The Pure Logic of Necessitation

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Abstract

In this paper we discuss the pure logic of necessitation **N**, a modal logic containing classical propositional calculus, with modus ponens and necessitation as inference rules, but without any axioms for manipulating modalities. We develop a theory of the logic **N**. We propose a sound and complete Kripke-like semantics for **N** and build a tableaux system for testing whether a formula is provable from a theory in the logic **N**. An alternative method to compute modal-free consequences of a finite theory is also given. Our main motivation to consider the logic **N** comes from the area of nonmonotonic reasoning. The nonmonotonic variant of **N** seems to be particularly useful in investigations of knowledge sets built when only partial information is available. In particular, this logic **N** is deeply connected with the default logic. In this paper, we apply our results to problems in nonmonotonic reasoning and we design algorithms for building the non-monotonic consequence operator associated with **N**.

1. Introduction

In this paper we study a certain subnormal modal logic. This logic contains classical propositional calculus, contains no specific axioms involving modality and allows for an unrestricted application of necessitation. We restrict our attention to the language which is the standard extension of some fixed language $\mathscr{L}$ of classical propositional calculus by a single modal operator L. This language will be denoted by $\mathscr{L}_L$.

Investigations of modal logics so far have mainly been concerned with normal modal logics. A modal logic is *normal* if it is closed under uniform substitution and

(1) contains classical propositional calculus, that is, contains all axioms for propositional calculus and is closed under modus ponens;
(2) is closed under necessitation rule;
(3) contains the axiom K: $L(\varphi \rightarrow \psi) \rightarrow (L\varphi \rightarrow L\psi)$.

Let us call a modal logic *subnormal* if at least one of these conditions fails.

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There are, then, three main ways to define a subnormal modal logic:

(1) Replace classical propositional calculus by some weaker logic.
(2) Restrict the applicability of the necessitation rule.
(3) Do not require that axiom K be contained in a logic.

In each of these cases there are problems with semantics for the resulting systems. For case (1) several generalizations of the standard Kripke semantics were proposed; see Plotkin and Stirling [18], Wijesekera [25], Nishimura [16], and Ono [17] for an exhaustive list of references.

The second approach has been studied extensively and good accounts of this research are given in the monographs by Segerberg [20], Chellas [1] and Fitting [2], where semantical issues are discussed in detail.

In this paper, we deal with the third approach. Speaking precisely, we study the following notion of provability. A proof of a formula φ from a set of formulae I in a modal logic $\mathscr{S}$ is a sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that $\varphi_n = \varphi$, and for all $i \leq n$, either φ_i is a substitution instance of an axiom of $\mathscr{S}$, or $\varphi_i \in I$, or φ_i is obtained from preceding formulae in the proof by modus ponens or the necessitation rule. In particular, the necessitation rule can be applied to formulae in I, not only to axioms of $\mathscr{S}$. The fact that a formula φ is provable from I using the above notion of proof is denoted by $I \vdash_{\mathscr{S}} \varphi$. The consequence operator $Cn_{\mathscr{S}}$ in logic $\mathscr{S}$ is defined by $Cn_{\mathscr{S}}(I) = \{\varphi : I \vdash_{\mathscr{S}} \varphi\}$.

Let us stress that this notion of provability is different from the notion of provability considered in standard monographs on modal logics, for example [1]. We allow application of necessitation to all formulae and not only to the axioms of $\mathscr{S}$.

In this paper, we are particularly interested in the case when $\mathscr{S}$ is the weakest modal logic containing propositional calculus and closed under modus ponens and necessitation. We call this logic the *pure logic of necessitation* and denote it by **N**.

There are at least three motivations for studying the logic **N** and the notion of provability introduced above.

Firstly, our treatment of necessitation corresponds to computational intuitions, where $L\varphi$ means that φ has been computed or derived. The logic **N** appears to be the most natural logic for studying provability in formal systems. Specifically, we have the following result which, for technical reasons, we prove at the end of Section 5.

PROPOSITION 1.1
Let R be a collection of inference rules of the form

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}. \tag{1}$$

Let T be the modal theory obtained by replacing each rule (1) in R by

$$L\varphi_1 \wedge \ldots \wedge L\varphi_n \rightarrow \psi.$$

Then for every set of formulae $I \subseteq \mathcal{L}$ and for every formula $\omega \in \mathcal{L}$, ω belongs to the least set of formulae containing I and closed under propositional consequence and the rules from R if and only if $I \cup T \vdash_{\mathbf{N}} \omega$.

Secondly, under our notion of provability, the logic **N** plays a special role among *all* modal logics closed under necessitation. Namely, we have the following straightforward result.

PROPOSITION 1.2
Let $\mathcal{S}$ be a modal logic closed under necessitation. For every theory $I \subseteq \mathcal{L}_L$

$$Cn_{\mathcal{S}}(I) = Cn_{\mathbf{N}}(I \cup Ax(\mathcal{S})),$$

where $Ax(\mathcal{S})$ is the set of all substitution instances of axioms of $\mathcal{S}$.

Thirdly, important applications of subnormal modal logics, such as **N**, have recently emerged from the efforts to formalize commonsense reasoning with incomplete information. Such reasoning is inherently nonmonotonic—if a fact p can be concluded from a theory I, it is not necessarily derivable from a theory I' which properly contains I. Most formalisms designed to describe nonmonotonic reasoning can be characterized by means of a fixed point construction applied to the consequence operator of some (monotone) logic. Two basic nonmonotonic systems, default logic and autoepistemic logic, can be characterized in such a way by means of the logic **N** [9, 21].

We will now briefly describe the use of modal logics and in particular of the logic **N** in nonmonotonic reasonings. Modal logics were first proposed as a means to formalize commonsense reasoning by McDermott and Doyle [14] and McDermott [13]. Let $\mathcal{S}$ be a modal logic. McDermott and Doyle described a construction which, for every modal logic $\mathcal{S}$, produces its nonmonotonic variant. They argued that in a nonmonotonic logic corresponding to $\mathcal{S}$, a theory T can be considered as a belief (knowledge) set associated with an initial theory I if and only if T is exactly the set of facts that can be derived from I and all modal facts of the form '$\neg\varphi$ is consistent'. The formula '$\neg\varphi$ is consistent' is expressed as $\neg L\varphi$. If a theory T is closed under $\mathcal{S}$-consequence, then $\neg\varphi$ is consistent with T precisely when $\varphi \notin T$. Consequently, McDermott and Doyle [14] and McDermott [13] introduced the fixed point equation:

$$T = Cn_{\mathcal{S}}(I \cup \{\neg L\varphi : \varphi \notin T\}), \tag{2}$$

and proposed to consider its consistent solutions as candidates for the belief sets of I. In equation (2), $Cn_{\mathcal{S}}$ stands for the consequence operator in the logic $\mathcal{S}$, as introduced above. They proposed the following crucial definition: *a theory T is an $\mathcal{S}$-expansion of I if T is consistent and satisfies* (2). The operator $Cn_{\mathcal{S}}$ is, of course, monotone. But T appears on both sides of equation (2) and the dependence of T on I is no longer monotone. What is more, a theory may have no $\mathcal{S}$-expansions, exactly one $\mathcal{S}$-expansion, or many $\mathcal{S}$-expansions.

Marek and Truszczynski [9] argued that **N**-expansions can be regarded as knowledge sets of an agent with full introspection capabilities and pointed out the close connection between extensions of default theories and **N**-expansions. (For all undefined notions related to default logic see the original paper by Reiter [19] or Marek and Truszczynski [9].) Consider a default rule:

$$d = \frac{\alpha : M\beta_1, \ldots, M\beta_k}{\omega}. \tag{3}$$

Such a rule, following Reiter, is interpreted informally as follows: if α is known, and if it is known that each β_i, $i \leq k$, is possible (this is the reason for the notation $M\beta_i$), then establish ω.

To capture this interpretation, it was proposed in [24] that a default d of the form (3) be encoded by the following modal formula:

$$tr(d) = L\alpha \wedge LM\beta_1 \wedge \ldots \wedge LM\beta_k \rightarrow \omega, \tag{4}$$

where M abbreviates $\neg L \neg$. Let us point out that this translation treats the premise part and justification part of a default rule differently. In fact the modalities L and LM are related in the logic **N** in a very loose fashion, and the interplay of the formulae of the form $L\varphi$ and $LM\psi$, with $\varphi, \psi \in \mathcal{L}$ corresponds precisely to that between the premise a and the premises Mb_i in deault logic.

For a default theory $\Delta = (D, W)$ (D is a set of defaults, W is a set of formulae) define

$$tr(\Delta) = W \cup \{tr(d) : d \in D\}.$$

The following result was proved in [24].

THEOREM 1.3
Let $\Delta = (D, W)$ be a default theory. A theory $S \subseteq \mathcal{L}$ is an extension of Δ if and only if $S = T \cap \mathcal{L}$ for an **N**-expansion T of $tr(\Delta)$.

Theorem 1.3 is a generalization of Proposition 1.1. The proof, although technically more involved, is similar to that of Proposition 1.1 given at the end of Section 5.

Let us mention here that no simple characterization of default extensions in terms of stable expansions [15] is known. Each known characterization of default logic by means of stable expansions uses extra-logical concepts like reducts [6, 8] or requires that new atoms (in general, infinitely many) are introduced and that the rules are interpreted by more complex modal formulae than those of the form (4) [22, 11].

We will briefly present now one application of Theorem 1.3 to default logic. Recall that under the translation $tr(\cdot)$ which assigns to a default

$$\frac{\alpha : M\beta, \ldots, M\beta_n}{\omega}$$

the formula of $\mathcal{L}_L : L\alpha \wedge LM\beta_1 \wedge \ldots \wedge LM\beta_n \rightarrow \omega$, default extensions of a default theory $\Delta = \langle D, W\rangle$ are in a one-to-one correspondence with the N-expansions of $tr(\Delta)$, that is solutions to the equation $T = Cn_{\mathbf{N}}(tr(\Delta) \cup \{\neg L\varphi : \varphi \notin T\})$.

We shall now define an entailment relation for defaults as follows: Let $\Delta = \langle D, W\rangle$ be a default theory, and let d be a default. We say that Δ entails d, in symbols $\Delta \rhd d$ if the extensions of Δ and $\Delta' = \langle D \cup \{d\}, W\rangle$ are exactly the same.

It is easy to see that this relation $\rhd$ has the properties of reflexivity and cumulative transitivity (cf. [7]).

The relation $\rhd$ can be characterized in the language $\mathcal{L}_{\omega_1,\omega}$ that is, the propositional language admitting denumerable conjunctions and disjunctions, using the methods of [12]. Here we give a finitary sufficient condition for the entailment $\Delta \rhd d$ to hold.

Proposition 1.4
Let d be a default rule, and Δ a default theory. Let $tr(d)$, $tr(\Delta)$ be translations into modal language. Then $tr(\Delta) \vdash_{\mathbf{N}} tr(d)$ implies $\Delta \rhd d$.

Proof. Assume $tr(\Delta) \vdash_{\mathbf{N}} tr(d)$. Set $\Delta' = \langle D \cup \{d\}, W\rangle$. Let S be any extension of Δ. Then, by Theorem 1.3, there exists an N-expansion T of $tr(\Delta)$ such that $T \cap \mathcal{L} = S$. This means that

$$T = Cn_{\mathbf{N}}(tr(\Delta) \cup \{\neg L\varphi : \varphi \notin T\}).$$

Since $tr(\Delta) \vdash_{\mathbf{N}} tr(d)$,

$$\begin{aligned} T &= Cn_{\mathbf{N}}(tr(\Delta) \cup \{tr(d)\} \cup \{\neg L\varphi : \varphi \notin T\}) \\ &= Cn_{\mathbf{N}}(tr(\Delta') \cup \{\neg L\varphi : \varphi \notin T\}). \end{aligned}$$

Thus T is an N-expansion of $tr(\Delta')$ and since $S = T \cap \mathcal{L}$, S is an extension of Δ' (by Theorem 1.3).
The converse implication is proved in a similar manner. □

To build nonmonotonic reasoning systems based on the logic **N**, algorithms constructing all N-expansions of a finite theory I are needed. In the paper, we develop a theory of the logic **N**. We propose a sound and complete Kripke-like semantics for **N** and build a tableaux system for testing whether a formula is provable from a theory in the logic **N**. An alternative method to compute modal-free consequences of a finite theory is also given. Finally, our results are applied to problems of nonmonotonic reasoning. Namely, we present algorithms to compute all **N**-expansions of a finite theory.

2. Basic properties of provability in the logic N

In this section we present examples and prove some simple properties of the provability operator in $\mathbf{N} - Cn_{\mathbf{N}}$.

EXAMPLE 2.1

(a) One of the basic properties of normal modal systems is that if $\varphi \to \psi$ is a theorem, so is $L\varphi \to L\psi$. This property fails for the logic **N**. For example, $(a \vee a) \to a$ is a theorem of **N** but $L(a \vee a) \to La$ is not; a formal argument for this claim will be given at the end of Section 3.

(b) Next we illustrate the concept of proof in the logic **N**. Consider the theory $I = \{La \to b, L\neg La \to Lb, \neg Lb \vee a, \neg b\}$. The formula a can be proved from I as follows:

(1) $\neg b$ and $La \to b$ yield $\neg La$ (in propositional calculus).
(2) Necessitation applied to $\neg La$ yields $L\neg La$.
(3) $L\neg La$ and $L\neg La \to Lb$ yield Lb (by modus ponens).
(4) Lb and $\neg Lb \vee a$ yield a (in propositional calculus).

We will now develop a convenient representation of the provability operator $Cn_{\mathbf{N}}$. Let us define an operator A as follows:

$$A_0(I) = Cn(I) \qquad A_{n+1}(I) = Cn(I \cup \{L\varphi : \varphi \in A_n(I)\})$$

and

$$A(I) = \bigcup_{n=0}^{\infty} A_n(I)$$

where Cn denotes the provability operator in classical propositional logic.

PROPOSITION 2.1

$Cn_{\mathbf{N}}(I) = A(I)$.

PROOF. First, we define an auxiliary operator A':

$$A'_0(I) = Cn(I) \qquad A'_{n+1}(I) = Cn(A'_n(I) \cup \{L\varphi : \varphi \in A'_n(I)\})$$

and

$$A'(I) = \bigcup_{n=0}^{\infty} A'_n(I).$$

Clearly, $Cn_{\mathbf{N}}(I) = A'(I)$. By induction on n it easily follows that

$$A'_n(I) = A_n(I).$$

Thus, $Cn_{\mathbf{N}}(I) = A(I)$.

By the *L-depth* of a formula φ we mean the maximum depth of nesting of occurrences of L in φ. In modal logics, because of the presence of modal axiom schemata, it is often the case that *any* proof of a formula φ from a theory I contains formulae of L-depth exceeding the maximum L-depth of any formula in $I \cup \{\varphi\}$. Proposition 2.1 implies that this is not the case for the logic **N**. Let $\mathscr{L}_{L,k}$ denote the set of all formulae in $\mathscr{L}_L$ with L-depth at most k.

Proposition 2.2
Let $I \subseteq \mathscr{L}_{L,k}$. For every formula $\varphi \in \mathscr{L}_{L,k}$, if $I \vdash_{\mathbf{N}} \varphi$, then there is a proof of φ from I in **N** where each formula is in $\mathscr{L}_{L,k}$.

Proof. The proof is by induction on the *level* of φ, defined as the minimum n such that $\varphi \in A_n(I)$. The proof is based on the following property of the propositional provability operator: if $\varphi \in Cn(I)$, then there exists a proof of φ from I with each formula built only of atoms occurring in $I \cup \{\varphi\}$. We omit the details. □

In a similar fashion a natural deduction system can be written for **N** and a cut-elimination result proved (see [4] for a definition of the cut-rule in natural deduction systems).

3. A semantics for the logic N

The logic **N** is subnormal. Consequently, there is no conventional Kripke semantics for **N**. In this section we introduce a variant of Kripke semantics that differs from the standard Kripke semantics in that infinitely many accessibility relations are required, one for each formula. An **N**-*structure* is a triple

$$\mathcal{M} = \langle M, \{R_\varphi\}_{\varphi \in \mathscr{L}_L}, V \rangle$$

where M is a nonempty set of objects called *worlds*, V gives valuations of propositional variables (atoms of $\mathscr{L}$) in the worlds from M, that is $V : M \times At \rightarrow \{0, 1\}$, and each R_φ is a binary relation on M.

Given an **N**-structure $\mathcal{M}$, the *satisfaction relation* $\mathcal{M}, m \models \varphi$, for $m \in M$ and $\varphi \in \mathscr{L}_L$, is defined by induction on the complexity of φ as follows:

(1) If p is an atom, then $\mathcal{M}, m \models p$ if $V(m, p) = 1$.
(2) If $\psi = \neg\varphi$, then $\mathcal{M}, m \models \psi$ if it is not true that $\mathcal{M}, m \models \varphi$ (in symbols: $\mathcal{M}, m \not\models \varphi$).
(3) If $\psi = \varphi_1 \wedge \varphi_2$, then $\mathcal{M}, m \models \psi$ if $\mathcal{M}, m \models \varphi_1$ and $\mathcal{M}, m \models \varphi_2$; the other Boolean connectives are dealt with similarly.
(4) If $\psi = L\varphi$, then $\mathcal{M}, m \models \psi$ if for every m' such that $(m, m') \in R_\varphi$, $\mathcal{M}, m' \models \varphi$.

We say that an N-structure $\mathcal{M}$ satisfies φ ($\mathcal{M} \models \varphi$) if for all $m \in M$, $\mathcal{M}, m \models \varphi$. We say that $\mathcal{M}$ satisfies a theory I ($\mathcal{M} \models I$) if $\mathcal{M} \models \varphi$ for every $\varphi \in I$.

The basic intuition behind **N**-structures is that, in the logic **N**, any possible relationship (other than identity) between formulae φ and ψ should have no effect on the mutual relationship between the formulae $L\varphi$ and $L\psi$. One way to ensure that is to use separate relations R_φ and R_ψ for verifying the validity of $L\varphi$ and $L\psi$ in the structure.

We now show that the logic **N** is sound with respect to the class of all **N**-structures.

Proposition 3.1
Let $I \vdash_{\mathbf{N}} \vartheta$. Then for every **N**-structure $\mathcal{M}$, if $\mathcal{M} \models I$ then $\mathcal{M} \models \vartheta$.

Proof. We proceed by induction on the length n of a derivation $\varphi_1, \ldots, \varphi_n$ of ϑ. Assume, as an induction hypothesis, that the proposition holds for every formula with a derivation from I of length less than n, and now consider a formula ϑ such that $I \vdash_{\mathbf{N}} \vartheta$ with a derivation of length n. There are several possibilities.

If $\theta \in I$ or is a tautology of the propositional calculus the assertion is evident. These two cases establish also the basis of induction.

If ϑ is derived from earlier terms φ and $\varphi \to \vartheta$ of the derivation by modus ponens then both φ and $\varphi \to \vartheta$ have derivations from I of length less than n. By the induction hypothesis, $\mathcal{M}, m \models \varphi$ and $\mathcal{M}, m \models \varphi \to \vartheta$ for every $m \in \mathcal{M}$. Then, by the definition of the relation of satisfiability, $\mathcal{M}, m \models \vartheta$, for every $m \in \mathcal{M}$.

Finally, if ϑ follows from an earlier term φ by necessitation, then φ has a derivation from I of length less than n and $\vartheta = L\varphi$. By the induction hypothesis for every $m' \in M$, $\mathcal{M}, m' \models \varphi$. Consequently, for an arbitrary $m \in M$ and every m' such that $(m, m') \in R_\varphi$, $\mathcal{M}, m' \models \varphi$. Then, $\mathcal{M}, m \models L\varphi$ that is, $\mathcal{M}, m \models \vartheta$. □

Next, we will prove the completeness of the semantics of **N**-structures with respect to provability in **N**. The proof is standard and follows the general scheme for such arguments. It is based on the construction of a 'canonical' structure whose worlds are complete theories in the language $\mathcal{L}_L$ (see [5] for examples of such proofs for several normal modal logics). We give the proof here for the convenience of readers not familiar with modal logics.

We begin by introducing two crucial notions. We say that S is *T-inconsistent* if there exists a finite set $\{\varphi_0, \ldots, \varphi_n\} \subseteq S$ such that $T \vdash_{\mathbf{N}} \neg\varphi_0 \vee \ldots \vee \neg\varphi_n$. We say that S is *T-consistent* if S is not T-inconsistent. Thus, S is T-consistent if and only if for every finite set $\{\varphi_0, \ldots, \varphi_n\} \subseteq S$, $T \nvdash_{\mathbf{N}} \neg\varphi_0 \vee \ldots \vee \neg\varphi_n$. (Because the empty disjunction is false, if $S = \varnothing$ is T-inconsistent then every set is T-inconsistent.)

LEMMA 3.2
If S is T-consistent, then S is propositionally consistent.

PROOF. Suppose that S is propositionally inconsistent. Then there exists a finite set $\{\varphi_0, \ldots, \varphi_n\} \subseteq S$ such that $\{\varphi_0, \ldots, \varphi_n\} \vdash \bot$. Hence, using the deduction theorem for propositional logic, $\vdash \neg\varphi_0 \vee \ldots \vee \neg\varphi_n$ and so $T \vdash_{\mathbf{N}} \neg\varphi_0 \vee \ldots \vee \neg\varphi_n$, a contradiction. □

Next we make an observation that allows us to produce T-consistent sets of formulae.

LEMMA 3.3
If $S \subseteq \mathcal{L}_L$ is T-consistent and $\neg L\varphi \in S$, then $\{\neg\varphi\}$ is T-consistent.

PROOF. Assume that $\{\neg\varphi\}$ is T-inconsistent. Then, $T \vdash_{\mathbf{N}} \neg\neg\varphi$. Consequently, $T \vdash_{\mathbf{N}} \varphi$ and so $T \vdash_{\mathbf{N}} L\varphi$. This, of course, implies that $T \vdash_{\mathbf{N}} \neg\neg L\varphi$, so S is T-inconsistent. □

REMARK
A related property used in the case of *normal* modal logics is: if S is T-consistent and $\neg L\varphi \in S$, then $\{\neg\varphi\} \cup \{\psi : L\psi \in S\}$ is T-consistent. The schema K plays a critical role in the proof of this property, and it is not available in **N**. This forces us to use the weaker statement, Lemma 3.3.

Now, we will prove the existence of maximal T-consistent sets of formulae.

LEMMA 3.4
If S is T-consistent then S is contained in a maximal T-consistent set.

PROOF. The union of every $\subseteq$-increasing sequence of T-consistent sets containing S is T-consistent and contains S. Consequently the Kuratowski–Zorn Lemma is applicable and so there exists a maximal T-consistent set extending S. □

Now we list some basic properties of maximal T-consistent sets. The proofs are standard [3] and are omitted.

LEMMA 3.5
If S is a maximal T-consistent set then S possesses these properties:
(a) For every $\vartheta \in \mathcal{L}_L$, $\neg\vartheta \in S$ if and only if $\vartheta \notin S$.
(b) For every $\vartheta_1, \vartheta_2 \in \mathcal{L}_L$, $\vartheta_1 \wedge \vartheta_2 \in S$ if and only if $\vartheta_1 \in S$ and $\vartheta_2 \in S$.
(c) If $T \vdash_{\mathbf{N}} \vartheta$ then $\vartheta \in S$. In particular, $T \subseteq S$.

Now, using Lemma 3.5 we prove the completeness of **N**-structures with respect to provability in the logic **N**. Although the argument is similar to the standard one, our definition of accessibility relations is different, and this is the reason why we provide the proof here.

THEOREM 3.6
For $T \subseteq \mathcal{L}_L$ and $\psi \in \mathcal{L}_L$, $T \vdash_{\mathbf{N}} \psi$ if and only if $T \models \psi$.

PROOF. The 'only if' part was proved in Proposition 3.1. To prove the 'if' part we assume that $T \nvdash_{\mathbf{N}} \psi$ and we build a *canonical* **N**-structure $\mathcal{M} = \langle M, \{R_\varphi\}_{\varphi \in \mathcal{L}_L}, V \rangle$ which satisfies T but does not satisfy ψ. Our assumption implies that $\{\neg\psi\}$ is T-consistent. Define M to consist of all maximal T-consistent sets S. Since $\{\neg\psi\}$ is T-consistent, it can be extended to a maximal T-consistent set. Thus $M \neq \emptyset$. For $m \in M$ and an atomic p, we set $V(m, p) = 1$ if and only if $p \in m$. Furthermore we define $m_1 R_\varphi m_2$ if and only if $\neg L\varphi \in m_1$ and $\neg\varphi \in m_2$. We first prove the following crucial claim:

CLAIM
For every $m \in M$, and every $\varphi \in \mathcal{L}_L$,

$$\mathcal{M}, m \models \varphi \quad \text{if and only if} \quad \varphi \in m.$$

We prove the claim by induction on the length of formula φ.

(1) If p is atomic, then the definition of V implies the assertion.
(2) If $\varphi = \neg\vartheta$ then $\mathcal{M}, m \models \varphi$ precisely when $\mathcal{M}, m \nvDash \vartheta$. By the induction hypothesis, this latter fact is equivalent to $\vartheta \notin m$. Since m is a maximal T-consistent set, by Lemma 3.5(a) this last statement is equivalent to $\varphi \in m$.
(3) If $\varphi = \vartheta_1 \wedge \vartheta_2$, then we reason as in (2), using Lemma 3.5(b).
(4) Finally, we need to consider the case $\varphi = L\vartheta$. First, assume that $L\vartheta \in m$. We need to prove that $\mathcal{M}, m \models L\vartheta$. Assume to the contrary that $\mathcal{M}, m \models \neg L\vartheta$. Then, for some m_1 such that $(m, m_1) \in R_\vartheta$, $\mathcal{M}, m_1 \models \neg\vartheta$. Since $(m, m_1) \in R_\vartheta$, $\neg L\vartheta \in m$. Thus both $L\vartheta$ and $\neg L\vartheta$ belong to m, and m is inconsistent, a contradiction with Lemma 3.2. Conversely, suppose $\mathcal{M}, m \models L\vartheta$ but $L\vartheta \notin m$. Then, by Lemma 3.5(a), $\neg L\vartheta \in m$, and by Lemma 3.3 the set $\{\neg\vartheta\}$ is T-consistent. Consequently, there exists a maximal T-consistent set m_1 such that $\neg\vartheta \in m_1$. Since m_1 is consistent, $\vartheta \notin m_1$. By the induction hypothesis, $\mathcal{M}, m_1 \models \neg\vartheta$. By the definition of R_ϑ, $(m, m_1) \in R_\vartheta$. Then, $\mathcal{M}, m \models \neg L\vartheta$. This is a contradiction and it completes the proof of the claim. Claim□

By Lemma 3.5(d), for all $m \in M$, $T \subseteq m$. By the claim, for every $m \in M$, $\mathcal{M}, m \models T$. Hence, $\mathcal{M} \models T$. On the other hand, there is a maximal T-consistent set m containing $\neg\psi$. Consequently, $\mathcal{M}, m \models \neg\psi$. Thus $\mathcal{M} \nvDash \psi$.

We conclude this section with some examples.

EXAMPLE 3.1
(a) First we will show that $L(a \vee a) \to La$ is not a theorem of **N**. To this end, consider the **N**-structure $\mathcal{M}$, such that $M = \{m\}$, $V(m, a) = 0$, $R_a = \{(m, m)\}$ and $R_{a \vee a} = \emptyset$. It is easy to see that $\mathcal{M} \nvDash L(a \vee a) \to La$. It is crucial that we have two different accessibility relations in this structure. If all

relations in an **N**-structure are identical then it collapses to a conventional Kripke structure and each standard Kripke structure satisfies $L(a \vee a) \rightarrow La$.
(b) Consider the theory $I = \{La, L(a \rightarrow b), \neg Lb\}$. Theory I is clearly inconsistent in logic **K**, the least normal modal system. We will show that I is **N**-consistent. To this end, consider the **N**-structure $\mathcal{M}$ defined as follows: $M = \{m\}$, $V(m, a) = V(m, b) = 0$, $R_a = R_{a \rightarrow b} = \emptyset$ and $R_b = \{(m, m)\}$. Again, it is easy to see that $\mathcal{M} \models I$.

4. A tableaux method for the logic N

In this section we introduce the notion of a *modal I-tableau* for a finite theory I and use it in an algorithm that, for a given formula φ, decides whether $I \vdash_{\mathbf{N}} \varphi$.

We begin by defining a *classical* I-tableau for φ. Such a tableau is a rooted binary tree, with formulae as node labels. If α occurs as a node label on a branch, we say simply that α occurs on the branch. We call a branch of such a tree *directly closed* if both α and $\neg\alpha$ occur on it, for some formula α. A classical I-tableau is *directly closed* if each of its branches is directly closed. A branch that is not directly closed is called *open*. In this section, to simplify the description of the tableaux method for **N**, we assume that the only classical connectives we use to build formulae in $\mathcal{L}$ are $\neg$ and $\wedge$. The tableau development rules for other connectives can easily be introduced as in [23] or [3].

Definition 4.1
Let $I = \{\theta_1, \ldots, \theta_n\}$. A *classical I-tableau* for a formula φ is defined recursively, as follows:

(1) The tree consisting of a single branch, with $n + 1$ nodes labelled $\theta_1, \ldots, \theta_n, \varphi$, is a classical I-tableau for φ.
(2) If T is a classical I-tableau for φ, then the result T' of applying one of the following *tableau development rules* to T is another classical I-tableau for φ.
   (a) If a formula $\neg\neg\alpha$ occurs on an open branch B of T but α does not, then extend branch B by adding a new node to the end of B and label it with α;
   (b) If a formula $\alpha \wedge \beta$ occurs on an open branch B of T but at least one of α or β does not, then extend B by adding two new nodes to the end of B, one following the other, and label them α and β;
   (c) If a formula $\neg(\alpha \wedge \beta)$ occurs on an open branch B of T but neither $\neg\alpha$ nor $\neg\beta$ occurs, then add two new nodes as left and right children of the last node of B, and label one with $\neg\alpha$, the other $\neg\beta$.

These tableau development rules are well-known and yield a tableaux method for classical propositional calculus. Specifically, $\varphi \in Cn(I)$ if and only if there is a closed (maximal) classical I-tableau for $\neg\varphi$. They will be referred to as classical tableau rules.

A classical I-tableau for φ is *maximal* if no classical tableau rule applies to it. Since I is finite, each classical I-tableau can be extended to a maximal one in a finite number of steps. An algorithm is straightforward and we omit the details.

Now we take the modal connective into account, and define a broader notion of tableaux.

DEFINITION 4.2
A *modal* I-tableau for a formula φ is defined recursively, as follows:

(1) A maximal classical I-tableau for a formula φ is a modal I-tableau for φ.
(2) Suppose T is a modal I-tableau for φ, with an open branch B containing a formula $\neg L\alpha$. Let T' be maximal classical I-tableau for $\neg\alpha$. Tableau T' is a modal I-tableau for φ; we call it a $\neg\alpha$-child of B.

Our tableaux method will construct maximal sets of tableaux (which will be called *φ-saturated* sets), and will use them to decide whether a formula is derivable in **N** from a theory I. We give now a precise definition of a φ-saturated set.

DEFINITION 4.3
A set S of modal I-tableaux for φ is called *φ-saturated* provided:

(1) S contains a classical I-tableau for φ; and
(2) for each $T \in S$, for each open branch B of T, and for each formula $\neg L\alpha$ on B, there is in S a classical I-tableau for $\neg\alpha$ (that is, a $\neg\alpha$-child of B).

Notice that if S is φ-saturated, then it consists only of maximal classical tableaux since, by the Definition 4.2, modal I-tableaux are maximal classical I-tableaux.

There is a straightforward algorithm for constructing φ-saturated sets. For the following, I is a fixed finite set, and φ is a formula.

$T :=$ a maximal classical I-tableau for φ;
$S := \{T\}$;
while S is not φ-saturated **do**
  select a tableau in S, with an open branch B,.containing a formula $\neg L\alpha$ with no classical I-tableau for $\neg\alpha$ in S;
  $T :=$ a maximal classical I-tableau for $\neg\alpha$;
  $S := S \cup \{T\}$.
od

It is evident that this algorithm will always terminate, and at termination, S will be φ-saturated. Note also that the φ-saturated set produced by the algorithm contains exactly one classical I-tableau for φ. In the remainder of this section, we restrict our considerations to φ-saturated sets that can be produced by the algorithm above and we call the unique classical I-tableau for φ in such a set the *root tableau.*

Now we extend the definitions of closure to the case of modal tableaux contained in a φ-saturated set.

DEFINITION 4.4

Let S be a φ-saturated set of modal I-tableaux for φ. Let C be the least set $X \subseteq S$ satisfying the following condition:

(*) $T \in X$ whenever, for each branch B of T, either B is directly closed, or B has an α-child in X, for some formula α.

Note that $X = S$ satisfies (*), and so the collection of sets satisfying (*) is nonempty. Since it is closed under intersections, the smallest set C satisfying (*) exists. Note also that C contains all directly closed tableaux in S. Each tableau in C is called S-closed. A branch of a tableau from S is S-closed if it is directly closed or if for some formula α it has an S-closed α-child in S. A branch is S-open if it is not S-closed.

The set C can be easily constructed by the following algorithm.

$C := \emptyset$;
repeat
  $C' := \emptyset$;
  for all $T \in S \backslash C$ **do**
    if each branch B of T is directly closed or, for some formula α, B has an α-child in C **then** $C' := \{T\} \cup C'$
  rof
  $C := C \cup C'$;
until $C' = \emptyset$.

The proof of correctness of this algorithm is simple and we omit the details. Note that the algorithm allows us to assign ranks to tableaux in C. Define the *rank* of a tableau in C to be the index of the iteration of the **repeat** loop in which the tableau was included in C. For example, each directly closed tableau is included into C in the first iteration. Thus, all directly closed tableaux of S have rank 1. It is easy to see that all other tableaux have ranks greater than 1.

We now have the following theorem.

THEOREM 4.5

Let S be a $\neg\varphi$-saturated set of modal I-tableaux for $\neg\varphi$. Then $I \vdash_{\mathrm{N}} \varphi$ if and only if the root tableau of S is S-closed.

This theorem proves correctness of the following algorithm for deciding whether a formula φ is a consequence in the logic **N** of a finite theory I: using the algorithms mentioned or outlined above construct a $\neg\varphi$-saturated set S of modal I-tableaux for $\neg\varphi$. Next compute the set C of S-closed tableaux. If C contains the root tableau of S, then $I \vdash_{\mathbf{N}} \varphi$. Otherwise, $I \nvdash_{\mathbf{N}} \varphi$.

In order to prove Theorem 4.5 we need some technical facts. We begin with a definition of the auxiliary concept of I-satisfiability. Loosely, J is I-satisfiable if J is satisfied at some world of a model in which I is valid (satisfied at every world).

Definition 4.6
Let $I, J \subseteq \mathcal{L}_L$. Theory J is *I-satisfiable* if there is an **N**-structure $\mathcal{M}$ and a world m of $\mathcal{M}$ such that $\mathcal{M} \models I$ and $\mathcal{M}, m \models J$. A branch B of a classical I-tableau is *I-satisfiable* if the set of all formulae on B is I-satisfiable. A classical tableau is *I-satisfiable* if it has an I-satisfiable branch.

We have the following simple lemma.

Lemma 4.7
Let I be a finite theory, $\{\varphi\}$ be I-satisfiable, and T be a classical I-tableau for φ. Then there is a branch B in T such that the set of formulae on B is I-satisfiable. In other words, T is I-satisfiable.

Proof. The proof is by induction on the number of applications of the classical tableau rules. It is standard and we omit the details. □

The next lemma plays a key role in the proof of the sufficiency part of Theorem 4.5.

Lemma 4.8
Let S be a φ-saturated set of modal I-tableaux for φ. An S-closed tableau in S is not I-satisfiable.

Proof. Let T be an S-closed tableau in S. We proceed by induction on the rank of T. If the rank of T is 1, then T is directly closed. Let B be any branch of T. Then there are formulae α and $\neg\alpha$ on B, for some $\alpha \in \mathcal{L}_L$. Consequently, B is not I-satisfiable. Thus, since B was an arbitrary branch of T, T is not I-satisfiable.

Suppose that the lemma holds for all S-closed tableaux with rank less than k, and consider an S-closed tableau $T \in S$ with rank k. Tableau T is a classical I-tableau for some formula ψ. Assume that T is I-satisfiable. Then there is a branch B in T such that the set F of formulae on B is I-satisfiable. It follows that B is not directly closed. Thus, since B is S-closed, there is a formula $\neg L\alpha$ on B and a classical I-tableau T' for $\neg\alpha$ such that T' is in S, T' is S-closed and T' has rank smaller than k. Since the set F of formulae on B is I-satisfiable, there is an **N**-structure $\mathcal{M}$ and a world m such that $\mathcal{M} \models I$ and $\mathcal{M}, m \models F$. In particular, $\mathcal{M}, m \models \neg L\alpha$. Consequently, there is a world m' such that $\mathcal{M}, m' \models \neg\alpha$. Thus, both $\{\neg\alpha\}$ and, by Lemma 4.7, T' are I-satisfiable. Tableau T' has a smaller rank than T and is S-closed. By the induction hypothesis it follows that T' is not I-satisfiable, a contradiction. Hence, T is not I-satisfiable. □

Next we show that if a root tableau of a $\neg\varphi$-saturated set of modal I-tableaux for $\neg\varphi$ has an S-open branch, then $I \nvdash_{\mathbf{N}} \varphi$.

LEMMA 4.9
Let $I \subseteq \mathcal{L}_L$ be finite and let S be a $\neg\varphi$-saturated set of modal I-tableaux for $\neg\varphi$. If the root tableau of S has an S-open branch, then there is an $\mathbf{N}$-structure $\mathcal{M}$, such that $\mathcal{M} \models I$, and $\mathcal{M} \not\models \varphi$.

PROOF. Define M to consist of the theories of the S-open branches of the tableaux from S. Let $m \in M$ and let p be an atom. Set $V(m, p) = 1$ if and only if $p \in m$. Finally, for a $\vartheta \in \mathcal{L}_L$ define a relation R_ϑ as follows: $(m_1, m_2) \in R_\vartheta$ if $\neg L\vartheta \in m_1$, and $\neg\vartheta \in m_2$. Put $\mathcal{M} = \langle M, \{R_\vartheta\}_{\vartheta \in \mathcal{L}_L}, V\rangle$. We first prove the following claim.

CLAIM
If $\vartheta \in m$, then $\mathcal{M}, m \models \vartheta$. (Note that in the canonical structure used in the proof of the completeness result in Section 3, all worlds were complete theories and we were able to prove the equivalence of these two statements. Here the worlds need not be complete and we can prove only implication one way.)

PROOF of the claim. Let m be the set of formulae on an S-open branch B. We proceed by induction on the length of ϑ. If ϑ is an atom, the claim holds by definition.

(1) Assume that $\vartheta = \neg\psi$. Since B is S-open, $\psi \notin m$. If ψ is an atom, then $\mathcal{M}, m \not\models \psi$, by the definition of V. If $\psi = \neg\psi_1$, then $\psi_1 \in m$ (by the definition of classical tableaux). By the induction hypothesis, $\mathcal{M}, m \models \psi_1$. Thus, $\mathcal{M}, m \models \vartheta$. If $\psi = \varphi_1 \wedge \varphi_2$ then for some $i, i = 1$ or 2, $\neg\varphi_i \in m$ (by the definition of classical tableaux). By the induction hypothesis, $\mathcal{M}, m \models \neg\varphi_i$. Thus, $\mathcal{M}, m \models \neg\psi$. The last possibility is $\psi = L\psi_1$. Since B is not directly closed and S is $\neg\varphi$-saturated, there is a $\neg\psi_1$-child T of B in S. The tableau T is a classical I-tableau for $\neg\psi_1$. Since B is S-open, there is an S-open branch B' in T. Let m' be the theory of B'. Then, $\neg\psi_1 \in m'$ and $(m, m') \in R_{\psi_1}$. By the induction hypothesis, $\mathcal{M}, m' \models \neg\psi_1$. Thus, $\mathcal{M}, m \models \neg L\psi_1$.
(2) Next suppose that $\vartheta = \psi_1 \wedge \psi_2$. It follows from the definition of the development rules that ψ_1 and $\psi_2 \in m$. By the induction hypothesis, $\mathcal{M}, m \models \psi_i$, $i = 1, 2$. Consequently, $\mathcal{M}, m \models \vartheta$.
(3) Finally, suppose $\vartheta = L\psi$. Since B is S-open, $\neg L\psi \notin m$. Thus, no world in M can be accessed from m via R_ψ. Consequently, $\mathcal{M}, m \models L\psi$. This completes the proof of the claim.

Now the assertion follows easily. Let B be an S-open branch in the root tableau of S. Clearly, $\neg\varphi \in B$. Then, the claim implies that there is a world $m \in M$ such that $\mathcal{M}, m \models \neg\varphi$. Consequently, $\mathcal{M}, m \not\models \varphi$. On the other hand, each branch of a tableau in S contains I. Then, again using the claim, we obtain that $\mathcal{M} \models I$.

PROOF of Theorem 4.5. We use Lemmas 4.8 and 4.9. Let S be a $\neg\varphi$-saturated set of modal tableaux for $\neg\varphi$. Suppose that the root tableau of S has an S-open branch. Then Theorem 3.6 and Lemma 4.9 imply that $I \nvdash_{\mathbf{N}} \varphi$. Conversely, suppose that the root tableau of S is S-closed. Then, by Lemmas 4.8 and 4.7, $\neg\varphi$ is not I-satisfiable. That means, that for each $\mathbf{N}$-structure $\mathcal{M}$ if $\mathcal{M} \models I$, then for each world m of $\mathcal{M}$, $\mathcal{M}, m \not\models \neg\varphi$ or, equivalently, $\mathcal{M}, m \models \varphi$. Then, again by Theorem 3.6, $I \vdash_{\mathbf{N}} \varphi$.

EXAMPLE 4.1
Consider the theory $I = \{La \rightarrow b, L\neg La \rightarrow b, \neg Lb \vee a, b\}$. Does $I \vdash_{\mathbf{N}} a$ hold? We have seen in Example 2.1 that the answer is yes. Now, we will show how to use the method of modal I-tableaux to resolve this problem. Figure 1 shows a $\neg a$-saturated set S of I-tableaux. The tableau $T1$ in the picture is the root tableau and the tableau $T2$ is its $\neg La$-child. Branches marked by $*$ are directly closed. Tableau $T1$ is a complete classical tableau for I and a. Levels of $T1$ 1–4 contain I. Level 5 is $\neg a$. At the level 6 we extend the tableau using

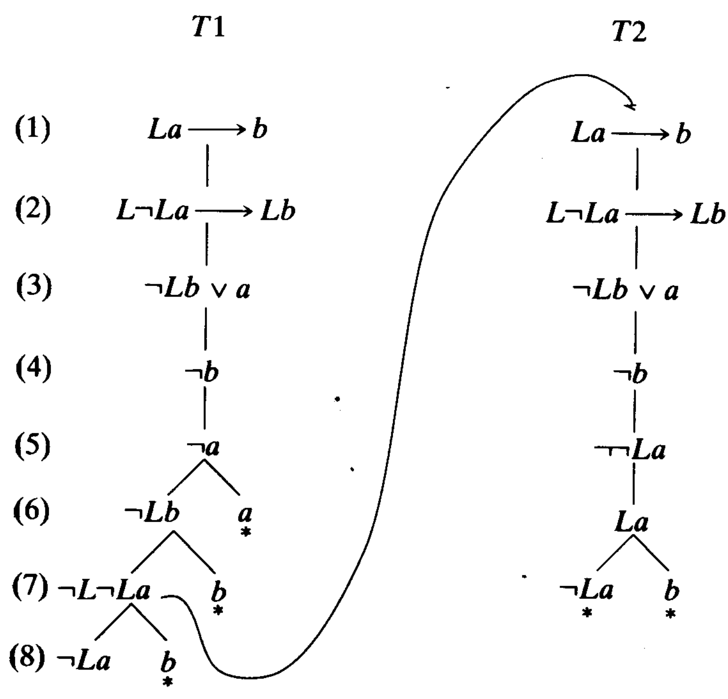

FIG. 1

the formula of level 3. The right branch is closed at this point. At the level 7 we extend the tableau using the formula of level 2. The right branch is closed at this point. At the level 8 we extend the tableau using the formula of level 1. Again the right branch is closed immediately. The left branch is *not* closed at this point and the tableau $T1$ is not closed. We select now a formula of the form $\neg L\Psi$ on an open branch ($T1$ has just one open branch). We build now a tableau for I and $\neg\Psi$. In our case it is the tableau $T2$. Thus, in the tableau $T2$, after initial four levels listing I we put $\neg\Psi$. In our case Ψ is $\neg La$. Thus at level 5 we put in the tableau $T2$ the formula $\neg\neg La$. The tableau $T2$ is then developed and it is a closed tableau—all its branches are closed. This closes the last non-closed branch of the tabeau $T1$. Thus, $I \vdash_{\mathbf{N}} a$ holds.

Several normal modal logics $\mathcal{S}$ possess the *finite model property*, that is, if I is finite and $I \nvdash_{\mathcal{S}} \varphi$, then there is a Kripke structure $\mathcal{M}$ for $\mathcal{S}$ with a finite universe such that $\mathcal{M} \models I$ and $\mathcal{M} \not\models \varphi$. The situation here is similar. We have the following theorem which can be proved by using the **N**-structure constructed in the proof of Lemma 4.9.

THEOREM 4.10 (Finite universe property)
Let $I \subseteq \mathcal{L}_L$ be finite and let $\varphi \in \mathcal{L}_L$ be such that $I \nvdash_{\mathbf{N}} \varphi$. Then, there is an **N**-structure $\mathcal{M}$ with finite universe such that $\mathcal{M} \models I$ and $\mathcal{M} \not\models \varphi$.

The proof of Lemma 4.9 provides a bound (in terms of the total length of formulae in I) for the size of the universe.

In the case of **N**-structures, even if the universe of an **N**-structure is finite there are infinitely many accessibility relations to deal with. So, after restricting the size of the universe the next task is to reduce the number of accessibility relations. This can be achieved by means of the following theorem. Its proof is standard and is omitted.

THEOREM 4.11
Let $I \in \mathcal{L}_L$ and let $\mathcal{M}_i = \langle M_i, \{R_{i,\varphi}\}_{\varphi\in\mathcal{L}_L}, V_i\rangle$, $i = 1, 2$, be **N**-structures such that $M_1 = M_2$, $V_1 = V_2$ and $R_{1,\varphi} = R_{2,\varphi}$ for every subformula φ of I. Then, for every $m \in M_1$ $(=M_2)$, $\mathcal{M}_1, m \models I$ if and only if $\mathcal{M}_2, m \models I$. In particular, $\mathcal{M}_1 \models I$ if and only if $\mathcal{M}_2 \models I$.

If I is finite, Theorems 4.10 and 4.11 allow a restriction to models with finite universes and finitely many accessibility relations. Furthermore, standard methods of Kripke structures allow us to restrict the domains of valuations to atoms actually appearing in formulae of I.

5. Modal-free consequences in the logic N

In the previous section we presented a method to solve the membership problem for the consequence operator of the logic **N**: given a finite theory

$I \subseteq \mathscr{L}_L$ and a formula $\alpha \in \mathscr{L}_L$, determine whether $I \vdash_{\mathbf{N}} \alpha$. In this section we will study a restricted variant of the problem in which α is modal-free, that is $\alpha \in \mathscr{L}$. In fact, for this restricted variant of the membership problem we will describe an algorithm which, given a finite theory $I \subseteq \mathscr{L}_L$ produces a finite set $S \subseteq \mathscr{L}$ such that

$$Cn_{\mathbf{N}}(I) \cap \mathscr{L} = Cn(S).$$

The first step is to replace an arbitrary theory $I \subseteq \mathscr{L}_L$ by a theory J which consists of formulae of L-depth at most 1. This requires introducing new propositional atoms. We now describe the basic step in the construction of J. For each modal atom $L\varphi$ of L-depth 1 occurring in I we introduce a new propositional atom a_φ. In each formula $\psi \in I$ we replace all occurrences of modal atoms of L-depth 1 by the corresponding new propositional atoms. The resulting formula will be denoted by $\bar{\psi}$. Finally, we add formulae $L\varphi \leftrightarrow a_\varphi$, for all new atoms a_φ. The resulting theory will be denoted by $e(I)$. The language obtained from $\mathscr{L}$ by adding to it atoms a_φ will be denoted by $\mathscr{L}^e$.

LEMMA 5.1
$Cn_N(I) \cap \mathscr{L} = Cn_{\mathbf{N}}(e(I)) \cap \mathscr{L}$.

PROOF. Let $\mathcal{M}^e = \langle M, \{R^e_\psi\}_{\psi \in \mathscr{L}^e_L}, V^e \rangle$ be an **N**-structure. For $\psi \in \mathscr{L}_L$ define $R_\psi = R^e_{\bar{\psi}}$. For an atom $p \in \mathscr{L}$ define $V(p, m) = V^e(p, m)$. Finally, define $\mathcal{M} = \langle M, \{R_\psi\}_{\psi \in \mathscr{L}_L}, V \rangle$. We will prove that for every $\psi \in \mathscr{L}_L$ and every $m \in M$,

$$\mathcal{M}^e, m \models \bar{\psi} \quad \text{if and only if} \quad \mathcal{M}, m \models \psi.$$

We proceed by induction on the length of ψ. If ψ is an atom of $\mathscr{L}$, then the claim follows from the definition of V and from the equality $\psi = \bar{\psi}$. Assume that $\psi = \neg\alpha$. Then $\bar{\psi} = \neg\bar{\alpha}$. By the induction hypothesis, $\mathcal{M}^e, m \models \bar{\alpha}$ if and only if $\mathcal{M}, m \models \alpha$. Thus, the equivalence $\mathcal{M}^e, m \models \bar{\psi}$ if and only if $\mathcal{M}, m \models \psi$ follows. The cases of other Boolean connectives can be dealt with in a similar fashion. Thus, assume that $\psi = L\alpha$. If α has L-depth at least 1, then $\bar{\psi} = L\bar{\alpha}$. Consequently, the statement $\mathcal{M}^e, m \models \bar{\psi}$ is equivalent to the statement: $\mathcal{M}^e, m' \models \bar{\alpha}$ for each m' such that $(m, m') \in R^e_{\bar{\alpha}}$. Since $R^e_{\bar{\alpha}} = R_\alpha$, by the induction hypothesis it follows that this last statement is equivalent to the statement: $\mathcal{M}, m' \models \bar{\alpha}$ for each m' such that $(m, m') \in R_\alpha$, which is equivalent to $\mathcal{M}, m \models L\alpha$.

If α has L-depth 0, then $\bar{\psi} = a_\alpha$. Since $\mathcal{M}^e \models L\alpha \leftrightarrow a_\alpha$, the statement $\mathcal{M}^e, m \models \bar{\psi}$ is equivalent to $\mathcal{M}^e, m \models L\alpha$. Since $\alpha \in \mathscr{L}$, $\bar{\alpha} = \alpha$. Thus, $R^e_{\bar{\alpha}} = R^e_\alpha = R_\alpha$. Therefore, the statement $\mathcal{M}^e, m \models L\alpha$ is equivalent to $\mathcal{M}, m \models L\alpha$.

Let now $\varphi \in Cn_{\mathbf{N}}(I) \cap \mathscr{L}$. Suppose that $\mathcal{M}^e$ satisfies $e(I)$. The claim we proved implies that $\mathcal{M}$ (defined as before) satisfies I. Consequently, $\mathcal{M} \models \varphi$.

Since $\varphi \in \mathscr{L}$, $\bar{\varphi} = \varphi$. Thus, again by the claim we proved, $\mathscr{M}^e \models \varphi$. By Theorem 3.6, $\varphi \in Cn_{\mathbf{N}}(e(I))$.

The converse inclusion can be proved in a similar fashion. Consider an arbitrary **N**-structure $\mathscr{M} = \langle M, \{R_\psi\}_{\psi \in \mathscr{L}_L}, V \rangle$. Define an **N**-structure $\mathscr{M}^e = \langle M, \{R^e_\psi\}_{\psi \in \mathscr{L}^e_L}, V^e \rangle$ as follows. For $\psi \in \mathscr{L}_L$ set $R^e_\psi = R^e_{\bar{\psi}} = R_\psi$. All other relations R^e_ψ are chosen arbitrarily. Finally, for an atom $p \in \mathscr{L}$ put $V^e(p, m) = V(p, m)$ and for each new atom a_φ define $V^e(a_\varphi, m) = 1$ if and only if $\mathscr{M}, m \models L\varphi$. Similarly as before, for each such $\mathscr{M}^e$ the following statements can be established (we omit the details):

(1) For every $\vartheta \in \mathscr{L}$, $\mathscr{M}^e, m \models \vartheta$ if and only if $\mathscr{M}, m \models \vartheta$.
(2) For every $m \in M$, $\mathscr{M}^e, m \models L\varphi \leftrightarrow a_\varphi$.
(3) For every $m \in M$, and $\psi \in \mathscr{L}_L$, $\mathscr{M}, m \models \psi$ if and only if $\mathscr{M}^e, m \models \bar{\psi}$.

Let now $\varphi \in Cn_{\mathbf{N}}(e(I)) \cap \mathscr{L}$. Consider an arbitrary **N**-structure $\mathscr{M}$ satisfying I. Then, by (2) and (3), $\mathscr{M}^e$ satisfies $e(I)$. Consequently, $\mathscr{M}^e \models \varphi$ and, by (1), $\mathscr{M} \models \varphi$. Thus, $\varphi \in Cn_{\mathbf{N}}(I)$. (Note that this reasoning proves a stronger inclusion: $Cn_{\mathbf{N}}(e(I)) \cap \mathscr{L}_L \subseteq Cn_{\mathbf{N}}(I)$.)

Thus we introduced an operator e which, given a theory $I \subseteq \mathscr{L}_L$, produces a theory $e(I)$ in a modal language with more propositional atoms. This theory $e(I)$ carries the information defining some propositional atoms as equivalent to modal atoms. There are two important features of the theory $e(I)$. First, as long as I is finite and its L-depth is bigger than 1, $e(I)$ is finite and has a smaller L-depth. Secondly, the consequences of $e(I)$ in the original propositional language $\mathscr{L}$ are the same as those of I.

Now we shall iterate this construction.

THEOREM 5.2
For every theory $I \subseteq \mathscr{L}_L$ (finite or not) there exists a theory J consisting of formulae of depth at most 1 and such that

$$Cn_{\mathbf{N}}(I) \cap \mathscr{L} = Cn_{\mathbf{N}}(J) \cap \mathscr{L}.$$

PROOF. Let I_n consist of all formulae of I that have L-depth at most n. By e^n denote the operator resulting from iterating n times the operator e. Define $J_0 = I_0$, $J_1 = I_1$ and $J_n = e^{n-1}(I_n)$ for $n \geq 2$, and assume that when constructing J_n the same propositional atoms are used for the modal atoms that occur in I_{n-1} that were used when constructing J_{n-1}. This additional assumption guarantees that $J_{n-1} \subseteq J_n$. Define $J \doteq \bigcup_{n=0}^{\infty} J_n$. Clearly, each formula in $e^{n-1}(I_n)$ has L-depth at most 1 and Lemma 5.1 implies that $Cn_{\mathbf{N}}(I_n) \cap \mathscr{L} = Cn_{\mathbf{N}}(J_n) \cap \mathscr{L}$. Since $I = \bigcup_{n=0}^{\infty} I_n$, $J = \bigcup_{n=0}^{\infty} J_n$, $I_0 \subseteq I_1 \subseteq \ldots$, and $J_0 \subseteq J_1 \subseteq \ldots$, the assertion follows. □

We now restrict attention to theories contained in $\mathscr{L}_{L,1}$. For such theory I define (see Section 2 for the definitions of the operators A_n and A)

$$B_n(I) = A_n(I) \cap \mathscr{L}$$

and

$$B(I) = \bigcup_{n=0}^{\infty} B_n(I).$$

It easily follows from the definition of the operator A that if $I \subseteq \mathcal{L}_{L,1}$ then

$$B_{n+1}(I) = Cn(I \cup LB_n(I)) \cap \mathcal{L}.$$

Consequently,

$$Cn_{\mathbf{N}}(I) \cap \mathcal{L} = B(I).$$

Thus, to find $Cn_{\mathbf{N}}(I) \cap \mathcal{L}$ it suffices to describe a method to compute $B(I)$.

First, we will prove a technical fact. Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two propositional languages with disjoint sets of atoms, At_1 and At_2, respectively. Let $\mathcal{L}$ be the language generated by the union $At = At_1 \cup At_2$.

Each formula φ in $\mathcal{L}$ can be represented as a conjunction of disjunctions $\alpha \vee \omega$ with $\alpha \in \mathcal{L}_1$ and $\omega \in \mathcal{L}_2$. Thus, each theory I has a propositionally equivalent theory consisting of such disjunctions.

Let $I = \{\alpha_i \vee \omega_i : i \in S\}$, where each $\alpha_i \in \mathcal{L}_1$ and each $\omega_i \in \mathcal{L}_2$. Define $\mathcal{H}_I = \{J : J$ is finite, $\{\alpha_i : i \in J\} \vdash \bot\}$. For a finite set $J \subseteq S$ define $\omega_J = \bigvee \{\omega_i : i \in J\}$.

Proposition 5.3
$Cn(I) \cap \mathcal{L}_2 = Cn(\{\omega_J : J \in \mathcal{H}_I\})$.

Proof. Clearly, if $\{\alpha_i : i \in J\} \vdash \bot$, then $\{\alpha_i \vee \omega_i : i \in J\} \vdash \omega_J$. Thus, for each $J \in \mathcal{H}_I$, $\omega_J \in Cn(I) \cap \mathcal{L}_2$.

We now prove the converse inclusion. Let $\omega \in \mathcal{L}_2$ and $\omega \notin Cn(\{\omega_J : J \in \mathcal{H}_I\})$. Then there is a valuation v_2 of At_2 such that $v_2(\omega) = 0$ and $v_2(\omega_J) = 1$ for every $J \in \mathcal{H}_I$. Let $J' = \{j : v_2(\omega_j) = 0\}$. Consider now $\{\alpha_j : j \in J'\}$. Since $v_2(\omega_{J'}) = 0$, $J' \notin \mathcal{H}_I$. Thus, $\{\alpha_j : j \in J'\}$ is consistent. Let v_1 be a valuation of At_1 such that $v_1(\alpha_j) = 1$ for all $j \in J'$. Combine v_1 and v_2 into a single valuation v of At. This is possible since $At_1 \cap At_2 = \varnothing$. Now, let $j \in S$. If $j \in J'$, then $v_1(\alpha_j) = 1$ and so $v(\alpha_j \vee \omega_j) = 1$. If $j \notin J'$, then $v_2(\omega_j) = 1$ and so $v(\alpha_j \vee \omega_j) = 1$. Consequently, v evaluates all I as 1. Since v coincides with v_2 on At_2, $\omega \notin Cn(I) \cap \mathcal{L}_2$. □

Now we will describe a method to compute $B(I)$ for a theory $I \subseteq \mathcal{L}_{L,1}$. First, without loss of generality, we may assume that each formula in I is of the form $\alpha \vee \omega$, where α is built only of modal atoms and ω is built only of propositional atoms, say $I = \{\alpha_i \vee \omega_i : i \in S\}$, for some set of indices S. For each n, we produce a set $U_n \subseteq \{\omega_J : J \subseteq S\}$ such that $B_n(I) = Cn(U_n)$. Proposition 5.3 implies that for U_0 we can take $\{\omega_J : J \in \mathcal{H}_I\}$. To compute U_{n+1}, we proceed as follows. First, we find all modal atoms $L\beta$ occurring in I such that $\beta \in B_n(I)$ $(=Cn(U_n))$. Define I' to be the union of such modal atoms and I. Then Proposition 5.3 implies that $U_{n+1} = \{\omega_J : J \in \mathcal{H}_{I'}\}$ satisfies $B_{n+1}(I) = Cn(U_{n+1})$. Clearly, $B(I) = Cn(\bigcup_{n=0}^{\infty} U_n)$. If I is finite, then for some n, $U_n = U_{n+1}$. At that point the construction can be stopped and $B(I) = Cn(U_n)$.

Clearly, the method just described, together with Theorem 5.2, allows one to find for every finite $I \subseteq \mathcal{L}_L$ a finite set U such that $Cn_{\mathbf{N}}(I) \cap \mathcal{L} = Cn(U)$.

As an illustration of the usefulness of the operator B, we will prove now Proposition 1.1.

PROOF of Proposition 1.1
Let $I \subseteq \mathcal{L}$ and let R be a collection of rules of the form 1. Define

$$Q_0(I, R) = Cn(I),$$

$$Q_{n+1}(I, R) = Cn\left(I \cup \left\{\psi : \frac{\varphi_1, \ldots, \varphi_n}{\psi} \in R \text{ and } \varphi_1, \ldots, \varphi_n \in Q_n(I, R)\right\}\right)$$

and

$$Q(I, R) = \bigcup_{n=0}^{\infty} Q_n(I, R).$$

It is easy to see that $Q(I, R)$ is precisely the least subset of $\mathcal{L}$ containing I and closed under propositional consequence and the rules from R.

Now, by induction on n, we prove that

$$B_n(I \cup T) = Q_n(I, R),$$

where T is the modal theory obtained by replacing in R each rule of the form (1) by a modal formula $L\varphi_1 \wedge \ldots \wedge L\varphi_n \rightarrow \psi$. The details are routine. □

6. Computing N-expansions

In this section we will use some of the previously obtained results to design a method of computing all consistent **N**-expansions of a finite theory $I \subseteq \mathcal{L}_L$. (Theory I has an inconsistent expansion if and only if I is **N**-inconsistent, which can be checked directly using the results of Section 4 or 5.) This, in view of Theorem 1.3, yields a method to compute extensions of default theories.

First, we recall several related notions and results. Moore [15] defined an *expansion* of a theory $I \subseteq \mathcal{L}_L$ to be any theory T satisfying

$$T = Cn(I \cup \{L\varphi : \varphi \in T\} \cup \{\neg L\varphi : \varphi \notin T\}).$$

Expansions of a theory I are *stable* (see [15, 10]). Let us recall that a theory $T \subseteq \mathcal{L}_L$ is *stable* if

(1) T is closed under propositional consequence;
(2) T is closed under necessitation;
(3) for every $\varphi \notin T$, $\neg L\varphi \in T$.

A stable theory T is uniquely determined by its objective, that is modal-free, part $T \cap \mathscr{L}$. For $U \subseteq \mathscr{L}$, let $E(U)$ be the unique stable theory such that $E(U) \cap \mathscr{L} = Cn(U)$.

Expansions of a finite theory I were characterized by Marek and Truszczynski [10]. Since propositionally equivalent theories have the same expansions, without loss of generality we may assume that I consists of formulae of the form $\alpha \vee \omega$, where α is built of modal literals only and ω is built of propositional literals, say $I = \{\alpha_i \vee \omega_i : i \in S\}$, for some finite set of indices S. For such a theory I we have the following result.

THEOREM 6.1 [10]
A consistent theory T is an expansion of I if and only if

$$T = E(\{\omega_i : i \in S_T\}),$$

where $S_T = \{i \in S : \neg\alpha_i \in T\}$.

In [9] the following results on **N**-expansions are proved.

THEOREM 6.2
Let $I \subseteq \mathscr{L}_L$.

(1) Every **N**-expansion of I is an expansion of I.
(2) Theory $E(U)$, where $U \subseteq \mathscr{L}$, is an **N**-expansion of I if and only if $I \subseteq E(U)$ and $U \subseteq Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : \varphi \notin E(U)\})$.

EXAMPLE 6.1
The class of **N**-expansions for a theory I is, in general, a proper subclass of the class of expansions. Consider a theory $I = \{Lp \rightarrow p\}$. It easily follows from Theorem 6.1 that I has two expansions $E(\varnothing)$ and $E(\{p\})$. On the other hand, it follows from Theorem 6.2 that only one of them, namely $E(\varnothing)$, is an **N**-expansion for I.

Thus, to compute all consistent **N**-expansions of I we need to consider all consistent sets $U \subseteq \{\omega_i : i \in S\}$ and for each such set we need to check whether $I \subseteq E(U)$ and $U \subseteq Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : \varphi \notin E(U)\})$. Algorithms to accomplish the first of these two tasks are given in [10]. Below we will describe how to check whether $U \subseteq Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : \varphi \notin E(U)\})$. Since $I \cup \{\neg L\varphi : \varphi \notin E(U)\}$ is infinite, algorithms developed in Sections 4 and 5 cannot be used directly. To overcome this difficulty, we prove the following general result.

THEOREM 6.3
Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be subsets of $\mathscr{L}_L$ such that if $\varphi \in \mathscr{L}_i$, $i = 1, 2$, then $\psi \in \mathscr{L}_i$ for each subformula ψ of φ. Let $A \subseteq \mathscr{L}_1$ and $B \subseteq \mathscr{L}_2$ meet the following conditions:

(1) B is **N**-consistent;
(2) $A \subseteq B$;
(3) if $\alpha \in B \setminus A$ then α is of the form $\neg L\beta$, where $L\beta \in \mathscr{L}_2 \setminus \mathscr{L}_1$.

Then, for $\alpha \in \mathcal{L}_1$, $A \vdash_{\mathbf{N}} \alpha$ if and only if $B \vdash_{\mathbf{N}} \alpha$.

PROOF. The implication from left to right is immediate from 2. Now, suppose $A \nvdash_{\mathbf{N}} \alpha$. Then there is an **N**-model $\mathcal{M}_1 = \langle M_1, \{R^1_\varphi\}_{\varphi \in \mathcal{L}_L}, V_1 \rangle$ such that $\mathcal{M}_1 \models A$ but $\mathcal{M}_1 \not\models \alpha$. Since B is **N**-consistent, there is an **N**-structure $\mathcal{M}_2 = \langle M_2, \{R^2_\varphi\}_{\varphi \in \mathcal{L}_L}, V_2 \rangle$ such that $\mathcal{M}_2 \models B$. Without loss of generality we can assume that $M_1 \cap M_2 = \varnothing$.

Construct a new structure $\mathcal{M} = \langle M, \{R_\varphi\}_{\varphi \in \mathcal{L}_L}, V \rangle$ as follows. Put $M = M_1 \cup M_2$,

$$R_\varphi = \begin{cases} (M_1 \times M_2) \cup R^2_\varphi & \text{if } L\varphi \notin \mathcal{L}_1 \\ R^1_\varphi \cup R^2_\varphi & \text{otherwise.} \end{cases}$$

and let V be the smallest valuation containing both V_1 and V_2.

The following two claims can be proved by induction on the complexity of φ.

Claim 1. If $\varphi \in \mathcal{L}_1$, then for each $m \in M_1$, $\mathcal{M}_1, m \models \varphi$ if and only if $\mathcal{M}, m \models \varphi$.
Claim 2. If $\varphi \in \mathcal{L}_2$, then for each $m \in M_2$, $\mathcal{M}_2, m \models \varphi$ if and only if $\mathcal{M}, m \models \varphi$.

The first claim implies that $\mathcal{M} \not\models \alpha$. Consider now $\varphi \in B$. If $\varphi \in A$, then both claims together imply that $\mathcal{M} \models \varphi$. If $\varphi \notin A$, then $\varphi = \neg L\psi$, where $L\psi \in \mathcal{L}_2 \backslash \mathcal{L}_1$. By the second claim, $\mathcal{M}, m \models \neg L\psi$ for each $m \in M_2$. In particular, it follows from that $\mathcal{M}, m_0 \models \neg\psi$, for some $m_0 \in M_2$. By the definition of R_ψ, we obtain that $\mathcal{M}, m \models \neg L\psi$, for each $m \in M_1$. Thus, $\mathcal{M} \models \neg L\psi (= \varphi)$. Consequently, $\mathcal{M} \models B$ and $B \nvdash_{\mathbf{N}} \alpha$.

Let $I \subseteq \mathcal{L}_L$. Every formula $L\varphi$ that occurs in a formula of I is called a *modal atom* of I. The collection of modal atoms of I is denoted $ma(I)$. Let us denote by $\mathcal{L}_{ma(I)}$ the language generated by the atoms of $\mathcal{L}$ and all the atoms in $ma(I)$.

COROLLARY 6.4
Let $U \subseteq \mathcal{L}$ and $I \subseteq E(U)$. Then $Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : L\varphi \in ma(I) \backslash E(U)\}) \cap \mathcal{L}_{ma(I)} = Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : \varphi \notin E(U)\}) \cap \mathcal{L}_{ma(I)}$.

PROOF. Directly from Theorem 6.3 by applying it to $\mathcal{L}_1 = \mathcal{L}_{ma(I)}$, $\mathcal{L}_2 = \mathcal{L}_L$, $A = I \cup \{\neg L\varphi : L\varphi \in ma(I) \backslash E(U)\}$ and $B = I \cup \{\neg L\varphi : \varphi \notin E(U)\}$. □

Now, to check whether $U \subseteq Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : \varphi \notin E(U)\})$ we check whether $U \subseteq Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : L\varphi \in ma(I) \backslash E(U)\})$. The set $I \cup \{\neg L\varphi : L\varphi \in ma(I) \backslash \{L\psi : \psi \in E(U)\}\}$ is finite. Thus, methods developed in Sections 4 and 5 can be used.

EXAMPLE 6.2
We will use methods of Sections 5 and 6 to compute **N**-expansions of $I = \{L\neg Lq \rightarrow p, L\neg Lp \rightarrow q\}$. There are four candidate theories for an **N**-expansion since there are four subsets of the set $\{p, q\}$. Consider one of these candidates: $E(\{p\})$. Clearly, $I \subseteq E(\{p\})$ (general algorithms to verify

membership in a stable set are described in [10]). According to Theorem 6.2(2), to determine whether $E(\{p\})$ is an **N**-expansion of I we have to determine whether

$$p \in Cn_{\mathbf{N}}(I \cup \{\neg L\varphi : \varphi \notin E(\{p\})\}).$$

From Corollary 6.4, it follows that it is enough to check whether

$$p \in Cn_{\mathbf{N}}(I \cup \{\neg Lq, \neg L\neg Lp\}).$$

To resolve this, we could use either the method of tableaux or the method described in Section 5. We take the latter approach here. First, we eliminate formulae of depth greater than 1 by introducing new atoms: a for Lq and b for Lp. Theory $I \cup \{\neg Lq, \neg L\neg Lp\}$ can be now replaced by $I' = \{Lq \rightarrow a, \neg Lq \rightarrow \neg a, Lp \rightarrow b, \neg Lp \rightarrow \neg b, L\neg a \rightarrow p, L\neg b \rightarrow q, \neg a, \neg L\neg b\}$ which has the same modal-free consequences in the language generated by p and q as $I \cup \{\neg Lq, \neg L\neg Lp\}$. Finally, using Proposition 5.3 and the definition of the operator B we establish that $p, \neg a, b$ are in $Cn_{\mathbf{N}}(I')$. Thus

$$p \in Cn_{\mathbf{N}}(I \cup \{\neg Lq, \neg L\neg Lp\})$$

and $E(\{p\})$ is an **N**-expansion of I.

7. Conclusions

In this paper we investigated both proof theory and semantics for the pure logic of necessitation **N**. The logic **N** is naturally related to various topics of current investigations in knowledge representations, in particular, default logic. Our results provide new methods for computing default extensions and a tool for studying possible entailment relations in default logic.

We believe that the logic **N** deserves further investigations. The natural rigour of provability in the logic **N** makes it suitable for formalization of various processes of computation. In particular, the logic **N** underlies all modal logics admitting the necessitation rule, for example, dynamic logic.

One subject not discussed in this paper is the complexity of membership for the consequence operator in the logic **N**. Proposition 2.2 and Theorem 5.2 can be used to obtain some estimates.

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