S4LP and Local Realizability

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Abstract. The logic S4LP combines the modal logic S4 with the justification logic LP, both axiomatically and semantically. We introduce a simple restriction on the behavior of constants in S4LP, having no effect on the LP sublogic. Under this restriction some powerful derived rules are established. Then these are used to show completeness relative to a semantics having what we call the *local realizability* property: at each world and for each formula true at that world there is a realization also true at that world, where a realization is the result of replacing all modal operators with explicit justification terms. This is a part of a project to understand the deeper aspects of Artemov's Realization Theorem.

1 Introduction

Logics of knowledge, Hintikka style, are familiar tools, [1]. Recently a family of *justification logics* has been created. In these, instead of a modal operator, *known*, there is an infinite family of explicit reasons by which something is known. There are justification logic analogs of several standard single-knower Hintikka style logics, and work proceeds on multiple-knower versions. Connections between Hintikka logics and explicit logics are quite close, via *Realization Theorems*. They say any theorem of a standard Hintikka logic of knowledge can be *realized*, its knowledge operators can be replaced with explicit justifications, to produce a theorem of the corresponding explicit logic of knowledge. Thus the usual knowledge operators carry hidden explicit content.

Justification logics began with an analog of S4, due to Sergei Artemov, [2]. The motivation was to create an arithmetic semantics for propositional intuitionistic logic, completing a project begun by Gödel. Artemov succeeded in this. The justification logic created was called LP, for "logic of proofs;" explicit justifications represent formal arithmetic proofs. It was soon realized that proofs were only one kind of justification, and LP was one of a family of similar logics. In order to keep the discussion manageable here, I will frame it in terms of LP, thinking of it as a representative member of the family but having historical precedence. The work, in fact, applies to a range of logics.

The Realization Theorem connects LP with S4. Each theorem of S4 has a realization—a replacement of modal operators with explicit justification terms that produces a theorem of LP. (The converse is also true, and trivial.) Indeed, a realization can be chosen that is *normal*, negative occurrences of necessity can be replaced with distinct variables. Moreover a realization can be extracted constructively from a proof in S4. Given this fundamental relationship between S4 and LP, it is natural to consider a logic that combines of LP and S4, so that both explicit and implicit notions of knowledge are present. This has been done, it is known as S4LP, [3, 4]. Axiomatically, one simply provides the machinery of S4, the machinery of LP, and a connecting axiom saying that explicit knowledge implies implicit knowledge (see Section 3.1). S4LP is a conservative extension of both S4 and LP. The Realization Theorem becomes a result about this single logic, rather than one connecting two different ones. Unfortunately, the only proofs known for the single-logic version of the Realization Theorem detour through the older proofs, via conservativity. (Unfortunately too, this paper does not shed any fresh light on this important issue.)

A Hintikka/Kripke semantics for S4 is standard. In [5, 6] a semantics for LP was presented, combining justification logic machinery originating in [7] with the usual S4 semantics. This semantics has been adapted to S4LP in two distinct ways. First, one can use the LP semantics without change, since there is an underlying Kripke structure for the interpretation of the modal operator. Axiomatic soudness is in [3, 4] and a completeness theorem is in [8]. The second semantics is the single agent version of an n-agent logic of knowledge with explicit common knowledge, [9, 10]. In this, separate accessibility relations are used for the modal operator and for explicit justification terms. Again, soundness and completeness results have been shown. We are not concerned here with the two-accessibility-relation version of S4LP semantics, but only the single accessibility version, as investigated in [3, 4, 8]. In this, justifications can be thought of as supplying an analysis of an individual's knowledge, and the connection between justifications and the modal/knowledge operator can be expected to be quite close.

We say an S4LP model meets the *local realizability condition* provided, at each possible world of the model, each formula that is true at that world has a realization that is also true at that world (normaliity is not required). The main result of this paper is that axiomatic S4LP is complete with respect to models meeting the local realizability condition, provided a certain condition is placed on the constant specifications allowed. What was called *strong* completeness for LP in [6] is an easy corollary.

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2 The Logic LP

We begin with a sketch of the oldest justification logic, LP, from [2]. First, the language and an axiom system, and then a standard semantics.

2.1 LP Axiomatically

Justification terms or *proof terms* are built up from *variables*, x_1, x_2, $\ldots$, and *constant symbols*, c_1, c_2, $\ldots$. They are built up using the following *operation* *symbols*: $+$ and $\cdot$, both binary infix, and $!$, unary prefix. The reader is referred to [2] and to [11] for a discussion of the intended meaning of these.

Formulas are built up from *propositional letters*, P_1, P_2, ..., and a *falsehood constant*, $\bot$, using $\supset$, together with an additional rule of formation, $t{:}X$ is a formula if t is a justification term and X is a formula. Read it as "t is a justification for X." Other propositional connectives are introduced as abbreviations.

Axiom (schemes) for LP, and rules, are as follows.

Classical Axioms:	all tautologies
Truth Axioms:	$t{:}X \supset X$
$+$ Axioms:	$t{:}X \supset (t+u){:}X$
	$u{:}X \supset (t+u){:}X$
$\cdot$ Axioms:	$t{:}(X \supset Y) \supset (u{:}X \supset (t \cdot u){:}Y)$
$!$ Axioms:	$t{:}X \supset !t{:}t{:}X$

$$\text{Modus Ponens: } \frac{X \qquad X \supset Y}{Y}$$

$$\text{Axiom Necessitation: If } X \text{ is an axiom and } c \text{ is a constant: } \frac{}{c{:}X}$$

A *constant specification* $\mathcal{C}$ is an assignment of axioms to constants; take it to be a set of formulas of the form $c{:}X$, where c is a constant and X is an axiom. A proof *meets constant specification* $\mathcal{C}$ provided that whenever $c{:}X$ is introduced using the Axiom Necessitation rule, then X is an axiom that $\mathcal{C}$ assigns to constant c. A constant specification can be given ahead of time, or can be created during the course of a proof. In this paper we will assume a constant specification has been fixed ahead of time. Various conditions can be imposed on constant specifications. A constant specification is *axiomatically appropriate* if all instances of axiom schemes have proof constants—here *this will always be assumed*. Another common condition is being *injective*: at most one formula is associated with each constant. We will need a condition, given in Section 4, that conflicts with injectivity, but which is nonetheless natural to consider.

2.2 LP Semantics

The usual semantics for LP comes from [6], and amounts to a blending of an earlier semantics from [7] with the usual Hintikka semantics for logics of knowledge. A model is $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$, where $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame, with $\mathcal{R}$ a reflexive and transitive relation on $\mathcal{G}$. $\mathcal{V}$ maps propositional variables to subsets of $\mathcal{G}$. The item not standard in Kripke models is $\mathcal{A}$, an *admissible evidence function.* For each justification term t and formula X, $\mathcal{A}(t, X)$ is some subset of $\mathcal{G}$. Intuitively, $\mathcal{A}(t, X)$ is the set of worlds at which t is admissible evidence for X. This does not mean conclusive evidence—just evidence that is relevant. Admissible evidence functions must meet certain conditions which we give next. (In earlier work a related mapping $\mathcal{E}$, called an evidence function, was used in place of $\mathcal{A}$. The change in notation is essentially cosmetic.)

Constant Specification Condition For the given constant specification $\mathcal{C}$, if $c{:}X \in \mathcal{C}$ then $\mathcal{A}(c, X) = \mathcal{G}$. If this condition is met, we say $\mathcal{A}$ *meets constant specification* $\mathcal{C}$.
$\cdot$ **Condition** $\mathcal{A}(s, X \supset Y) \cap \mathcal{A}(t, X) \subseteq \mathcal{A}(s \cdot t, Y)$.
$+$ **Condition** $\mathcal{A}(s, X) \cup \mathcal{A}(t, X) \subseteq \mathcal{A}(s + t, X)$.
$\mathcal{R}$ **Closure Condition** $\Gamma\mathcal{R}\Delta$ and $\Gamma \in \mathcal{A}(t, X)$ imply $\Delta \in \mathcal{A}(t, X)$.
! **Condition** $\mathcal{A}(t, X) \subseteq \mathcal{A}(!t, t{:}X)$.

Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$ be an LP model. $\mathcal{M}, \Gamma \Vdash X$ is read: formula X is true at world $\Gamma \in \mathcal{G}$, of LP model $\mathcal{M}$. The conditions for it are as follows.

Atomic Condition For a propositional letter P, $\mathcal{M}, \Gamma \Vdash P$ if $\Gamma \in \mathcal{V}(P)$.
Classical Conditions $\mathcal{M}, \Gamma \Vdash X \supset Y$ iff $\mathcal{M}, \Gamma \nVdash X$ or $\mathcal{M}, \Gamma \Vdash Y$. Also $\mathcal{M}, \Gamma \nVdash \bot$.
Justification Condition $\mathcal{M}, \Gamma \Vdash t{:}X$ iff $\Gamma \in \mathcal{A}(t, X)$ and $\mathcal{M}, \Delta \Vdash X$ for all $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$.

We say X is *true at world* Γ if $\mathcal{M}, \Gamma \Vdash X$, and otherwise X is *false at* Γ. X is *valid* in a model $\mathcal{M}$ if X is true at every world of it.

The Justification Condition says we have $t{:}X$ at Γ if X is knowable at Γ in the Hintikka sense, and t is admissible evidence for X at Γ. If Hintikka semantics captures *true belief*, then the present machinery captures is *justified* true belief.

The semantics just given is the *weak model semantics*; there is a stronger version. A model $\mathcal{M}$ is *fully explanatory* provided, if $\mathcal{M}, \Delta \Vdash X$ for all $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ then there is some justification t such that $\mathcal{M}, \Gamma \Vdash t{:}X$. That is, $\mathcal{M}$ is fully explanatory provided Hintikka-knowability of X at Γ implies there is a justification for X at Γ. Fully explanatory is examined in Section 6.

In [6] soundness and completeness was shown. If $\mathcal{C}$ is an axiomatically appropriate constant specification, then X has an axiomatic proof using $\mathcal{C}$ if and only if X is valid in every weak LP model meeting $\mathcal{C}$ if and only if X is valid in every fully explanatory LP model meeting $\mathcal{C}$. Being fully explanatory is an interesting condition that has not yet found applications. This is a puzzling circumstance, which the results of this paper will only make more puzzling.

3 The Logic S4LP

LP and S4 are connected intimately via the Realization Theorem, as noted in the Introduction. So it is natural to consider a logic combining the two, S4LP, originating in [3, 4].

3.1 S4LP Axioms

First, the language of LP is extended with the formation rule: if X is a formula, so is $\Box X$. Next, the axiomatization of LP as given in Section 2.1 is extended with S4 machinery, and a connecting axiom.

$\Box$ Axioms: $\Box X \supset X$
$\Box(X \supset Y) \supset (\Box X \supset \Box Y)$
$\Box X \supset \Box\Box X$
Connecting Axiom: $t{:}X \supset \Box X$

The usual necessitation rule is added.

$$\Box \text{ Necessitation: } \frac{X}{\Box X}$$

Finally, it is assumed that the LP Axiom Necessitation rule also applies to the new axioms just added, and that constant specifications also take these new axioms into account.

$\Box$ Necessitation can be shown to be a redundant rule, but doing so involves proving an Internalization Theorem, whose statement and proof we skip here.

3.2 An S4LP Semantics

As was noted in the Introduction, there are two semantics for S4LP, with different motivations. One, from [9, 10], allows not just one but multiple agents, each with its own knowledge operator, K_i, but with justifications meaningful to all and playing the role of justified common knowledge. In this semantics one has multiple accessibility relations, one for each agent, and one for justification terms. If there is a single agent the logic reduces to S4LP; there are two relations, one Hintikka style to supply an interpretation for $\Box$, the other is combined with an admissible evidence function as in Section 2.2. Both weak and strong completeness theorems are provable. However, this is not the semantics that will concern us here, and it will not be mentioned further.

The semantics examined in this paper understands knowledge as having an explicit (justification term) aspect and an implicit (modal) aspect. Justification terms provide an analysis of our knowledge, rather than being the items of knowledge we share with other agents. This approach is from [3, 4, 8]. There is a *single* accessibility relation for both implicit and explicit knowledge. Now details.

First, LP models are fundamental. These are as in Section 2.2. This provides the semantics for justification terms. But since we are using an extended language now, we can also adopt the following, familiar from modal logic.

Necessitation Condition $\mathcal{M}, \Gamma \Vdash \Box X$ iff $\mathcal{M}, \Delta \Vdash X$ for all $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$.

In other words, justification terms are interpreted using the accessibility relation and the admissible evidence function, while the modal operator uses the accessibility relation but does not take the admissible evidence function into account. The Justification Condition now can be given a somewhat simpler expression: $\mathcal{M}, \Gamma \Vdash t{:}X$ iff $\Gamma \in \mathcal{A}(t, X)$ and $\mathcal{M}, \Gamma \Vdash \Box X$.

As with LP, we can introduce notions of weak and strong models, but now the Fully Explanatory condition is simpler to state: for each Γ and for each

X, if $\mathcal{M}, \Gamma \Vdash \Box X$ then for some justification term t we have $\mathcal{M}, \Gamma \Vdash t{:}X$. Equivalently, if $\mathcal{M}, \Gamma \Vdash \Box X$ then there is some t such that $\Gamma \in \mathcal{A}(t, X)$.

Axiomatic soundness comes from [3, 4]. Completeness of S4LP with respect to the weak model semantics for S4LP is in [8]. Completeness with respect to the strong model semantics, with models satisfying the fully explanatory condition, is an open problem. Here it will be a special case of a more general result, but the general result requires a special condition on constant specifications.

4 Some Derived S4LP Rules

The rules presented here will be used in our S4LP completeness proof in the next section. But we must impose a restriction on constant specifications that is at odds with injectivity. We begin with the replacement of terms containing a variable with a $\Box$ operator.

Definition 1. *Let Z be a formula, and let x be a variable. $Z(x/\Box)$ is the result of replacing every justification term in Z that contains x with $\Box$:*

$$\begin{aligned} A(x/\Box) &= A \textit{ for } A \textit{ atomic} \\ [X \supset Y](x/\Box) &= [X(x/\Box) \supset Y(x/\Box)] \\ [\Box X](x/\Box) &= \Box[X(x/\Box)] \\ [t{:}X](x/\Box) &= \begin{cases} t{:}[X(x/\Box)] & \textit{if } x \textit{ does not occur in } t \\ \Box[X(x/\Box)] & \textit{if } x \textit{ occurs in } t \end{cases} \end{aligned}$$

If we replace justification terms containing variable x with $\Box$ this turns axioms into axioms except for the $\cdot$ Axiom, where the results are theorems but not axioms. *We now modify the formulation of* S4LP *by adding these theorems to our axiom list.* This affects the role of constants in applications of the Axiom Necessitation Rule, and keeps things simpler than they otherwise would be.

New Axioms From now on our axiomatization of S4LP also contains the following two schemes: $t{:}(X \supset Y) \supset (\Box X \supset \Box Y)$ and $\Box(X \supset Y) \supset (u{:}X \supset \Box Y)$

Lemma 1. *If Z is an axiom of* S4LP*, so is $Z(x/\Box)$ for every variable x.*

Proof. We give one case as an example. Consider the axiom $t{:}X \supset (t + u){:}X$. There are three subcases.

1. x does not occur in either t or u. Then $[t{:}X \supset (t+u){:}X](x/\Box) = [t{:}X(x/\Box) \supset (t + u){:}X(x/\Box)]$, which is also a $+$ axiom.
2. x occurs in u but not in t. Then $[t{:}X \supset (t + u){:}X](x/\Box) = [t{:}X(x/\Box) \supset \Box X(x/\Box)]$, which is a connecting axiom.
3. x occurs in t. Then $[t{:}X \supset (t + u){:}X](x/\Box) = [\Box X(x/\Box) \supset \Box X(x/\Box)]$, a classical axiom.

Definition 2 ($\Box$ Closed). *Let $\mathcal{C}$ be a constant specification for* S4LP*. $\mathcal{C}$ is $\Box$* closed *provided, whenever $c{:}Z \in \mathcal{C}$ then also $c{:}Z(x/\Box) \in \mathcal{C}$, for each variable x.*

Note that a constant specification that is $\Box$ closed cannot be injective (though it may be when restricted to LP formulas, not containing $\Box$). Here is the main result concerning $\Box$ closed constant specifications.

Theorem 1. *Suppose the constant specification* $\mathcal{C}$ *is* $\Box$ *closed. If* Z *is provable in* S4LP *using constant specification* $\mathcal{C}$*, so is* $Z(x/\Box)$*, for each variable* x.

Proof. By induction on axiomatic proof length. Lemma 1 takes care of axioms. Modus ponens and $\Box$ Necessitation are straightforward. Axiom Necessitation is covered by the assumption that the constant specification is $\Box$ closed.

If we do not assume the constant specification is $\Box$ closed, it is still true that if Z is provable in S4LP so is $Z(x/\Box)$, but using a modified constant specification.

Corollary 1. *Assuming a constant specification that is* $\Box$ *closed, the following is a derived rule of* S4LP*. If* x *does not occur in* A *or* B*:*

$$\frac{x{:}A \supset B}{\Box A \supset B}$$

This corollary points out a similarity in behavior between the $\Box$ operator in S4LP and the existential quantifier in first-order logic. This similarity has been an important motivating factor in the development of justification logics.

5 Completeness

We begin with a definition of local realizability and a statement of the completeness theorem. The completeness proof has similarities with one often used for first-order modal logic with the Barcan formula.

Definition 3. *For a formula* X *of* S4LP*, a* realization *is a formula* X' *whose structure is like that of* X*, but in which every occurrence of* $\Box$ *has been replaced with a justification term:*

$$\begin{aligned} A &\text{ realizes } A \text{ if } A \text{ is atomic} \\ X' \supset Y' &\text{ realizes } X \supset Y \text{ if } X' \text{ realizes } X \text{ and } Y' \text{ realizes } Y \\ t{:}X' &\text{ realizes } t{:}X \text{ if } X' \text{ realizes } X \\ t{:}X' &\text{ realizes } \Box X \text{ if } t \text{ is a term and } X' \text{ realizes } X \end{aligned}$$

This extends the usual notion of LP realization. Ordinarily it is S4 formulas that are realized, while here we include a case covering justification terms which are not part of the language of S4. We are not considering *normal* realizations.

Definition 4. *Let* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$ *be an* S4LP *model.* $\mathcal{M}$ meets the local realizability condition *provided: for every world* $\Gamma \in \mathcal{G}$ *and for every formula* X*, if* $\mathcal{M}, \Gamma \Vdash X$ *then there is some realization* X' *of* X *such that* $\mathcal{M}, \Gamma \Vdash X'$.

Theorem 2 (Completeness). *Let $\mathcal{C}$ be a constant specification that is $\Box$ closed (Definition 2). If Z_0 is not provable using $\mathcal{C}$ then Z_0 is false at some world of an* S4LP *model meeting $\mathcal{C}$ and* meeting the local realizability condition.

The rest of this section contains the proof of Completeness. *$\mathcal{C}$ is a $\Box$ closed constant specification fixed for the sectio*n (the first few results don't use $\Box$ closure). For a set S of formulas and a formula X, we write $S \vdash X$ if there is some finite subset $\{Y_1, \ldots, Y_n\}$ of S such that $(Y_1 \wedge \ldots \wedge Y_n) \supset X$ is a theorem of S4LP using $\mathcal{C}$. With this definition the deduction theorem and compactness are immediate. We say S is *inconsistent* if $S \vdash \bot$, and *consistent* if it is not inconsistent. *Maximal* consistency has its usual meaning.

Definition 5. *Let X be a formula, and assume an occurrence of $\Box$ in X has been designated. Let t be a justification term. By $X(t)$ we mean the result of replacing the designated occurrence of $\Box$ in X with t.*

Say X is $\Box(P \supset x{:}\dot{\Box}Q) \supset \Box R$ and the designated occurrence of $\Box$ is marked with a dot. Then $X(t) = \Box(P \supset x{:}t{:}Q) \supset \Box R$. The notation $X(t)$ is incomplete since which occurrence of $\Box$ in X is designated is understood, and is not represented in the notation itself.

Definition 6. *A set S of formulas has the $\Box$* instantiation property *provided, for every formula X with a designated occurrence of $\Box$, if $S \cup \{X\}$ is consistent then there is some term t such that $S \cup \{X(t)\}$ is also consistent.*

In canonical models, maximally consistent sets are possible worlds. We will require the $\Box$ instantiation property too. For maximally consistent sets the $\Box$ instantiation property becomes: if $X \in S$ then there is some t such that $X(t) \in S$. This is what we need, but the more general version comes in along the way.

Definition 7. *Let S be a set of formulas. $S^\sharp$ is $\{X \mid \Box X \in S\}$.*

Proposition 1. *Suppose S is a maximally consistent set of formulas that has the $\Box$ instantiation property. Then $S^\sharp$ also has the $\Box$ instantiation property.*

Proof. Assume the hypothesis. Let X be a formula with a designated occurrence of $\Box$. Suppose $S^\sharp \cup \{X\}$ is consistent. We show that for some t, $S^\sharp \cup \{X(t)\}$ is consistent. For convenience we use the defined modal operator $\Diamond$.

Since $S^\sharp \cup \{X\}$ is consistent, so is $S \cup \{\Diamond X\}$ by the following argument. If it were not consistent, $S, \Diamond X \vdash \bot$, and so $S \vdash (\Diamond X \supset \bot)$, that is, $S \vdash \Box\neg X$. Since S is maximal, $\Box\neg X \in S$, hence $\neg X \in S^\sharp$, so $S^\sharp \cup \{X\}$ is not consistent.

Since S has the $\Box$ instantiation property, for some justification term t, $S \cup \{\Diamond X(t)\}$ is consistent.

Since $S \cup \{\Diamond X(t)\}$ is consistent, so is $S^\sharp \cup \{X(t)\}$, which finishes the argument. Again we have a proof by contradiction. Suppose $S^\sharp \cup \{X(t)\}$ is not consistent. Then $S^\sharp, X(t) \vdash \bot$, so $S^\sharp \vdash \neg X(t)$. Then for some $Y_1, \ldots, Y_n \in S^\sharp$, the formula $(Y_1 \wedge \ldots \wedge Y_n) \supset \neg X(t)$ is provable. Using the Rule of Necessitation and standard modal theorems, $(\Box Y_1 \wedge \ldots \wedge \Box Y_n) \supset \Box\neg X(t)$ is provable. Since $Y_1, \ldots, Y_n \in S^\sharp$, we must have $\Box Y_1, \ldots, \Box Y_n \in S$, and since S is maximally consistent, we must also have $\Box\neg X(t) \in S$, contradicting the fact that $S \cup \{\neg\Box\neg X(t)\}$ is consistent.

Next we address the problem of extending a set that has the $\Box$ instantiation property to a maximally consistent set that still has this property.

Lemma 2. *Let S be a set of formulas with the $\Box$ instantiation property, and F be a finite set of formulas. If $S \cup F$ is consistent then $S \cup F$ also has the $\Box$ instantiation property.*

Proof. Let X be a formula with a designated occurrence of $\Box$. Assume $S \cup F \cup \{X\}$ is consistent. We show that for some t, $S \cup F \cup \{X(t)\}$ is also consistent. The argument is very simple.

Say $F = \{Y_1, \ldots, Y_n\}$. Then $S \cup \{Y_1 \wedge \ldots \wedge Y_n \wedge X\}$ is consistent. Since S has the $\Box$ instantiation property for some t, $S \cup \{Y_1 \wedge \ldots \wedge Y_n \wedge X(t)\}$ is consistent, using the original designated occurrence of $\Box$ in X. But this implies that $S \cup F \cup \{X(t)\}$ is consistent.

Proposition 2. *If S is a consistent set of formulas that has the $\Box$ instantiation property, then S can be extended to a set that is maximally consistent and has the $\Box$ instantiation property.*

Proof. Assume S is consistent and has the $\Box$ instantiation property. We extend S using a modified Lindenbaum construction. Enumerate all formulas , say X_0, X_1, ... Then define a sequence of sets of formulas, S_0, S_1, ..., in which each set extends its predecessor, is consistent, and has the $\Box$ instantiation property.

To start, $S_0 = S$.

Suppose S_n has been defined, is consistent, and has the $\Box$ instantiation property. If $S_n \cup \{X_n\}$ is not consistent, set $S_{n+1} = S_n$. Otherwise, proceed as follows. X_n has a finite number of $\Box$ occurrences, say k of them. Choose one of them as designated. $S_n \cup \{X_n\}$ is consistent and, by Lemma 2, it has the $\Box$ instantiation property. Then, using the designated occurrence of $\Box$, there must be some justification term t such that $S_n \cup \{X_n, X_n^1\}$ is consistent, where $X_n^1 = X_n(t)$. By Lemma 2 this set too has the $\Box$ instantiation property. Now repeat this with a different designated occurrence of $\Box$ in X_n, getting a set $S_n \cup \{X_n, X_n^1, X_n^2\}$, consistent and with the $\Box$ instantiation property. And so on for each of the k occurrences of $\Box$ in X_n. Let $S_{n+1} = S_n \cup \{X_n, X_n^1, X_n^2, \ldots, X_n^k\}$.

Let $S = S_0 \cup S_1 \cup S_2 \cup \ldots$. As usual, S is maximally consistent. But also it has the $\Box$ instantiation property by the following argument. Suppose X is a formula with a designated occurrence of $\Box$, and suppose $S \cup \{X\}$ is consistent. Say $X = X_n$. Then $S \cup \{X_n\}$ is consistent and so at stage n of the construction above, not only is $X_n = X$ in S_{n+1}, but also $X(t)$ is in S_{n+1} for some justification term t, and hence $X(t)$ is in S, so trivially $S \cup \{X(t)\}$ is consistent.

All results so far say: some set has the $\Box$ instantiation property provided some other set does. We do not yet know there are any such sets at all (except for inconsistent ones). This is taken care of by the following Lemma and Proposition.

Lemma 3. *Let F be a consistent finite set of formulas, and let X be a single formula with a designated occurrence of $\Box$. Then $F \cup \{X \supset X(x)\}$ is consistent, where x is a variable that does not occur in F or in X.*

Proof. Assume the hypothesis, but also assume $F \cup \{X \supset X(x)\}$ is not consistent, where x does not occur in F or in X. Then $[\bigwedge F \wedge (X \supset X(x))] \supset \bot$ is provable. By Theorem 1 we also have provability of $\{[\bigwedge F \wedge (X \supset X(x))] \supset \bot\}(x/\Box)$, but this is just $[\bigwedge F \wedge (X \supset X)] \supset \bot$, and it follows that F is not consistent.

Proposition 3. *If F is a finite, consistent set then F extends to a consistent set with the $\Box$ instantiation property.*

Proof. Enumerate the formulas, X_0, X_1, X_2, ... We define a chain F_0, F_1, F_2, ... of consistent finite sets, as follows.

$F_0 = F$.

Assume F_n has been defined. Consider formula X_n. Choose an occurrence of $\Box$ in X_n and take it to be designated (if there are none, this step is vacuous). Let x be a variable that does not occur in the finite set F_n or in X_n. Then $F_n \cup \{X_n \supset X_n(x)\}$ is consistent, by Lemma 3. Extend this set by consistently adding an instantiation implication for a different designated occurrence of $\Box$ in X_n, and so on until each occurrence has had a corresponding implication added. Call the resulting set F_{n+1}. Clearly it is finite and consistent.

Set F^* to be $\cup_n F_n$. Then F^* is consistent, has the $\Box$ instantiation property.

Completeness for LP was proved in [6], and that proof was extended to S4LP, as usually formulated, in [8]. Now we modify that proof to establish our main result, which we restate here for convenience.

Theorem 2 Let $\mathcal{C}$ be a constant specification that is $\Box$ closed (Definition 2). If Z_0 is not provable using $\mathcal{C}$ then Z_0 is false at some world of an S4LP model meeting $\mathcal{C}$ and *meeting the local realizability condition.*

Proof. Since Z_0 is not provable, $\{\neg Z_0\}$ is consistent. Using Proposition 3, this set extends to a set that has the $\Box$ instantiation property, and by Proposition 2, this further extends to a set that is maximally consistent and has the $\Box$ instantiation property. Call this set Γ_0.

Construct a model as follows. Let $\mathcal{G}$ be the set of all maximally consistent sets of formulas that have the $\Box$ instantiation property. (Note that $\Gamma_0 \in \mathcal{G}$.) If $\Gamma \in \mathcal{G}$, let $\Gamma^\sharp = \{X \mid \Box X \in \Gamma\}$, and set $\Gamma \mathcal{R} \Delta$ if $\Gamma^\sharp \subseteq \Delta$. This gives us a frame, $\langle \mathcal{G}, \mathcal{R} \rangle$. It is easily shown to be reflexive and transitive. Define a mapping $\mathcal{A}$ by setting $\mathcal{A}(t, X) = \{\Gamma \in \mathcal{G} \mid t{:}X \in \Gamma\}$. Finally, define a mapping $\mathcal{V}$ by specifying that for an atomic formula P, $\Gamma \in \mathcal{V}(P)$ if and only if $P \in \Gamma$. This gives us a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$. We begin by showing that $\mathcal{M}$ is an S4LP model.

First we verify that $\mathcal{A}$, meets the $\cdot$ Condition. Suppose we have $\Gamma \in [\mathcal{A}(s, X \supset Y) \cap \mathcal{A}(t, X)]$. By the definition of $\mathcal{A}$, we must have $t{:}X \in \Gamma$ and $s{:}(X \supset Y) \in \Gamma$. Since $s{:}(X \supset Y) \supset (t{:}X \supset (s \cdot t){:}Y)$ is an S4LP axiom, and Γ is maximally consistent, it follows that $(s \cdot t){:}Y \in \Gamma$, and hence $\Gamma \in \mathcal{A}(s \cdot t, Y)$.

Next we verify the $\mathcal{R}$ Closure Condition. Suppose $\Gamma, \Delta \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$. Also assume $\Gamma \in \mathcal{A}(t, X)$. By definition of $\mathcal{A}$, we have $t{:}X \in \Gamma$. But $t{:}X \supset !t{:}t{:}X$ is an S4LP axiom, and so is $!t{:}t{:}X \supset \Box t{:}X$, so we have $\Box t{:}X \in \Gamma$, and hence $t{:}X \in \Gamma^\sharp \subseteq \Delta$. Then $\Delta \in \mathcal{A}(t, X)$.

Verifying that $\mathcal{A}$ meets the $+$ and ! Conditions is similar, and is omitted. Likewise it is straightforward to check that $\mathcal{M}$ meets constant specification $\mathcal{C}$.

We have now verified that $\mathcal{M}$ is an S4LP model.

Next, a Truth Lemma can be shown: for each formula X and each $\Gamma \in \mathcal{G}$

$$X \in \Gamma \Longleftrightarrow \mathcal{M}, \Gamma \Vdash X \tag{1}$$

Many of the cases are familiar from standard S4 completeness proofs. I'll verify only one. Suppose (1) is known for X, and we are considering the formula $t{:}X$.

Suppose first that $t{:}X \in \Gamma$. Then, using the Connecting Axiom, $t{:}X \supset \Box X$, $\Box X \in \Gamma$, and so $X \in \Gamma^\sharp$. Then if Δ is an arbitrary member of $\mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ we have $\Gamma^\sharp \subseteq \Delta$ and hence $X \in \Delta$. By the induction hypothesis, $\mathcal{M}, \Delta \Vdash X$. Also since $t{:}X \in \Gamma$, we have $\Gamma \in \mathcal{A}(t, X)$. It follows that $\mathcal{M}, \Gamma \Vdash t{:}X$.

Next, suppose $\mathcal{M}, \Gamma \Vdash t{:}X$. This case is trivial. Part of the definition of $\Vdash$ tells us $\Gamma \in \mathcal{A}(t, X)$, and by definition of $\mathcal{A}$ for $\mathcal{M}$, we must have $t{:}X \in \Gamma$.

Thus we have the Truth Lemma. It follows immediately that $\mathcal{M}$ meets the local realizability condition. Here is the argument. Suppose $\mathcal{M}, \Gamma \Vdash X$. Then by the Truth Lemma, $X \in \Gamma$. Designate an occurrence of $\Box$ in X. Since members of $\mathcal{G}$ have the $\Box$ instantiation property, for some t, $\Gamma \cup \{X(t)\}$ is consistent, hence $X(t) \in \Gamma$ since Γ is maximal. If there are occurrences of $\Box$ in $X(t)$ repeat this step, eliminating a second occurrence. And so on. When all occurrences of $\Box$ are gone, we have a realization X' of X, with $X' \in \Gamma$. But then $\mathcal{M}, \Gamma \Vdash X'$, by the Truth Lemma again.

Since $\neg Z_0 \in \Gamma_0$, and $\Gamma_0 \in \mathcal{G}$, we have $\mathcal{M}, \Gamma_0 \nVdash Z_0$, completing the proof.

6 Conclusion

S4LP is an explicit/implicit analog of S4. In a similar way explicit/implicit analogs of weaker logics can be introduced, essentially by dropping axioms from S4LP. The results of this paper carry over to the analogs of K4, T, and K. In the other direction one can strengthen LP from an S4 counterpart to an S5 one, but it requires adding machinery. This affects what is needed for a completeness proof [12, 13], and the status of an explicit/implicit version of S5 has not been checked yet.

The LP semantics in Section 2.2 comes from [5, 6] in two versions, weak and strong. The weak version is basic in this paper. For the strong version an additional condition is placed on models. An LP model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$ is *fully explanatory* provided, for each world $\Gamma \in \mathcal{G}$, if $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, then $\mathcal{M}, \Gamma \Vdash t{:}X$ for some justification term t. Strong models are weak models that are fully explanatory. LP is complete with respect to strong models, using a proof that very closely mirrors the usual proof of completeness for modal logics. Conventionally one shows that if $\{\Box X_1, \ldots, \Box X_n, \neg\Box Y\}$ is consistent, then so is $\{X_1, \ldots, X_n, \neg Y\}$. In the LP case one shows that, for fixed choice of $t_1, \ldots, t_n$, if $\{t_1{:}X_1, \ldots, t_n{:}X_n, \neg u{:}Y\}$ is consistent for every choice of u, then $\{X_1, \ldots, X_n, \neg Y\}$ is also consistent.

The work of this paper provides a second proof of strong completeness for LP, along completely different lines, as follows. Suppose X is a formula of LP, and X is not LP provable using an axiomatically appropriate constant specification $\mathcal{C}$. It is simple to extend $\mathcal{C}$ to a constant specification that is axiomatically appropriate for all of S4LP and is $\Box$ closed. Call the extension $\mathcal{C}^*$. Then X is not provable in S4LP using $\mathcal{C}^*$ either, since an LP counter model for X easily extends to an S4LP countermodel for X, using the weak notion of S4LP model from [8]. Since X is not provable in S4LP using $\mathcal{C}^*$, by the completeness proof of the present paper, X is falsified at some world of an S4LP model meeting the local realizability condition. If we ignore $\Box$ in such a model we have an LP model, and local realizability immediately gives us the fully explanatory condition.

It is curious that the fully explanatory condition can be approached from such seemingly different directions—via a generalization of a standard modal argument, and via a generalization of a Henkin completeness argument. It is also curious that no use has been found for the condition. This report ends on a note of genuine puzzlement.

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