neg$, $\wedge$, $\Box$, and quantifier $\forall$ binding each of the two types of variables. Other connectives, quantifiers, and modal operators are defined as usual, as are free variable occurrences and substitutions.

More than one version of propositional quantification is available for propositional **S5**. First order quantification can be constant domain or varying domain. Clearly it is necessary to be quite precise about the semantics that will be used. I'll give three versions, beginning with the most general one. Note that first order quantification is taken to be constant domain.

Definition 3.1. A **QS5**$\pi-$ *model* is a structure $\langle \mathcal{G}, \mathcal{D}, \mathcal{P}, \mathcal{I} \rangle$ where: $\mathcal{G}$ is a non-empty set (of possible worlds), $\mathcal{D}$ is a non-empty set (the first-order domain), $\mathcal{P}$ is a non-empty collection of subsets of $\mathcal{G}$ (the propositional domain), and $\mathcal{I}$ is a mapping, assigning to each n-place relation symbol and each possible world some n-place relation on $\mathcal{D}$. Members of $\mathcal{P}$ are called the *propositions* of the model, and $\mathcal{I}$ is the *interpretation*.

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DEFINITION 3.2. A *valuation* in a $\mathbf{QS5}\pi-$ model $\langle\mathscr{G},\mathscr{D},\mathscr{P},\mathscr{I}\rangle$ is a mapping v that assigns to each first-order variable a member of $\mathscr{D}$ and to each propositional variable a member of $\mathscr{P}$. Valuation w is an α-*variant* of v if w agrees with v on all variables except possibly α.

Now the key notion of truth in a model. The notation used is $\mathscr{M},\Gamma \Vdash_v \Phi$: formula Φ is true at possible world Γ of model $\mathscr{M}$, with respect to valuation v.

DEFINITION 3.3. Let $\mathscr{M} = \langle\mathscr{G},\mathscr{D},\mathscr{P},\mathscr{I}\rangle$ be a $\mathbf{QS5}\pi-$ model, let $\Gamma \in \mathscr{G}$, and let v be a valuation.

1. Atomic cases.
   (a) $\mathscr{M},\Gamma \Vdash_v R(x_1,\ldots,x_n) \Leftrightarrow \langle v(x_1),\ldots,v(x_n)\rangle \in \mathscr{I}(R,\Gamma)$.
   (b) For a propositional variable X, $\mathscr{M},\Gamma \Vdash_v X \Leftrightarrow \Gamma \in v(X)$.
   (c) $\mathscr{M},\Gamma \Vdash_v \top$ and $\mathscr{M},\Gamma \not\Vdash_v \bot$.
2. $\mathscr{M},\Gamma \Vdash_v \neg\Phi \Leftrightarrow \mathscr{M},\Gamma \not\Vdash_v \Phi$.
3. $\mathscr{M},\Gamma \Vdash_v \Phi\wedge\Psi \Leftrightarrow \mathscr{M},\Gamma \Vdash_v \Phi$ and $\mathscr{M},\Gamma \Vdash_v \Psi$.
4. $\mathscr{M},\Gamma \Vdash_v \Box\Phi \Leftrightarrow \mathscr{M},\Delta \Vdash_v \Phi$ for all $\Delta \in \mathscr{G}$.
5. $\mathscr{M},\Gamma \Vdash_v (\forall\alpha)\Phi \Leftrightarrow \mathscr{M},\Gamma \Vdash_w \Phi$ for every α-variant w of v.

An α-variant of a valuation is another valuation, and so assigns members of $\mathscr{P}$ to propositional variables. Thus item 5 makes propositional quantifiers range over members of $\mathscr{P}$.

DEFINITION 3.4. Let $\mathscr{M} = \langle\mathscr{G},\mathscr{D},\mathscr{P},\mathscr{I}\rangle$ be a $\mathbf{QS5}\pi-$ model. An *atom* is a non-empty member of $\mathscr{P}$ with no non-empty proper subset in $\mathscr{P}$.

Now two stronger notions of model are defined. In the first, formulas determine propositions. In the second, besides this, there must be lots of atoms.

DEFINITION 3.5. Let $\mathscr{M} = \langle\mathscr{G},\mathscr{D},\mathscr{P},\mathscr{I}\rangle$ be a $\mathbf{QS5}\pi-$ *model*. $\mathscr{M}$ is a $\mathbf{QS5}\pi$ model if, for every valuation v and for every formula Φ of L, $\{\Gamma \in \mathscr{G} \mid \mathscr{M},\Gamma \Vdash_v \Phi\}$ is a member of $\mathscr{P}$. $\mathscr{M}$ is a $\mathbf{QS5}\pi+$ model if it is a $\mathbf{QS5}\pi$ model, and every possible world belongs to an atom.

A formula Φ of L is *valid* in $\mathscr{M}$ provided $\mathscr{M},\Gamma \Vdash_v \Phi$ for every v and every $\Gamma \in \mathscr{G}$. Φ is $\mathbf{QS5}\pi-$ ($\mathbf{QS5}\pi$, $\mathbf{QS5}\pi+$) *valid* if it is valid in every $\mathbf{QS5}\pi-$ ($\mathbf{QS5}\pi$, $\mathbf{QS5}\pi+$) model.

There is an obvious parallel between $\mathbf{S5}\pi$ and $\mathbf{QS5}\pi$ that seems to be lost when moving to $\mathbf{S5}\pi+$ and $\mathbf{QS5}\pi+$. In $\mathbf{S5}\pi+$ models, the set of propositions is the entire powerset of the set of worlds, but in $\mathbf{QS5}\pi+$ models this need not be the case. There are two reasons for this discrepancy. First, it was shown in [4] that $\mathbf{S5}\pi+$ has the same theory as the propositional logic characterized by a semantics imposing a propositional analog of $\mathbf{QS5}\pi+$, so things are not so far apart after all. Second, a referee for this paper notes that the direct analog of $\mathbf{S5}\pi+$ (with the set of propositions always being the full powerset of the set of possible worlds) is not axiomatizable, since second-order arithmetic embeds in it. This is more than enough to account for the version of $\mathbf{QS5}\pi+$ we use.

LEMMA 3.6. *Suppose* $\mathscr{M} = \langle\mathscr{G},\mathscr{D},\mathscr{P},\mathscr{I}\rangle$ *is a* $\mathbf{QS5}\pi+$ *model,* X *is a propositional variable, and* v *is a valuation such that* $v(X)$ *is an atom. If* $\Gamma,\Delta \in v(X)$, *then for every formula* Φ, $\mathscr{M},\Gamma \Vdash_v \Phi \Longleftrightarrow \mathscr{M},\Delta \Vdash_v \Phi$.

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PROOF. Assume the hypothesis, and suppose $\Gamma, \Delta \in v(X)$, $\mathscr{M}, \Gamma \Vdash_v \Phi$ but $\mathscr{M}, \Delta \not\Vdash_v \Phi$. Let $S = \{\Omega \in \mathscr{G} \mid \mathscr{M}, \Omega \Vdash_v (\Phi \wedge X)\}$. Since $\mathscr{M}$ is a **QS5**$\pi+$ model, it is a **QS5**π model, and so $S \in \mathscr{P}$. Clearly $S \subseteq v(X)$. $S \neq \emptyset$ since $\Gamma \in S$. But also, $\Delta \notin S$, so S is a proper subset of $v(X)$, which means $v(X)$ is not an atom. ⊣

A classical logic model is specified by giving a domain and an interpretation, and these can be chosen essentially arbitrarily. The situation is similar with a **QS5**$\pi-$ model: $\mathscr{G}$, $\mathscr{D}$, $\mathscr{P}$, and $\mathscr{I}$ can be specified more-or-less arbitrarily. But whether the result is a **QS5**π, or **QS5**$\pi+$ model is not, in general, easy to determine. One case in which it is obvious is when $\mathscr{P}$ consists of *all* subsets of $\mathscr{G}$. In such a case, atoms are just singleton sets.

§4. **Statement of results.** The following is straightforward to verify.

THEOREM 4.1. *The logic* **QS5**π *extends the logic* **QS5**$\pi-$, *and is extended by* **QS5**$\pi+$. *All of* **QS5**$\pi-$, **QS5**π, *and* **QS5**$\pi+$ *are conservative extensions of* **QS5**, *where* **QS5** *is the usual constant domain first order version of* **S5** (*without propositional variables or quantifiers*).

In Section 2 it was noted that $(\exists X)\big[X \wedge (\forall Y)\big(Y \supset \Box(X \supset Y)\big)\big]$ played a role in axiomatizing **S5**$\pi+$. We need an analog for the first order version.

DEFINITION 4.2.

$$Atom(X) =_{\text{def}} \Diamond X \wedge (\forall Y)\{[\Diamond Y \wedge \Box(Y \supset X)] \supset \Box(X \supset Y)\}.$$
$$Atom =_{\text{def}} (\exists X)\{X \wedge Atom(X)\}.$$

Clearly $\mathscr{M}, \Gamma \Vdash_v Atom(X)$ if and only if $v(X)$ is an atom. The $\Diamond X$ part says that $v(X)$ is non-empty. The quantified part says that $v(X)$ is minimal. Then $\mathscr{M}, \Gamma \Vdash_v Atom$ says that Γ belongs to an atom, so $Atom$ is valid in a **QS5**π model $\mathscr{M}$ if and only if $\mathscr{M}$ is a **QS5**$\pi+$ model. Thus we have the following.

THEOREM 4.3. *A formula* Φ *of* L *is* **QS5**$\pi+$ *valid if and only if* $\Box Atom \supset \Phi$ *is* **QS5**π *valid.*

The remaining results require much more complex proofs.

THEOREM 4.4. *Both* **QS5**$\pi-$ *and* **QS5**π *have sound and complete tableau systems.*

THEOREM 4.5. *Let* Φ *and* Ψ *be closed formulas of* L. *If* $\Phi \supset \Psi$ *is* **QS5**$\pi-$ *valid, then* $\Phi \supset \Psi$ *has an interpolant in* **QS5**$\pi+$; *that is, there is a closed formula* Ω *such that all relation symbols of* Ω *are common to* Φ *and* Ψ, *and both* $\Phi \supset \Omega$ *and* $\Omega \supset \Psi$ *are* **QS5**$\pi+$ *valid.*

§5. **Prefixed tableaus.** In this section I sketch prefixed tableau systems for **QS5**$\pi-$ and **QS5**π. See [7, 9] for more details. A *prefixed formula* is $n.\Phi$, where n is a positive integer, called a *prefix*, and Φ is a formula. Intuitively, think of n as a “name” for a possible world, and $n.\Phi$ as asserting that Φ is true at that world. This use of ‘external’ names for possible worlds provides a connection with methods of [2].

A prefixed tableau is a tree, constructed using certain *branch extension rules*. A proof of Φ is a *closed* tableau, with $1.\neg\Phi$ at its root. A tableau is *closed* if each branch is closed. A branch is closed if it contains $n.\Psi$ and also $n.\neg\Psi$, for some prefix n and some formula Ψ, or if it contains $n.\bot$.

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As usual with tableaus, the language is expanded to provide existential witnesses for use in constructing proofs.

DEFINITION 5.1. L^+ is the extension of L with a new countable set of first order variables, called *first order parameters*, and a disjoint new set of propositional variables, called *propositional parameters*. We do not quantify parameters. A formula of L^+ that may contain parameters, but that contains no free variables of L, will be called *grounded*.

Tableau proofs are of closed formulas of L, but in them grounded formulas of L^+ will appear. Parameters will be used to instantiate existential quantifiers, during the course of tableau proofs. Since parameters never occur bound, accidental capture of free variables by quantifiers is impossible.

DEFINITION 5.2. A *grounded substitution* is a mapping from first order variables of L to first order parameters of L^+, and from propositional variables of L to grounded formulas of L^+. The action of a substitution is extended from variables to formulas in the usual way. If formula Φ has only α free, and substitution σ maps α to τ, I'll generally write Φ as $\Phi(\alpha)$ and $\Phi\sigma$ as $\Phi(\tau)$.

Branch extension rules for propositional connectives are as usual, except that the prefix is carried along. For instance, from $n.\neg\neg\Phi$ one infers $n.\Phi$. Likewise the first-order quantifier rules are as usual, again with prefixes carried along. The modality rules are standard, but are stated for those who may be unfamiliar with them.

Necessity Rule: If prefix k already occurs on the branch, $\dfrac{n.\Box\Phi}{k.\Phi}$.

Possibility Rule: If prefix k is new to the branch, $\dfrac{n.\neg\Box\Phi}{k.\neg\Phi}$.

Finally we have the propositional quantifier rules.

Propositional Universal QS5$\pi-$ Rule: If P is any propositional parameter,

$$\frac{n.(\forall X)\Phi(X)}{n.\Phi(P)}.$$

Propositional Universal QS5π Rule: If F is any grounded formula of L^+,

$$\frac{n.(\forall X)\Phi(X)}{n.\Phi(F)}.$$

Propositional Existential Rule: If propositional parameter P is new to the branch,

$$\frac{n.\neg(\forall X)\Phi(X)}{n.\neg\Phi(P)}.$$

Two tableau systems have been presented simultaneously, differing only in the branch extension rules for the propositional universal quantifier. If the **QS5$\pi-$** propositional universal rule is used, the tableau system is called the **QS5$\pi-$** system, and similarly for the **QS5π** system.

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§6. **Completeness.** Soundness is proved as usual for tableaus, and the argument is omitted. Arguments of this kind can be found in [6, 7, 9]. Completeness for **QS5**$\pi-$ also follows standard lines, and once more I omit details. Completeness for **QS5**π is less straightforward, and key portions are given here. The proof amounts to a simplified version of a higher type argument from [16, 12].

Call a set S of grounded, prefixed formulas *consistent* if no tableau starting with a finite subset of S closes. Call S *E-complete* if it has appropriate witnesses. Specifically:

1. $n.\neg(\forall x)\Phi(x) \in S$ implies $n.\neg\Phi(p) \in S$ for a first order parameter p,
2. $n.\neg(\forall X)\Phi(X) \in S$ implies $n.\neg\Phi(P) \in S$ for a propositional parameter P,
3. $n.\neg\Box\Phi \in S$ implies $k.\neg\Phi \in S$ for a prefix k.

As usual, any consistent set of prefixed, grounded formulas that omits infinitely many parameters of each kind, and omits infinitely many prefixes, can be extended to a maximal consistent, E-complete set. When dealing with axiom systems, maximal consistent, E-complete sets typically satisfy equivalences such as: $X \wedge Y$ is present if and only if both X and Y are present. With tableaus (without cut) only implications downward are obtainable: if $n.X \wedge Y$ is present, so are both $n.X$ and $n.Y$. Fortunately, these implications are enough to establish completeness.

Now, suppose H is a maximal consistent, E-complete set. A model will be constructed in which H is satisfied in a natural way.

Definition 6.1. An *entity* is a pair $\langle F, S\rangle$, where F is a grounded formula and S is a set of prefixes (positive integers).

A *possible value* is any entity $\langle F, S\rangle$ such that, if $n.F \in H$ then $n \in S$, and if $n.\neg F \in H$ then $n \notin S$.

A *possible extension* is a set S of prefixes such that $\langle F, S\rangle$ is a possible value, for some formula F.

A possible value is an entity that is, in some sense, compatible with the information in H.

Definition 6.2. A **QS5**$\pi-$ model $\mathscr{M} = \langle \mathscr{G}, \mathscr{D}, \mathscr{P}, \mathscr{I}\rangle$ (called the *H-model*) is constructed as follows: $\mathscr{G}$ is the set of prefixes (positive integers); $\mathscr{D}$ is the set of first order parameters; $\mathscr{P}$ is the set of possible extensions; for R a k-place relation symbol, $\mathscr{I}(R, n) = \{\langle p_1, \dots, p_k\rangle \in \mathscr{D}^k \mid n.R(p_1, \dots, p_k) \in H\}$.

It is obvious that the H-model is a **QS5**$\pi-$ model. It will be shown that it is a **QS5**π model.

Definition 6.3. Let v be a valuation in $\mathscr{M}$, and let σ be a grounded substitution (Definition 5.2). σ is an *associate* of v provided,

1. for any propositional variable X, $\langle X\sigma, v(X)\rangle$ is a possible value;
2. for any first order variable x, $v(x) = x\sigma$.

Every valuation v in $\mathscr{M}$ has an associate (in general, many). For a propositional variable X, $v(X)$ is a possible extension, so there is a grounded formula F such that $\langle F, v(X)\rangle$ is a possible value. Choose one such F and set $X\sigma = F$. For first order variable x, set $x\sigma = v(x)$. Then σ is a grounded substitution that is an associate of v.

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PROPOSITION 6.4. *Let v be a valuation in $\mathscr{M}$, and let σ be any associate of v. For any formula Φ of L:*

1. *If $n.(\Phi\sigma) \in H$ then $\mathscr{M}, n \Vdash_v \Phi$.*
2. *If $n.(\neg\Phi\sigma) \in H$ then $\mathscr{M}, n \nVdash_v \Phi$.*

PROOF. The two items are shown simultaneously, by induction on the complexity of Φ. Most of the cases are standard and are omitted. I give just the Propositional Quantifier ones. Suppose Φ is $(\forall X)\Psi(X)$, and the result is known for formulas of lower degree than Φ.

1. Suppose $n.[(\forall X)\Psi(X)]\sigma \in H$. Let v' be an arbitrary X-variant of v. We show that $\mathscr{M}, n \Vdash_{v'} \Psi(X)$; it follows that $\mathscr{M}, n \Vdash_v (\forall X)\Psi(X)$.

   Say $v'(X) = S$. S is a possible extension, so there is a grounded formula F such that $\langle F, S\rangle$ is a possible value—let σ' be the substitution that is like σ, except that $X\sigma' = F$. Clearly σ' is an associate of v'. Since $n.[(\forall X)\Psi(X)]\sigma \in H$, then $n.[\Psi(F)]\sigma \in H$. This is equivalent to $n.[\Psi(X)]\sigma' \in H$. Since $\Psi(X)$ is of lower degree than $(\forall X)\Psi(X)$, the induction hypothesis applies, and so $\mathscr{M}, n \Vdash_{v'} \Psi(X)$.
2. Suppose $n.[\neg(\forall X)\Psi(X)]\sigma \in H$. Then $n.[\neg\Psi(P)]\sigma \in H$, for some propositional parameter P. Let σ' be like σ, except that $X\sigma' = P$. Then $n.[\neg\Psi(X)]\sigma' \in H$.

   Let $S = \{k \mid k.P \in H\}$. It is easy to see that $\langle P, S\rangle$ is a possible value. Let v' be the X-variant of v such that $v'(X) = S$. σ' is an associate of v'. Since $\Psi(X)$ is of lower degree than $(\forall x)\Psi(X)$, $\mathscr{M}, n \nVdash_{v'} \Psi(X)$. Since v' is an X-variant of v, it follows that $\mathscr{M}, n \nVdash_v (\forall X)\Psi(X)$. ⊣

COROLLARY 6.5. *For any formula Φ of L, and for any valuation v in $\mathscr{M}$, $\{n \mid \mathscr{M}, n \Vdash_v \Phi\} \in \mathscr{P}$, the set of possible extensions.*

PROOF. Let substitution σ be any associate of v. $\Phi\sigma$ is a grounded formula of L^+. For convenience let $S = \{n \mid \mathscr{M}, n \Vdash_v \Phi\}$. We show that $\langle \Phi\sigma, S\rangle$ is a possible value, from which it follows that S is a possible extension.

Suppose $n.\Phi\sigma \in H$. By Proposition 6.4, $\mathscr{M}, n \Vdash_v \Phi$, hence $n \in S$. Similarly if $n.\neg\Phi\sigma \in H$, $\mathscr{M}, n \nVdash_v \Phi$, and so $n \notin S$. This concludes the proof. ⊣

Completeness for the **QS5**π tableau system follows. If Φ is a closed formula of L that has no **QS5**π tableau proof, then $\{1.\neg\Phi\}$ is consistent, and so extends to a maximal consistent, E-complete set H. Let $\mathscr{M}$ be the corresponding H-model, a **QS5**$\pi-$ model by construction. Also, let v be any valuation, and let σ be any associate of v. Since Φ is closed, $\Phi\sigma = \Phi$. Since $1.\neg\Phi \in H$, $\mathscr{M}, 1 \nVdash_v \Phi$. And finally, by Corollary 6.5, $\mathscr{M}$ is actually a **QS5**π model.

§7. **Quasi-interpolants.** The interpolation theorem, 4.5, will be proved by showing how to extract an interpolant from a closed tableau. I'll follow the methodology of [8], which is a variant of that in [14]. In a tableau proof of $\Phi \supset \Psi$, one begins with the negation of the formula, which yields Φ and $\neg\Psi$. From here on we keep track of which tableau entries descend from Φ and which from $\neg\Psi$ by marking them with an L (for *left* side of implication) or R (for *right* side).

A *biased formula* is either $L(n.\Phi)$ or $R(n.\Phi)$, where $n.\Phi$ is a prefixed formula. A *biased tableau* is one in which only biased formulas appear, and for which branch

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extension rules are modified by adding L's and R's in the obvious way. For instance, the conjunction rule gives us the following two biased rules.

$$\frac{L(n.\Phi \wedge \Psi)}{\begin{array}{c} L(n.\Phi) \\ L(n.\Psi) \end{array}} \qquad \frac{R(n.\Phi \wedge \Psi)}{\begin{array}{c} R(n.\Phi) \\ R(n.\Psi) \end{array}}.$$

Similarly all the other tableau rules split into L versions and R versions. Now a branch is closed if it contains a syntactic contradiction, *ignoring the L and R symbols*. If $\Phi \supset \Psi$ has a **QS5**$\pi-$ tableau proof, it can obviously be converted into a closed, biased tableau beginning with $L(1.\Phi)$ and $R(1.\neg\Psi)$.

A *quasi-interpolant* (definition below) will be assigned to each finite set of biased formulas that can generate a closed biased tableau. If $\Phi \supset \Psi$ is a **QS5**$\pi-$ valid closed formula, the set $\{L(1.\Phi), R(1.\neg\Psi)\}$ must then have a quasi-interpolant. Eventually I'll show how to convert this quasi-interpolant into a real interpolant for the formula $\Phi \supset \Psi$.

In order to define the notion of a quasi-interpolant, we first introduce a distinguished family of propositional variables. Recall that Definition 3.4 requires that a **QS5**$\pi+$ model must have lots of atoms.

Definition 7.1. Let $A_1, A_2, A_3, \ldots$ be a new infinite list of propositional parameters, called *atom parameters*. A valuation v in a **QS5**$\pi+$ model $\langle \mathscr{G}, \mathscr{D}, \mathscr{P}, \mathscr{I} \rangle$ is called *atom respecting* provided it assigns atoms in $\mathscr{P}$ to atom parameters.

In effect, atom parameters supply names for atoms from inside the modal language. They are distinct from the propositional parameters we use to instantiate propositional existential quantifiers. Atom parameters will eventually be eliminated.

Definition 7.2. Call a formula Φ *Atom*-**QS5**$\pi+$ valid if $\mathscr{M}, \Gamma \Vdash_v \Phi$ for every **QS5**$\pi+$ model $\mathscr{M}$, for every world Γ of $\mathscr{M}$, and for every *atom-respecting* valuation v.

Definition 7.3. Let $\{L(i_1.\Phi_1), \ldots, L(i_m.\Phi_m), R(j_1.\Psi_1), \ldots, R(j_k.\Psi_k)\}$ be a finite set of biased formulas. A *quasi-interpolant* for this set is a formula Ω meeting the following conditions.

1. The only free variables in Ω are propositional and first order parameters, including atom parameters.
2. Every relation symbol or parameter that occurs in Ω must also occur in $\{A_{i_1}, \ldots, A_{i_m}, \Phi_1, \ldots, \Phi_m\}$ and in $\{A_{j_1}, \ldots, A_{j_k}, \Psi_1, \ldots, \Psi_k\}$.
3. $[\Box(A_{i_1} \supset \Phi_1) \wedge \cdots \wedge \Box(A_{i_m} \supset \Phi_m)] \supset \Omega$ is *Atom*-**QS5**$\pi+$ valid.
4. $\Omega \supset [\Box(A_{j_1} \supset \neg\Psi_1) \vee \cdots \vee \Box(A_{j_k} \supset \neg\Psi_k)]$ is *Atom*-**QS5**$\pi+$ valid.

Note: in the definition above, A_{i_n} and A_{j_n} are the atom parameters having the explicit indices shown, the prefixes of Φ_n and Ψ_n respectively.

Theorem 7.4. *If there is a closed* **QS5**$\pi-$ *tableau for* $\{L(i_1.\Phi_1), \ldots, L(i_m.\Phi_m), R(j_1.\Psi_1), \ldots, R(j_k.\Psi_k)\}$, *then the set has a quasi-interpolant.*

Proof. The proof is constructive, and is similar to the classical version. A quasi-interpolant is assigned to each closed tableau branch. Then, each tableau rule application is undone, and quasi-interpolants are produced for the resulting shorter branches based on those for the original longer ones. In this way, a quasi-interpolant is finally produced for the initial set.

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I'll write $T \xrightarrow{int} \Omega$ to indicate that Ω is a quasi-interpolant for a set T of biased formulas. In what follows I'll assume $S = \{L(i_1.\Phi_1), \ldots, L(i_m.\Phi_m), R(j_1.\Psi_1), \ldots, R(j_k.\Psi_k)\}$. S_L is the set of parameters and prefixes occurring in L-formulas of S. Similarly for S_R. $\Omega\{\gamma/\beta\}$ is the result of replacing all occurrences of γ in Ω with occurrences of the variable β.

Closed Branch Cases:

$$S \cup \{L(n.\Phi), L(n.\neg\Phi)\} \xrightarrow{int} \bot \qquad S \cup \{R(n.\Phi), R(n.\neg\Phi)\} \xrightarrow{int} \top$$
$$S \cup \{L(n.\bot)\} \xrightarrow{int} \bot \qquad S \cup \{R(n.\bot)\} \xrightarrow{int} \top$$
$$S \cup \{L(n.\Phi), R(n.\neg\Phi)\} \xrightarrow{int} \Box(A_n \supset \Phi)$$
$$S \cup \{R(n.\Phi), L(n.\neg\Phi)\} \xrightarrow{int} \Box(A_n \supset \neg\Phi).$$

Negation Cases:

$$\frac{S \cup \{L(n.\top)\} \xrightarrow{int} \Omega}{S \cup \{L(n.\neg\bot)\} \xrightarrow{int} \Omega} \qquad \frac{S \cup \{L(n.\bot)\} \xrightarrow{int} \Omega}{S \cup \{L(n.\neg\top)\} \xrightarrow{int} \Omega}$$
$$\frac{S \cup \{R(n.\top)\} \xrightarrow{int} \Omega}{S \cup \{R(n.\neg\bot)\} \xrightarrow{int} \Omega} \qquad \frac{S \cup \{R(n.\bot)\} \xrightarrow{int} \Omega}{S \cup \{R(\neg\top.)\} \xrightarrow{int} \Omega}$$
$$\frac{S \cup \{L(n.\Phi)\} \xrightarrow{int} \Omega}{S \cup \{L(n.\neg\neg\Phi)\} \xrightarrow{int} \Omega} \qquad \frac{S \cup \{R(n.\Phi)\} \xrightarrow{int} \Omega}{S \cup \{R(n.\neg\neg\Phi)\} \xrightarrow{int} \Omega}.$$

Conjunctive Cases:

$$\frac{S \cup \{L(n.\Phi), L(n.\Psi)\} \xrightarrow{int} \Omega}{S \cup \{L(n.\Phi \wedge \Psi)\} \xrightarrow{int} \Omega} \qquad \frac{S \cup \{R(n.\Phi), R(n.\Psi)\} \xrightarrow{int} \Omega}{S \cup \{R(n.\Phi \wedge \Psi)\} \xrightarrow{int} \Omega}.$$

Disjunctive Cases:

$$\frac{S \cup \{L(n.\neg\Phi)\} \xrightarrow{int} \Theta \qquad S \cup \{L(n.\neg\Psi)\} \xrightarrow{int} \Omega}{S \cup \{L(n.\neg(\Phi \wedge \Psi))\} \xrightarrow{int} \Theta \vee \Omega}$$
$$\frac{S \cup \{R(n.\neg\Phi)\} \xrightarrow{int} \Theta \qquad S \cup \{R(n.\neg\Psi)\} \xrightarrow{int} \Omega}{S \cup \{R(n.\neg(\Phi \wedge \Psi))\} \xrightarrow{int} \Theta \wedge \Omega}.$$

Existential Cases: Let γ be a parameter not occurring in S or in $\neg(\forall\alpha)\varphi(\alpha)$, first order if α is a first order variable, and propositional otherwise.

$$\frac{S \cup \{L(n.\neg\varphi(\gamma))\} \xrightarrow{int} \Omega}{S \cup \{L(n.\neg(\forall\alpha)\varphi(\alpha))\} \xrightarrow{int} \Omega} \qquad \frac{S \cup \{R(n.\neg\varphi(\gamma))\} \xrightarrow{int} \Omega}{S \cup \{R(n.\neg(\forall\alpha)\varphi(\alpha))\} \xrightarrow{int} \Omega}.$$

Universal Cases: γ is a parameter of the same type as the variable α, and the variable β is new to Ω. (β is introduced to get around the restriction that

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parameters aren't quantified.)

$$\frac{S \cup \{L(n.\varphi(\gamma))\} \xrightarrow{int} \Omega}{S \cup \{L(n.(\forall\alpha)\varphi(\alpha))\} \xrightarrow{int} \Omega} \quad \text{if } \gamma \in S_L$$

$$\frac{S \cup \{L(n.\varphi(\gamma))\} \xrightarrow{int} \Omega}{S \cup \{L(n.(\forall\alpha)\varphi(\alpha))\} \xrightarrow{int} (\forall\beta)\Omega\{\gamma/\beta\}} \quad \text{if } \gamma \notin S_L$$

$$\frac{S \cup \{R(n.\varphi(\gamma))\} \xrightarrow{int} \Omega}{S \cup \{R(n.(\forall\alpha)\varphi(\alpha))\} \xrightarrow{int} \Omega} \quad \text{if } \gamma \in S_R$$

$$\frac{S \cup \{R(n.\varphi(\gamma))\} \xrightarrow{int} \Omega}{S \cup \{R(n.(\forall\alpha)\varphi(\alpha))\} \xrightarrow{int} (\exists\beta)\Omega\{\gamma/\beta\}} \quad \text{if } \gamma \notin S_R.$$

Modal Possibility Cases: In these, the prefix n must not appear in S, and must be distinct from h.

$$\frac{S \cup \{L(n.\neg\Phi)\} \xrightarrow{int} \Omega}{S \cup \{L(h.\neg\Box\Phi)\} \xrightarrow{int} \Omega} \qquad \frac{S \cup \{R(n.\neg\Phi)\} \xrightarrow{int} \Omega}{S \cup \{R(h.\neg\Box\Phi)\} \xrightarrow{int} \Omega}.$$

Finally the cases that will be presented in some detail. Recall the definition of $Atom(X)$, Definition 4.2.

Modal Necessity Cases: The propositional variable Q is new to Ω.

$$\frac{S \cup \{L(n.\Phi)\} \xrightarrow{int} \Omega}{S \cup \{L(h.\Box\Phi)\} \xrightarrow{int} \Omega} \quad \text{if } n \in S_L$$

$$\frac{S \cup \{L(n.\Phi)\} \xrightarrow{int} \Omega}{S \cup \{L(h.\Box\Phi)\} \xrightarrow{int} (\forall Q)[Atom(Q) \supset \Omega\{A_n/Q\}]} \quad \text{if } n \notin S_L$$

$$\frac{S \cup \{R(n.\Phi)\} \xrightarrow{int} \Omega}{S \cup \{R(h.\Box\Phi)\} \xrightarrow{int} \Omega} \quad \text{if } n \in S_R$$

$$\frac{S \cup \{R(n.\Phi)\} \xrightarrow{int} \Omega}{S \cup \{R(h.\Box\Phi)\} \xrightarrow{int} (\exists Q)[Atom(Q) \wedge \Omega\{A_n/Q\}]} \quad \text{if } n \notin S_R.$$

Soundness for these rules is generally similar to the usual classical and modal arguments. I'll do the Necessity Rules in detail.

Two of the quantifier cases use the validity $(\forall x)\varphi(x) \supset \varphi(p)$. A modal analog is: $\Box(A_h \supset \Box\Phi) \supset \Box(A_n \supset \Phi)$, an $Atom$-**QS5**$\pi+$ valid formula, as I will show in a moment. With it the soundness of the first and third rules above is straightforward. Suppose, in a **QS5**$\pi+$ model, that $\Box(A_h \supset \Box\Phi)$ is true at world Γ, with respect to atom respecting valuation, v. A_h is an atom parameter, so $v(A_h)$ is an atom and hence non-empty. Say $\Delta \in v(A_h)$, so A_h is true at Δ. But $A_h \supset \Box\Phi$ is true at Δ since it is necessary at Γ. Hence $\Box\Phi$ is true at Δ, and so Φ is true at every world. Then $A_n \supset \Phi$ is true at every world, so $\Box(A_n \supset \Phi)$ is true at Γ.

Finally I'll show soundness of the second rule above (the fourth is similar). Assume $S \cup \{L(n.\Phi)\} \xrightarrow{int} \Omega$, where $n \notin S_L$. I'll show $S \cup \{L(h.\Box\Phi)\} \xrightarrow{int} (\forall Q)[Atom(Q) \supset \Omega\{A_n/Q\}]$.

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By assumption, both of the following are *Atom*-**QS5**$\pi+$ valid.

$$[\Box(A_{i_1} \supset \Phi_1) \wedge \cdots \wedge \Box(A_{i_m} \supset \Phi_m) \wedge \Box(A_n \supset \Phi)] \supset \Omega$$
$$\Omega \supset [\Box(A_{j_1} \supset \neg\Psi_1) \vee \cdots \vee \Box(A_{j_k} \supset \neg\Psi_k)].$$

Also every relation symbol or parameter that occurs in Ω occurs in $\{A_{i_1}, \ldots, A_{i_m}, A_n, \Phi_1, \ldots, \Phi_m, \Phi\}$ and in $\{A_{j_1}, \ldots, A_{j_k}, \Psi_1, \ldots, \Psi_k\}$.

First, I'll show the *Atom*-**QS5**$\pi+$ validity of the following.

$$(\forall Q)[Atom(Q) \supset \Omega\{A_n/Q\}] \supset [\Box(A_{j_1} \supset \neg\Psi_1) \vee \cdots \vee \Box(A_{j_k} \supset \neg\Psi_k)].$$

But this is easy. $(\forall Q)[Atom(Q) \supset \Omega\{A_n/Q\}] \supset [Atom(A_n) \supset \Omega\{A_n/A_n\}]$ is **QS5**π valid and, since A_n is an atom parameter, $Atom(A_n)$ is *Atom*-**QS5**$\pi+$ valid. Consequently $(\forall Q)[Atom(Q) \supset \Omega\{A_n/Q\}] \supset \Omega$ is *Atom*-**QS5**$\pi+$ valid, and the desired formula follows.

Next, I'll show the *Atom*-**QS5**$\pi+$ validity of the following.

$$[\Box(A_{i_1} \supset \Phi_1) \wedge \cdots \wedge \Box(A_{i_m} \supset \Phi_m) \wedge \Box(A_h \supset \Box\Phi)] \supset (\forall Q)[Atom(Q) \supset \Omega\{A_n/Q\}].$$

Suppose otherwise; say that at world Γ of model $\mathscr{M}$, and for some atom respecting valuation v we have

$$\mathscr{M}, \Gamma \Vdash_v \Box(A_{i_1} \supset \Phi_1) \wedge \cdots \wedge \Box(A_{i_m} \supset \Phi_m) \wedge \Box(A_h \supset \Box\Phi)$$
$$\mathscr{M}, \Gamma \nVdash_v (\forall Q)[Atom(Q) \supset \Omega\{A_n/Q\}].$$

There must be a Q-variant v' of v such that $\mathscr{M}, \Gamma \Vdash_{v'} Atom(Q)$ but $\mathscr{M}, \Gamma \nVdash_{v'} \Omega\{A_n/Q\}$. By the first of these, $v'(Q)$ must be an atom. Let v'' be the A_n variant of v' such that $v''(A_n) = v'(Q)$. Then v'' is another atom respecting valuation. By the second, $\mathscr{M}, \Gamma \nVdash_{v''} \Omega$.

Q was a "new" propositional variable, and $n \notin S_L$, so A_n does not occur in any of $\Box(A_{i_1} \supset \Phi_1), \ldots, \Box(A_{i_m} \supset \Phi_m)$. Then v and v'' agree on all the free variables of these formulas, so,

$$\mathscr{M}, \Gamma \Vdash_{v''} \Box(A_{i_1} \supset \Phi_1) \wedge \cdots \wedge \Box(A_{i_m} \supset \Phi_m).$$

Also $\mathscr{M}, \Gamma \Vdash_v \Box(A_h \supset \Box\Phi)$. $v(A_h)$ is an atom, hence non-empty. If $\Delta \in v(A_h)$, at Δ we have $\Box\Phi$, hence Φ is true at *every* world with respect to v. Φ is a grounded formula and so cannot contain A_n or Q free, so v and v'' agree on the free variables of Φ. It follows that we have $\mathscr{M}, \Gamma \Vdash_{v''} \Box(A_n \supset \Phi)$. Since we have $[\Box(A_{i_1} \supset \Phi_1) \wedge \cdots \wedge \Box(A_{i_m} \supset \Phi_m) \wedge \Box(A_n \supset \Phi)] \supset \Omega$, it follows that $\mathscr{M}, \Gamma \Vdash_{v''} \Omega$, and we have a contradiction.

The condition on relation symbol and parameter occurrences is straightforward to verify, and is left to the reader.

This concludes the proof. $\dashv$

§**8. The interpolation theorem.** Theorem 7.4 is not quite what we want, but it is almost there. With it available, here is the proof of Theorem 4.5.

Proof. Let Φ and Ψ be closed formulas of L such that $\Phi \supset \Psi$ is **QS5**$\pi-$ valid. Then there is a proof of $\Phi \supset \Psi$ in the **QS5**$\pi-$ tableau system, and consequently a quasi-interpolant, call it Ω.

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1. Since Φ and Ψ are closed formulas, Ω cannot contain any first order parameters or propositional parameters other than atom parameters.
2. The only atom parameter Ω can contain is A_1.
3. $[\Box(A_1 \supset \Phi)] \supset \Omega$ is *Atom*-**QS5**π+ valid.
4. $\Omega \supset [\Box(A_1 \supset \neg\neg\Psi)]$ is *Atom*-**QS5**π+ valid.

Here is the candidate for an interpolant, where propositional variable Q is new to Ω.

$$\Omega^* = (\exists Q)[Q \wedge Atom(Q) \wedge \Omega\{A_1/Q\}].$$

From the quasi-interpolant conditions, any relation symbol in Ω^* must be common to Φ and Ψ. And since A_1 was the only free variable in Ω, either propositional or first order, Ω^* is a closed formula. It remains to show $\Phi \supset \Omega^*$ and $\Omega^* \supset \Psi$ are both **QS5**π+ valid formulas. Let $\mathscr{M}$ be a **QS5**π+ model, let Γ be a possible world of the model, and let v be an arbitrary valuation, not necessarily atom respecting.

Suppose first that $\mathscr{M}, \Gamma \Vdash_v \Phi$; I'll show $\mathscr{M}, \Gamma \Vdash_v \Omega^*$. Since $\mathscr{M}$ is a **QS5**π+ model, there is an atom S such that $\Gamma \in S$. Let v' be any valuation that is like v except that it assigns atoms to atom parameters, and $v'(A_1) = S$. Since Φ contains no free variables, Φ is true at the same worlds with respect to v' and v. Then Φ is true at Γ with respect to v', and hence at every member of S, by Lemma 3.6. Consequently $A_1 \supset \Phi$ is true at every member of S, with respect to v'. Also, $A_1 \supset \Phi$ is true at every non-member of S, with respect to v', because A_1 will be false. Thus $A_1 \supset \Phi$ is true at every world, with respect to v', and so $\mathscr{M}, \Gamma \Vdash_{v'} \Box(A_1 \supset \Phi)$. Now by item 3 above, $\mathscr{M}, \Gamma \Vdash_{v'} \Omega$. Of course $\mathscr{M}, \Gamma \Vdash_{v'} A_1 \wedge Atom(A_1)$. From all this it follows that $\mathscr{M}, \Gamma \Vdash_v (\exists Q)[Q \wedge Atom(Q) \wedge \Omega\{A_1/Q\}]$, that is, $\mathscr{M}, \Gamma \Vdash_v \Omega^*$.

Finally, suppose that $\mathscr{M}, \Gamma \Vdash_v \Omega^*$; I'll show that $\mathscr{M}, \Gamma \Vdash_v \Psi$. Since $\mathscr{M}, \Gamma \Vdash_v (\exists Q)[Q \wedge Atom(Q) \wedge \Omega\{A_1/Q\}]$, for some Q-variant v' of v, $\mathscr{M}, \Gamma \Vdash_{v'} Q \wedge Atom(Q) \wedge \Omega\{A_1/Q\}$. Since $\mathscr{M}, \Gamma \Vdash_{v'} Atom(Q)$, $v'(Q)$ is an atom. Let v'' be like v except that it assigns atoms to atom parameters, and $v''(A_1) = v'(Q)$. Since $\mathscr{M}, \Gamma \Vdash_{v'} \Omega\{A_1/Q\}$, it follows that $\mathscr{M}, \Gamma \Vdash_{v''} \Omega$. Using item 4 above, $\mathscr{M}, \Gamma \Vdash_{v''} \Box(A_1 \supset \Psi)$. Since $\mathscr{M}, \Gamma \Vdash_{v'} Q$ then $\mathscr{M}, \Gamma \Vdash_{v''} A_1$, and it follows that $\mathscr{M}, \Gamma \Vdash_{v''} \Psi$. Finally, since Ψ contains no free variables, $\mathscr{M}, \Gamma \Vdash_v \Psi$.

This concludes the proof. $\dashv$

The interpolant Ω^* for a valid $\Phi \supset \Psi$, constructed above, will not only have its relation symbols common to Φ and Ψ, but polarity will be respected as well: a relation symbol occurring positively in Ω will occur positively in Φ and Ψ, and similarly for negative occurrences. We actually have a version of the Craig-Lyndon theorem.

§9. **Conclusion.** In [5] Fine showed the Beth Definability Theorem fails for **QS5** with the Barcan formula. From his proof one can extract the following example. Let

$$\Phi = (\exists x)A(x) \wedge (\forall y)\Diamond(\forall x)\big[F(x) \supset \Box\big(A(y) \supset \neg F(x)\big)\big]$$
$$\Psi = (\exists x)B(x) \supset (\exists y)\Diamond(\forall x)\big[F(x) \supset \Diamond\big(B(y) \wedge \neg F(x)\big)\big].$$

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Then $\Phi \supset \Psi$ is **QS5**$\pi-$ valid, but has no interpolant in this logic. Using the construction above, the following is an interpolant for it in **QS5**$\pi+$.

$$(\exists Q)\Big\{Q \wedge Atom(Q) \wedge (\exists R)\Big[Atom(R) \wedge \\ (\forall x)\big[\Box\big(Q \supset \neg F(x)\big) \vee \Box\big(R \supset \neg F(x)\big)\big]\Big]\Big\}.$$

This formula is not equivalent to one without propositional quantifiers, since interpolation fails in **QS5**$\pi-$. Now consider the following **QS5**$\pi-$ valid formula.

$$\{(\forall x)\Diamond[A(x) \vee B(x)] \wedge (\exists x)\Box\neg B(x)\} \supset \{\Box(\forall x)C(x) \supset \Diamond(\exists x)[A(x) \wedge C(x)]\}.$$

The construction above gives $(\exists Q)\big[Atom(Q) \wedge (\exists x)\Box\big(Q \supset A(x)\big)\big]$ as an interpolant in **QS5**$\pi+$. It is not hard to see this is equivalent to $\Diamond(\exists x)A(x)$. Question: is there some way of recognizing which formulas admit elimination of propositional quantifiers and which do not? Is there some systematic procedure for eliminating them when possible?

One of the referees suggested an alternative method of proof for Theorem 4.5. Propositional **S5** translates into classical first-order logic in a standard way: treat modal operators as quantifiers over possible worlds. With first-order and propositional quantifiers added to the modal language, a translation can be made into a classical three-sorted language. Suppose we denote the translate of Φ by Φ^*. There is an interpolation theorem for classical many-sorted logic, so if $\Phi \supset \Psi$ is **QS5**$\pi-$ valid, there is a classical interpolant, Ω, for $\Phi^* \supset \Psi^*$. Ordinarily a formula like Ω need not be the translate of any modal formula, but with propositional quantifiers available, atoms are definable and serve within the language as counterparts of possible worlds, making a 'reverse' translation possible. Thus Ω gives us a modal interpolant.

I have not worked through the details sketched in the previous paragraph, but I suspect this approach simply amounts to shifting the construction of the interpolant given in this paper to an alternative arena, with no essential changes. What is significant, though, is that the referee also suggests a potential extension of Theorem 4.5 might be proved along these lines. Suppose $\Phi \supset \Psi$ is valid not in **QS5**$\pi-$, but in **QS5**$\pi+$. Again, $\Phi^* \supset \Psi^*$ will be valid in a classical three-sorted logic. The counterpart of propositional quantifiers, in this logic, will essentially be second-order quantifiers in a Henkin model. The referee proposes that a classical interpolation theorem might be provable in such a setting that constrains second-order quantification in the interpolant to quantifiers that will 'translate back' to propositional quantifiers in the original modal setting. If this is the case, a classical interpolant for $\Phi^* \supset \Psi^*$ could still be used to get a modal interpolant for $\Phi \supset \Psi$. This would mean **QS5**$\pi+$ itself has the interpolation property, and would avoid the 'mixed logic' flavor of Theorem 4.5. I do not know if the project can, in fact, be carried out. I welcome investigation by others.

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