Realization Using the Model Existence Theorem*†

Melvin Fitting
e-mail: melvin.fitting@lehman.cuny.edu
web page: comet.lehman.cuny.edu/fitting

May 15, 2013

Abstract

Justification logics refine modal logics by replacing the usual necessity operator with a family of justification terms that embody reasons for the necessity of a formula, rather than simply recording the fact of necessity. Many common modal logics have justification counterparts. The connection between a modal logic and its justification counterpart is through a Realization Theorem, which says that modal operators can be replaced in a precise way with justification terms so that modal theorems turn into justification logic theorems. In this paper we present a new proof of Realization. We use the familiar machinery of consistency properties to prove a weak version, we call it Quasi-Realization. Then we show how to convert Quasi-Realizations into Realizations proper. Unlike most other treatments in the literature, the work here is not propositional, but first-order. Only one modal/justification logic is discussed, but the methods easily extend to other standard systems.

1 Introduction

One can think of the modal operator of epistemic logics as representing *implicit* knowledge. $\Box A$ asserts that A is known/knowable, but no reason is given for this knowledge. *Justification* logics are modal-like logics of *explicit* knowledge. In place of $\Box$ is an infinite family of *justification terms*, and $\Box A$ is replaced with $t{:}A$, asserting that A is known for reason t, a justification term. The family of justification terms has operators defined on it. Choice and behavior of operators determines the particular justification logic. This paper does not start at the beginning of the subject. For appropriate background we recommend [3, 1, 2].

*I want to thank an anonymous referee for remarkably detailed and appropriate criticism of earlier drafts, improving the presentation and saving the author from embarrassment. In particular the formulation of, and the need for, Lemma 8.2 are due to this referee.

†This paper is dedicated to Arnon Avron, a friend of many years, and a researcher whose works I deeply admire.

The first propositional justification logic was LP, the Logic of Proofs, corresponding to the familiar logic S4. It was introduced by Artemov as part of a project to provide an arithmetic semantics for propositional intuitionistic logic, [1]. Propositional intuitionistic logic embeds into propositional S4 via the well-known Gödel translation. Propositional S4 in turn embeds into LP via a Realization Theorem. Propositional LP embeds into arithmetic, Artemov's *Arithmetical Completeness Theorem.*

What does it mean to say that S4 embeds into LP? The justification logic LP is most commonly characterized axiomatically (see Section 6 for a first-order version of this). A quick glance at the standard axiomatization reveals that, if every justification term is replaced with $\Box$, LP axioms become S4 validities and LP inference rules become S4 inference rules. So this replacement, called the *forgetful functor*, turns LP proofs into S4 proofs and hence turns LP theorems into S4 theorems. The Realization Theorem tells us the forgetful functor is onto. More precisely, for every S4 theorem there is some way of replacing $\Box$ occurrences by justification terms to produce a theorem of LP. Thus every S4 theorem, involving implicit knowledge, has an explicit counterpart. Indeed, one can always manage so that negative $\Box$ occurrences are replaced with distinct justification variables—in a real sense, these are inputs in a kind of explicit information flow. Generally speaking, every justification logic corresponds to some modal logic in this way, via a Realization Theorem. In this paper we concentrate on LP and S4, but our approach easily carries over to a much more general setting.

Recently, in [4], propositional LP was extended to a quantified version, FOLP. The motivation was to provide an arithmetic semantics for first-order intuitionistic logic, something that was successfully accomplished. A first-order Realization Theorem was proved, as part of this project, with a proof along the lines of the original one for LP. Generally, but not always, Realization Theorems are given constructive proofs, making use of cut-free sequent derivations. By now there are several algorithms known. It is our intention here to give a rather different proof of realization for first-order LP, a non-constructive version. In a subsequent paper we will convert it into a constructive argument, but the present approach is one of the simplest currently available, even when restricted to the propositional case. The argument proceeds in two stages. First a *quasi-realization* theorem is proved (Section 9). Our work is somewhat related to a result first established in [5], though the present proof is different and includes the first-order case. Then quasi-realizations are turned into proper realizations (Section 12). It is an interesting point that the first part, introducing quasi-realizations, makes use of S4 proofs while the second part, converting quasi-realizations away, is by induction on formula complexity. This division will be further explored in a later paper. The two sections just mentioned are the heart of the paper. The rest presents the background that is needed. It is hoped that this background material does not obscure the essence of the argument.

As noted, we begin with a proof of quasi-realization that is non-constructive. This introduces a familiar tool into the array of machinery available for justification logics, the *Model Existence Theorem.* The Model Existence Theorem for classical first-order logic was introduced by Smullyan in [13], and again in

[14]. It elegantly isolates those properties of consistency needed to carry out a Henkin construction, thus ensuring the existence of a classical model once it is known that an abstract set of syntactic criteria obtains. Rather remarkably then, completeness for axiom systems, for Gentzen systems, for tableaus, for natural deduction systems, as well as interpolation theorems, all reduce to mechanical exercises in the verification of the abstract criteria. Keisler extended the Model Existence Theorem to $L_{\omega_1,\omega}$ in [12], where it served as a fundamental tool. I extended it to various modal logics in [8], and again in [9], and it served the same broad purposes that it did classically. In this paper quasi-realization becomes yet one more application.

The only justification logic examined here is the first-order logic of proofs, FOLP, from [4]. This uses the Model Existence Theorem for first-order S4, but Model Existence Theorems exist for several first-order modal logics, and the proof given here extends to those logics with little difficulty. We leave aside the question of quantified S5, and other modal logics for which the Barcan Formula is valid. Quantified modal logics whose semantics is constant domain do not have consistency properties in the usual sense. This is something that merits further exploration, but is beyond the scope of the present work.

Finally, we reiterate our opening disclaimer: *this paper is not self-contained.* For background and motivation on justification logics generaly, see [3]. For other proofs of the Realization Theorem, see [1, 5, 6, 11]. For first-order justification logic, specifically FOLP, see [4, 10].

2 Languages Used

There are several different first-order languages used in this paper. We collect together information about them in this section.

L_M is a standard first-order modal language. Atomic formulas are composed from relation symbols and individual variables $x_1, x_2, \ldots$, but there are no constant symbols or function symbols. (This restriction could be relaxed, but we want to keep things simple.) Formulas are built up from atomic formulas using $\bot$, $\supset$, $\Box$ and $\forall$. It is easy to introduce other connectives, modal operators, and quantifiers, and we will do so as needed.

We will be mapping modal formulas to formulas of justification logic, and to do this we need to keep track of the various *occurrences* of $\Box$. In [6] we introduced *annotated formulas* to address this issue; for present purposes a simpler version than the one of that paper will do. The *annotated modal language*, L_{AM} is like the standard first-order modal language L_M except for the following.

1. Instead of a single modal operator $\Box$ there is an infinite family, $\Box_1$, $\Box_2$, $\ldots$, called *indexed* modal operators. Formulas are built up as usual, but using indexed modal operators instead of $\Box$. The resulting formulas will be referred to as *annotated formulas*. We assume that in an annotated formula, *no index occurs twice.*

2. If A is an annotated formula, and A' is the result of replacing all indexed modal operators, $\Box_n$, with $\Box$, regardless of index, then A' is a conventional modal formula. We say A is an *annotated version* of A', and A' is an *unannotated version* of A.

3. Annotations are purely for bookkeeping purposes, keeping track of $\Box$ occurrences. Semantically annotations are ignored.

We begin the discussion of a justification logic language with a propositional version, suitable for LP. *Justification terms* are built up from justification constants and justification variables. The constants serve to justify "accepted" facts—typically axioms. One can think of variables as place-holders for justifications that could be given as inputs. Then there are build-up rules. If t and u are justification terms, so are $t \cdot u$, $t + u$, and $!t$. The first, $t \cdot u$ is supposed to justify those formulas B such that there is a formula A justified by u with $A \supset B$ justified by t. The second, $t + u$ justifies anything that is justified by either t or by u. As to the third, if t justifies A then $!t$ is supposed to justify that fact—it is sometimes called *proof checker*. There is a rule of formation: if A is a formula and t is a justification term, then $t{:}A$ is a formula, read "t justifies A."

Next we move to the first-order version, FOLP. A detailed definition can be found in [4] or [10]—here we present a basic sketch. Atomic formulas now are relation symbols applied to individual variables, in the usual way. (Again, we do not have constant or function symbols.) Quantifiers are introduced, also as usual. One more justification term formation rule is added. If x is an individual variable and t is a justification term, then $\mathsf{gen}_x(t)$ is a justification term. The intention is that if t justifies A then $\mathsf{gen}_x(t)$ justifies $(\forall x)A$. It should be noted that in $\mathsf{gen}_x(t)$ the operation symbol is gen_x; we are assuming an infinite family of such operation symbols, one for each x. The individual variable x does not have a free occurrence in $\mathsf{gen}_x(t)$—more generally, justification terms do not contain occurrences of individual variables.

The propositional notation $t{:}A$ is replaced with something more elaborate. If t is a justification term, X is a finite set of individual variables, and A is a formula, then $t{:}_X A$ is a formula. The members of X specify the free individual variables in this formula, while free variables of A that do not occur in X are understood as bound. Thus in $t{:}_{\{x,y,u\}}A(x, y, z, w)$, the occurrences of z and w are bound while those of x, y, and u are free. For formal details see [4, 10].

In an axiomatic proof free individual variables play two different roles. One is simply that of a formal symbol. The Universal Generalization rule says we have a proof of $(\forall x)A(x)$ provided we have a proof of $A(x)$. Here x is just a syntactic object. But in addition, an individual variable can serve as a place-holder that can be substituted for. Suppose we have an axiomatic proof of $A(x)$, and 3 is a constant symbol of our language. We can produce a proof of $A(3)$ by going through the proof of $A(x)$ and replacing all free occurrences of x with occurrences of 3 *provided universal generalization was never used with* x. Similarly for $A(4)$ and so on. Then a proof is something like a proof template — we can stamp out many concrete proofs from it. But we had to put in a *caveat* about non-use of universal generalization—the two roles of variables are not compatible. The notation $t{:}_X A$ is intended to separate the two roles individual variables can play. In $t{:}_{\{x,y,z\}}A(x, y, u, v)$ the individual variables x and y are free, and can freely be substituted for. (Similarly for z, but since it does not occur in A doing so would not be particularly useful.) When the axiom system for FOLP is given in Section 6 there will be a restriction on quantification that involves them, in axiom **B5**. The individual variables u and v, while free in A, are bound in the overall formula and are not subject to substitution.

The formal language of first-order LP will be referred to as L_{FOLP}.

The construction behind the proof of the Model Existence Theorem is the familiar one of Henkin. Such a construction requires enlarging the set of individual variables, to provide witnesses for existential statements. For this purpose it simplifies things to introduce what are often called *parameters*. These are new individual variables that follow all the rules appropriate to individual variables, except that we will never consider formulas in which they are bound. Then any substitution introducing a parameter is automatically a free substitution. Calling a variable a parameter is not a fundamental change—it is simply a restriction on what we will permit ourselves to do with the variable. Any formula with parameters is still a formula; any proof with parameters involved is still a proof.

Introducing parameters into L_M or L_{AM} is unproblematic—they are additional individual variables, but we do not quantify them. But introducing parameters into L_{FOLP} requires some comment. Suppose $t{:}_X A$ contains parameters—say v is a parameter that occurs in A. Since parameters cannot occur bound, then v must also appear in X. (Of course a parameter may occur in X without being in A.) We also assume there are no terms of the form $\mathsf{gen}_v(t)$ where v is a parameter.

Definition 2.1 *If L is any of the first-order languages discussed above, L_M, L_{AM}, or L_{FOLP}, then L^* is the language that is like L but with the addition of a countable set of parameters, new individual variables that are never bound. We say a formula of L^* is* p-closed *if the only free variables in it are parameters. (Note that a closed formula of L is automatically a p-closed formula of L^*.)*

Definition 2.2 (Substitution Notation) *When necessary, we use (y/v) to represent the substitution that replaces free occurrences of individual variable y with individual variable v. Application of (y/v) to formula φ is written $\varphi(y/v)$. In fact we will mostly be interested in the case where v is a parameter, which guarantees the substitution is free for y. Whenever possible we adopt a common and convenient simplification. We may write a quantified formula, such as $(\forall y)A(y)$, and later write $A(v)$, understanding it as short for $A(y/v)$. The notation is directional; $A(v)$ has no meaning unless $(\forall y)A(y)$ has appeared first, to specify the variable being substituted for.*

Finally we make use of *signed formulas*. This simplifies things, though it is possible to avoid their use if desired. We use T and F as signs (intuitively representing *true* and *false*). If A is a formula of language L then $T\,A$ and $F\,A$ are signed formulas of L. Note that signs do not iterate; $T\,F\,A$ is not a legal expression, for instance. We extend signing to sets in the obvious way. If S is a set of formulas, $T\,S = \{T\,A \mid A \in S\}$, and similarly for $F\,S$.

Our use of signed formulas comes from tableaus, but tableaus themselves play no role in this paper. Primarily, signs provide a mechanical device for tracking positive and negative subformulas. Even though tableaus do not come up here, a few words about them may make what is happening a bit clearer. Tableaus and sequent calculi are inter-translatable. Corresponding to a sequent $A_1, \ldots, A_n \rightarrow B_1, \ldots, B_k$ is the set $\{T\,A_1, \ldots, T\,A_n, F\,B_1, \ldots, F\,B_k\}$, which one can think of as the members of a tableau branch. The point is that the left side of a sequent arrow corresponds to the sign T, while the right side corresponds to the sign F. Tableaus are refutation systems, while sequent systems are forward reasoning. Then $A_1, \ldots, A_n \rightarrow B_1, \ldots, B_k$ should be so (provable, valid) just in case $\{T\,A_1, \ldots, T\,A_n, F\,B_1, \ldots, F\,B_k\}$ is not so (leads to closure, is not satisfiable). This further suggests a connection between the formula $(A_1 \wedge \ldots \wedge A_n) \supset (B_1 \vee \ldots \vee B_k)$ and the set $\{T\,A_1, \ldots, T\,A_n, F\,B_1, \ldots, F\,B_k\}$, and as a special case, between B_1 and $\{F\,B_1\}$, and indeed there is one. These comments may make following some of the subsequent formal material a bit easier.

3 First-Order Modal Semantics

The Model Existence Theorem is about modal satisfiability and so we provide a definition of satisfiability for formulas of L_M and L_M^*, for reference purposes. There are several versions of quantified modal logic around—we will be using a semantics that assumes *monotonicity.*

Definition 3.1 *A* monotonic S4 model *is a structure* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ *meeting the following conditions.*

1. $\mathcal{G}$ *is a non-empty set (of possible worlds).*
2. $\mathcal{R}$ *is a binary* accessibility relation *on* $\mathcal{G}$*, reflexive and transitive.*
3. $\mathcal{D}$ *is a* domain function*, mapping members of* $\mathcal{G}$ *to non-empty sets, and meeting the* monotonicity *condition: if* $\Gamma, \Delta \in \mathcal{G}$ *and* $\Gamma \mathcal{R} \Delta$*, then* $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$*. We call* $\cup\{\mathcal{D}(\Gamma) \mid \Gamma \in \mathcal{G}\}$ *the* domain of the model.
4. $\mathcal{I}$ *is an interpretation function, mapping each* n*-place relation symbol and each member of* $\mathcal{G}$ *to an* n*-place relation on the domain of the model.*

A valuation *in the model is a mapping from variables of* L_M *(or of* L_M^**) to members of the domain of the model.*

Truth of formula φ*, in model* $\mathcal{M}$*, at possible world* Γ*, with respect to valuation* v*, is symbolized by* $\mathcal{M}, \Gamma \Vdash_v \varphi$*. It meets the following conditions.*

1. *Atomic case,* $\mathcal{M}, \Gamma \Vdash_v R(x_1, \ldots, x_n)$ *provided* $\langle v(x_1), \ldots, v(x_n) \rangle \in \mathcal{I}(R, \Gamma)$.
2. $\mathcal{M}, \Gamma \nVdash_v \bot$.
3. $\mathcal{M}, \Gamma \Vdash_v A \supset B$ *iff* $\mathcal{M}, \Gamma \nVdash_v A$ *or* $\mathcal{M}, \Gamma \Vdash_v B$.

4. *$\mathcal{M}, \Gamma \Vdash_v \Box A$ iff $\mathcal{M}, \Delta \Vdash_v A$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$.*

5. *$\mathcal{M}, \Gamma \Vdash_v (\forall x)\varphi$ iff $\mathcal{M}, \Gamma \Vdash_w \varphi$ for every valuation w that agrees with v on all variables except possibly x, and $w(x) \in \mathcal{D}(\Gamma)$.*

Notice that the atomic case allows truth of $R(x_1, \ldots, x_n)$ at a possible world under a valuation that assigns some variables values not in the domain of that world. This is a minor point however, because of the way we restrict things in our definition of validity. (There are several essentially equivalent ways this issue could have been dealt with.)

Definition 3.2 *A valuation v* lives in possible world Γ on formula A *provided, for each individual variable x that occurs free in A, $v(x) \in \mathcal{D}(\Gamma)$. We say formula A is* valid *in the model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ if, for every $\Gamma \in \mathcal{G}$, and for every valuation v that lives in Γ on A, $\mathcal{M}, \Gamma \Vdash_v A$.*

Note that, by monotonicity, if a valuation lives in a possible world on A it also does so in any world accessible from that one.

Truth of a formula φ, at a possible world Γ, under a valuation v, only depends on the behavior of v on free variables of φ. More precisely, if v and w agree on the free variables of φ then $\mathcal{M}, \Gamma \Vdash_v \varphi$ iff $\mathcal{M}, \Gamma \Vdash_w \varphi$.

4 Consistency Properties and the Model Existence Theorem

An S4 consistency property is a collection of sets of signed modal formulas involving parameters, meeting certain conditions, [8, 9]. Notation: if S is a set of signed formulas, by $S^\sharp$ is meant $\{T\,\Box A \mid T\,\Box A \in S\}$.

Definition 4.1 *Let $\mathcal{C}$ be a collection of non-empty sets of signed p-closed formulas in the language L_M^*. $\mathcal{C}$ is an* S4 consistency property *provided, for each $S \in \mathcal{C}$:*

0. *not both $T\,A \in S$ and $F\,A \in S$ for any atomic formula A; also $T\,\bot \notin S$;*
1. *if $T\,A \supset B \in S$ then at least one of $S \cup \{F\,A\} \in \mathcal{C}$ or $S \cup \{T\,B\} \in \mathcal{C}$;*
2. *if $F\,A \supset B \in S$ then $S \cup \{T\,A, F\,B\} \in \mathcal{C}$;*
3. *if $T\,(\forall x)\varphi(x) \in S$ then $S \cup \{T\,\varphi(v)\} \in \mathcal{C}$ for every parameter v;*
4. *if $F\,(\forall x)\varphi(x) \in S$ then $S \cup \{F\,\varphi(v)\} \in \mathcal{C}$ for some parameter v;*
5. *if $T\,\Box A \in S$ then $S \cup \{T\,A\} \in \mathcal{C}$;*
6. *if $F\,\Box A \in S$ then $S^\sharp \cup \{F\,A\} \in \mathcal{C}$.*

Definition 4.2 *A set S of signed formulas (with or without parameters) is* S4 satisfiable *if there is some model $\mathcal{M}$, some possible world Γ in the model, and some valuation v that lives in Γ on each formula in S and:*

1. *if* $T\,A \in S$ *then* $\mathcal{M}, \Gamma \Vdash_v A$*;*
2. *if* $F\,A \in S$ *then* $\mathcal{M}, \Gamma \not\Vdash_v A$*.*

And now we have everything we need to state the Model Existence Theorem.

Theorem 4.3 (Model Existence Theorem for First-Order S4) *If* $\mathcal{C}$ *is an S4 consistency property and* $S \in \mathcal{C}$ *then* S *is S4 satisfiable.*

We do not prove this here. Full proofs can be found in [8, 9]. Formulas appearing in consistency properties are in the language L_M^*. However, closed formulas of L_M are p-closed formulas of L_M^*, so we immediately have the following.

Corollary 4.4 *Suppose* $\mathcal{C}$ *is an S4 consistency property,* S *is a set of signed closed formulas of* L_M*, and* $S \in \mathcal{C}$*. Then* S *is S4 satisfiable.*

5 An Example—Axiomatic Completeness

Consistency properties can be used to prove completeness for axiomatic formulations, sequent calculus formulations, and tableau formulations. In addition they can be used to provide proofs of compactness, Löwenheim-Skolem, and interpolation theorems. In this section we briefly sketch a proof of completeness for an axiomatic formulation of first-order S4, since it is a natural lead-in to the proof of quasi-realization that follows later.

The axiom system we use has been designed to make application of the Model Existence Theorem straightforward. Completeness of more conventional axiom systems, not involving parameters, then follows using a translation process. Since this is not the subject of the present paper, we omit these details.

The language involved is L_M^*. Propositional axioms are all *p-closed* tautologies (or enough of them to derive the rest). Modal axioms are all *p-closed* formulas of the forms: $\Box(A \supset B) \supset (\Box A \supset \Box B)$, $\Box A \supset A$, and $\Box A \supset \Box\Box A$. Quantifier axioms are all *p-closed* formulas of the form: $(\forall x)A(x) \supset A(v)$, where v is a parameter but x is not. Rules of inference are: *modus ponens*, $A, A \supset B \Rightarrow B$; *necessitation*, $A \Rightarrow \Box A$; and *universal generalization* in the following form, $A \supset B(v) \Rightarrow A \supset (\forall x)B(x)$, where v is a parameter that does not occur in $A \supset (\forall x)B(x)$.

All axioms are p-closed, and the rules of inference preserve this. Consequently all theorems, indeed, all formulas occurring in proofs, are p-closed.

Call a set S of p-closed signed formulas of L_M^* inconsistent if there are finite sets Γ and Δ such that $T\,\Gamma \cup F\,\Delta \subseteq S$ and $\bigwedge \Gamma \supset \bigvee \Delta$ is axiomatically provable. Call S consistent if it is not inconsistent. Let $\mathcal{C}_{ax}$ be the collection of all sets that: 1) are consistent in this axiomatic sense and 2) omit infinitely many parameters. It is an exercise to check that $\mathcal{C}_{ax}$ is an S4 consistency property. Once this is done, completeness is an easy consequence. Suppose A is a closed formula of L_M that is not axiomatically provable. It follows that the set $\{F\,A\}$ is consistent, and hence in $\mathcal{C}_{ax}$. By Corollary 4.4 to the Model Existence Theorem this set is S4 satisfiable, and hence A is not S4 valid. This establishes completeness for closed formulas of L_M.

6 The Justification Logic FOLP

First-order justification logic, FOLP, has a standard axiomatic formulation. Here are the axioms and rules; we refer to [4] for a detailed discussion. We will need a few explicit consequences, and we present them in this section.

In the following X denotes a finite set of individual variables. If y is an individual variable, then Xy is short for $X \cup \{y\}$, and it is assumed that $y \notin X$. Axioms are instances of axiom schemes, listed below. A and B are formulas, s and t are justification terms, and y is an individual variable.

A1 Classical axioms of first order logic. We assume these consist of all tautologies, and instances of the two axiom schemes, $(\forall x)A(x) \supset A(y)$ (where y is free for x in A), and $(\forall x)(A \supset B) \supset (A \supset (\forall x)B)$ (where x is not free in A).

A2 $t{:}_{Xy}A \supset t{:}_X A$, provided y does not occur free in A

A3 $t{:}_X A \supset t{:}_{Xy}A$

B1 $t{:}_X A \supset A$

B2 $s{:}_X(A \supset B) \supset (t{:}_X A \supset (s \cdot t){:}_X B)$

B3 $t{:}_X A \supset (t+s){:}_X A, \quad s{:}_X A \supset (t+s){:}_X A$

B4 $t{:}_X A \supset !t{:}_X t{:}_X A$

B5 $t{:}_X A \supset \mathsf{gen}_x(t){:}_X \forall x A$, provided $x \notin X$

R1 $\vdash A, A \supset B \;\Rightarrow\; \vdash B$

R2 $\vdash A \;\Rightarrow\; \vdash \forall x A$

R3 $\vdash c{:}_\emptyset A$, where A is an axiom and c is a proof constant

The final rule, **R3**, is called *axiom necessitation*. It is often restricted through the use of *constant specifications*, but we do not discuss this machinery here. It can be introduced into our constructions, but it adds complexity without changing anything essential. We do, however, make a few additional comments concerning constant specifications in Section 11.

We will be interested in FOLP with parameters—recall the languages L_{FOLP} and L^*_{FOLP} that were defined in Section 2. Formally, let FOLP^* be the logic characterized using the axiom schemes and rules above, *but in which each line of a proof must be a formula of* L^*_{FOLP}, that is, must be a formula possibly containing parameters, but in which parameters cannot occur bound. Since every L_{FOLP} formula is also an L^*_{FOLP} formula, the logic FOLP is a sublogic of FOLP^*. But in a certain sense the converse is also true, since L^*_{FOLP} formulas are just L_{FOLP} formulas with additional free individual variables. Nonetheless, the use of parameters is restricted, and we need to be careful with them. Here are a few items that will be used in what follows, and which address the issue of how parameters behave.

Theorem 6.1 *Suppose $\mathcal{P}$ is an axiomatic proof in FOLP^* and v is a parameter that occurs in the proof. Let x be an individual variable, not a parameter, that does not occur in the proof. Replace all occurrences of v in $\mathcal{P}$ with occurrences of x—call the result $\mathcal{Q}$. Then $\mathcal{Q}$ is also a proof in FOLP^*.*

Proof Straightforward induction on proof length. ■

The following shows that FOLP^* is a conservative extension of FOLP.

Corollary 6.2 *Suppose A is a formula of L_{FOLP}, but A has a proof in FOLP^*. Then A has a proof in FOLP itself.*

Proof Begin with an axiomatic proof of A in FOLP^* and, by successive application of Theorem 6.1, eliminate all occurrences of parameters. The result will be a proof in FOLP. Since A contained no parameters the process does not change it, so the result is a proof of A in FOLP. ■

Universal generalization requires some care when parameters are involved, since parameters cannot be quantified. Fortunately we have the following workaround.

Corollary 6.3 *Suppose $(\forall x)A(x)$ is a formula of L^*_{FOLP}, v is a parameter that does not occur in it, and $A(v)$ is the result of substituting v for all free occurrences of x in $A(x)$. If $A(v)$ is provable in FOLP^* then so is $(\forall x)A(x)$.*

Proof Begin with a proof of $A(v)$ in FOLP^*. Using Theorem 6.1 replace v throughout the proof with z, an individual variable that is not a parameter and that does not occur in the proof. The result is a proof of $A(z)$. Then by Rule **R2**, $(\forall z)A(z)$ is provable. But $(\forall z)A(z) \supset A(x)$ is an instance of universal instantiation, and we conclude $A(x)$ by **R1**, and then $(\forall x)A(x)$ using **R2**. (The detour through variable z was necessary because x might have occurred somewhere in the proof of $A(v)$. ■

The Internalization Theorem (and its closely related Lifting Lemma) plays a fundamental role in justification logics. A standard treatment for FOLP can be found in [4]. We need variants that pay attention to the role of parameters. Only one step in what follows is different from the usual arguments but since the results play a critical role, it is best to display the details. Recall that, since parameters cannot be bound, if any parameters appear in A, and $t{:}_X A$ is an FOLP^* formula, those parameters must also appear in X.

It is common to state versions of the Internalization Theorem using a notion of deduction from premises, rather than just that of proof. A deduction in axiomatic FOLP or FOLP^* is like a proof, but allowing premises, and restricting universal generalization (**R2**) so that it cannot involve an individual variable that is free in any premise. As usual, we write $A_1, \ldots, A_n \vdash B$ to mean that B has a derivation from $A_1, \ldots, A_n$.

Theorem 6.4 *Let $p_1, \ldots, p_n$ be justification variables, and suppose $p_1{:}_Z A_1, \ldots, p_n{:}_Z A_n \vdash B$ in FOLP^*, where Z is the set of all parameters that occur in $A_1, \ldots, A_n$, or B. Then there is a justification term t such that $p_1{:}_Z A_1, \ldots, p_n{:}_Z A_n \vdash t{:}_Z B$ in FOLP^*.*

Proof The argument is by induction on the derivation of B from the premises $p_1{:}_Z A_1, \ldots, p_n{:}_Z A_n$.

1. B is an axiom of FOLP^*. This case requires some adjustment from the argument in [4]. That argument used an application of Rule **R3** directly but now, if B has parameters $c{:}_\emptyset B$ isn't an L^*_{FOLP} formula. Instead we take a small detour.

   Say the parameters that occur in B are $u_1, \ldots, u_k$ (there may be non-parameter individual variables as well, but they don't concern us now). Let us write B as $B(u_1, \ldots, u_k)$. Let $x_1, \ldots, x_k$ be distinct individual variables that are not parameters and that don't occur in B, and substitute them for the parameters, getting $B(x_1, \ldots, x_k)$. Since we have axiom *schemes*, this is also an axiom, and contains no parameters. Now using Rule **R3**, for some justification constant $c{:}_\emptyset B(x_1, \ldots, x_k)$ is provable. By repeated use of **A3**, $c{:}_{Z'} B(x_1, \ldots, x_k)$ is also provable, where Z' is like Z except that parameters $u_1, \ldots, u_k$ have been replaced with variables $x_1, \ldots, x_k$. Now using Rule **R2**, $(\forall x_1) \ldots (\forall x_k) c{:}_{Z'} B(x_1, \ldots, x_k)$ is provable. Then by repeated use of Universal Instantiation, $c{:}_Z B(u_1, \ldots, u_k)$ is provable, hence derivable from any set of premises. Let $t = c$.

2. B is one of the premises, say $p_i{:}_Z A_i$. Using **B4**, this implies $!p_i{:}_Z p_i{:}_Z A_i$. Let $t = !p_i$.

3. B follows by Rule **R1** from earlier steps in the derivation, say from A and $A \supset B$. By the induction hypothesis both $s_1{:}_Z(A \supset B)$ and $s_2{:}_Z A$ are derivable, for some A, s_1, and s_2. Using **B2**, $(s_1 \cdot s_2){:}_Z B$ is derivable. Let $t = s_1 \cdot s_2$.

4. B follows by Rule **R2**. This is similar to case 2, but uses **B5**. Note that any variable generalized on cannot be in Z since members of Z are parameters.

5. B follows by axiom necessitation. This is also similar to case 2, but uses **A3** and **B4**.

■

Corollary 6.5 (Internalization) *Let $p_1, \ldots, p_n$ be justification variables, and suppose that in FOLP^*, $p_1{:}_{X_1} A_1, \ldots, p_n{:}_{X_n} A_n \vdash B$, where X_i is exactly the set of parameters that occur in A_i, for each $i = 1, \ldots, n$. Then there is a justification term t such that $p_1{:}_{X_1} A_1, \ldots, p_n{:}_{X_n} A_n \vdash t{:}_X B$ in FOLP^*, where X is exactly the set of parameters that occur in B.*

Proof Suppose $p_1{:}_{X_1} A_1, \ldots, p_n{:}_{X_n} A_n \vdash B$ in FOLP^*, where X_i is exactly the set of parameters that occur in A_i. Let Z be the set of parameters occurring in any of $A_1, \ldots, A_n$ or B. Making use of both **A2** and **A3**, $\vdash p_i{:}_{X_i} A_i \equiv p_i{:}_Z A_i$, for $i = 1, \ldots, n$. Then $p_1{:}_Z A_1, \ldots, p_n{:}_Z A_n \vdash B$ in FOLP^*, so by Theorem 6.4, $p_1{:}_Z A_1, \ldots, p_n{:}_Z A_n \vdash t{:}_Z B$ in FOLP^*, for some t. Again using **A2** and **A3**, $\vdash t{:}_Z B \equiv t{:}_X B$, where X is exactly the set of parameters occurring in B. It follows that $p_1{:}_{X_1} A_1, \ldots, p_n{:}_{X_n} A_n \vdash t{:}_X B$ in FOLP^*. ■

We note that, in the usual way, the results above concerning deductions can be converted into results about provability. In particular, here is the version we will use later on. Suppose $(p_1{:}_{X_1}A_1 \wedge \ldots \wedge p_n{:}_{X_n}A_n) \supset B$ is provable in FOLP^*, where X_i is exactly the set of parameters that occur in A_i. Then there is some t so that $(p_1{:}_{X_1}A_1 \wedge \ldots \wedge p_n{:}_{X_n}A_n) \supset t{:}_X B$ is provable in FOLP^*, where X is the set of parameters that occur in B.

Corollary 6.6 (Lifting Lemma, parameter version) *Suppose B is provable in FOLP^*, and X is exactly the set of parameters that occur in B. Then there is a justification term t such that $t{:}_X B$ is provable in FOLP^*. We can take t to have no justification variables.*

Proof The first part of this is simply Corollary 6.5 with no premises involved. The second part follows from an examination of the proof of Theorem 6.4—the only step that introduces justification variables into t is step 2, which cannot arise now since there are no premises involved. ■

Corollary 6.7 (Lifting Lemma, no parameter version) *Suppose B is provable in FOLP, and X is exactly the set of free variables that occur in B. Then there is a justification term t such that $t{:}_X B$ is provable in FOLP. Again we can take t to have no justification variables.*

Proof This has an uncomplicated direct proof, but at this point we can derive it from work above, and it might be fun to do so. Let us say the free variables of B are $x_1, \ldots, x_n$, and write $B(x_1, \ldots, x_n)$. Since this is provable in FOLP, so is $(\forall x_1) \ldots (\forall x_n) B(x_1, \ldots, x_n)$. Trivially this is also provable in FOLP^*. Then using universal instantiation, so is $B(v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are distinct parameters. By Corollary 6.6, $t{:}_{\{v_1,\ldots,v_n\}} B(v_1, \ldots, v_n)$ is provable in FOLP^* for some justification term t without justification variables. Then by Corollary 6.3, $(\forall x_1) \ldots (\forall x_n) t{:}_{\{x_1,\ldots,x_n\}} B(x_1, \ldots, x_n)$ is provable in FOLP^*, and hence in FOLP by Corollary 6.2. Then by universal instantiation, $t{:}_{\{x_1,\ldots,x_n\}} B(x_1, \ldots, x_n)$ follows. ■

We informally say that t *internalizes the proof of* B when $t{:}_X B$ is provable as in the last two Corollaries above.

7 Annotated Consistency Properties

In Section 2 we defined a language, L_{AM}, that is like L_M except that modal operators are annotated. This is so that we can keep track of particular occurrences of $\Box$. We will be using signed annotated formulas throughout this section.

The notion of an $\mathsf{S4}$ consistency property was given in Definition 4.1. We now modify that to an *annotated* $\mathsf{S4}$ consistency property. Formulas now are in the annotated language L^*_{AM}. The sharp operation needs to be modified. If S is a set of signed *annotated* formulas, by $S^\sharp$ is meant $\{T\,\Box_n X \mid T\,\Box_n X \in S \text{ for some } n\}$. Conditions $0 - 4$ from Definition 4.1 formally read the same as before, except that formulas are annotated. Conditions 5 and 6 have the following obvious changes:

5. if $T\,\Box_n A \in S$ then $S \cup \{T\,A\} \in \mathcal{C}$;

6. if $F\,\Box_n A \in S$ then $S^\sharp \cup \{F\,A\} \in \mathcal{C}$.

Here are two examples. Suppose $\mathcal{C}$ is an annotated consistency property and $S \in \mathcal{C}$. Suppose $F\,\Box_2 P \supset \Box_1 Q \in S$. Then by condition 2, $S \cup \{T\,\Box_2 P, F\,\Box_1 Q\} \in \mathcal{C}$. Again, suppose $\mathcal{C}$ is an annotated consistency property, $S \in \mathcal{C}$, and $S = \{T\,\Box_3 A, F\,\Box_5 B, T\,\Box_2 C, F\,\Box_7 D\}$. By condition 6 above, both $\{T\,\Box_3 A, F\,B, T\,\Box_2 C\}$ and $\{T\,\Box_3 A, T\,\Box_2 C, F\,D\}$ must be in $\mathcal{C}$.

Being an annotated consistency property is a stronger requirement than just being a consistency property. If we happen to have an annotated consistency property, and we erase the indexes, we have a consistency property and hence members are satisfiable.

8 Quasi-Realizations

We now make use of FOLP^* to define an $\mathsf{S4}$ annotated consistency property. How we do this has a clear relationship with the axiomatic completeness argument sketched in Section 5. The result gives us the first part of a proof of realization for FOLP—we do not get a full realization theorem, but we get what we call a *quasi-realization* theorem. In Section 12 this will be built upon to produce a proper realization result.

We begin by defining a map from L^*_{AM}, signed annotated modal formulas, to sets of signed L^*_{FOLP} formulas. From now on we assume that p_1, p_2, … is an enumeration of all justification variables of FOLP with no variable repeated, fixed once and for all.

Definition 8.1 *The mapping $\langle\!\langle\cdot\rangle\!\rangle$ associates with each signed formula of L^*_{AM} a set of signed formulas in the language L^*_{FOLP}. It is defined recursively, as follows.*

1. *If A is atomic, $\langle\!\langle T\,A\rangle\!\rangle = \{T\,A\}$ and $\langle\!\langle F\,A\rangle\!\rangle = \{F\,A\}$.*

2. $\langle\!\langle T\,A \supset B\rangle\!\rangle = \{T\,U \supset V \mid F\,U \in \langle\!\langle F\,A\rangle\!\rangle, T\,V \in \langle\!\langle T\,B\rangle\!\rangle\}$
   $\langle\!\langle F\,A \supset B\rangle\!\rangle = \{F\,U \supset V \mid T\,U \in \langle\!\langle T\,A\rangle\!\rangle, F\,V \in \langle\!\langle F\,B\rangle\!\rangle\}$.

3. $\langle\!\langle T\,(\forall x)A\rangle\!\rangle = \{T\,(\forall x)U \mid T\,U \in \langle\!\langle T\,A\rangle\!\rangle\}$
   $\langle\!\langle F\,(\forall x)A\rangle\!\rangle = \{F\,(\forall x)(U_1 \vee \ldots \vee U_k) \mid F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,A\rangle\!\rangle\}$.

4. *In the following, X is the set of free variables of A (which includes parameters, if present).*
   $\langle\!\langle T\,\Box_n A\rangle\!\rangle = \{T\,p_n{:}_X U \mid T\,U \in \langle\!\langle T\,A\rangle\!\rangle\}$
   $\langle\!\langle F\,\Box_n A\rangle\!\rangle = \{F\,t{:}_X(U_1 \vee \ldots \vee U_k) \mid F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,A\rangle\!\rangle$ *and t is any justification term*$\}$.

5. *The mapping is extended to* sets *of signed annotated formulas by letting* $\langle\!\langle S\rangle\!\rangle = \cup\{\langle\!\langle Z\rangle\!\rangle \mid Z \in S\}$.

Comments This is too simple to be a formal theorem, but it plays a significant role in our work. It is an easy verification by induction on formula complexity that, for an annotated formula A in L^*_{AM}, the signed formula $T\,A$ and every member of $\langle\!\langle T\,A\rangle\!\rangle$ have the same free individual variables. In particular, they have the same parameters, and the same non-parameters. Similarly for $F\,A$.

As a special case, if A is p-closed so are all members of $\langle\!\langle T\,A\rangle\!\rangle$ and $\langle\!\langle F\,A\rangle\!\rangle$. Similarly if A is in the language L_{FOLP}, and so has no parameters, the members of $\langle\!\langle T\,A\rangle\!\rangle$ and $\langle\!\langle F\,A\rangle\!\rangle$ will also be in the language L_{FOLP}. For quasi-realization we will primarily be interested in the p-closed case, but it is still necessary to consider free variables generally, to cover things like step 3 in the example below. Also, for converting quasi-realizations to realizations, free variables that are not parameters are needed.

The following Lemma is needed for the proof of Theorem 9.1, but can be stated and proved at this point. Recall the substitution notation given in Definition 2.2.

Lemma 8.2 *Let φ be a formula of L^*_{AM}, and v be a parameter. If $T\,\psi \in \langle\!\langle T\,\varphi(y/v)\rangle\!\rangle$ then there exists a signed formula $T\,\psi' \in \langle\!\langle T\,\varphi\rangle\!\rangle$ such that $\psi = \psi'(y/v)$. Similarly for the F-signed case.*

Proof The proof is by induction on the complexity of φ.

1. φ is atomic. If $T\,\psi \in \langle\!\langle T\,\varphi(y/v)\rangle\!\rangle$ then $\psi = \varphi(y/v)$, so we can take $\psi' = \varphi$. Similarly for the F case.

2. $\varphi = A \supset B$, and the result is known for A and for B. This case is straightforward and the argument is omitted.

3. $\varphi = (\forall x)A$ and the result is known for A. We consider the T-signed case—the F-signed one is slightly more complicated and we leave it to the reader.

   If $x = y$, $\varphi(y/v) = [(\forall x)A](y/v) = (\forall x)A = \varphi$ so things are simple; we can take $\psi' = \psi$.

   Now suppose $x \neq y$, and $T\,\psi \in \langle\!\langle T\,\varphi(y/v)\rangle\!\rangle$. Then $T\,\psi \in \langle\!\langle T\,[(\forall x)A](y/v)\rangle\!\rangle = \langle\!\langle T\,(\forall x)[A(y/v)]\rangle\!\rangle$ so $\psi = (\forall x)B$ where $T\,B \in \langle\!\langle T\,A(y/v)\rangle\!\rangle$. By the induction hypothesis, there exists $T\,B' \in \langle\!\langle T\,A\rangle\!\rangle$ so that $B = B'(y/v)$. Now set $\psi' = (\forall x)B'$. Then $T\,\psi' = T\,(\forall x)B' \in \langle\!\langle T\,(\forall x)A\rangle\!\rangle = \langle\!\langle T\,\varphi\rangle\!\rangle$ and $\psi'(y/v) = [(\forall x)B'](y/v) = (\forall x)[B'(y/v)] = (\forall x)B = \psi$.

4. $\varphi = \Box_n A$, and the result is known for A. Again we consider the T-signed case and leave the F-signed one to the reader.

   If y is not free in A, $\varphi(y/v) = [\Box_n A](y/v) = \varphi$, so just take $\psi' = \psi$.

   Suppose y is free in A, and $T\,\psi \in \langle\!\langle T\,\varphi(y/v)\rangle\!\rangle$. Then $T\,\psi \in \langle\!\langle T\,[\Box_n A](y/v)\rangle\!\rangle$, so $\psi = p_n{:}_X B$ where $T\,B \in \langle\!\langle T\,A(y/v)\rangle\!\rangle$ and X is the set of free variables of $A(y/v)$. Note that X doesn't include y, but must include v since y is free in A. By the induction hypothesis there exists $T\,B' \in \langle\!\langle T\,A\rangle\!\rangle$ with $B = B'(y/v)$. Now let $\psi' = p_n{:}_{Xy}B'$. We have $T\,\psi' = T\,p_n{:}_{Xy}B' \in \langle\!\langle T\,\Box_n A\rangle\!\rangle$ (recall that y is free in A and hence also in B'). And also, $\psi'(y/v) = [p_n{:}_{Xy}B'](y/v) = p_n{:}_X[B'(y/v)] = p_n{:}_X B = \psi$.

∎

Example 8.3 *We use Definition 8.1 to compute* $\langle\!\langle F\,\Box_2\bot \supset (\forall x)\Box_1 A(x,v)\rangle\!\rangle$, *where* v *is a parameter and* $A(x,v)$ *is an atomic formula.*

1. $\langle\!\langle T\,\bot\rangle\!\rangle = \{T\,\bot\}$

2. $\langle\!\langle T\,\Box_2\bot\rangle\!\rangle = \{T\,p_2{:}_\emptyset\bot\}$

3. $\langle\!\langle F\,A(x,v)\rangle\!\rangle = \{F\,A(x,v)\}$

4. $\langle\!\langle F\,\Box_1 A(x,v)\rangle\!\rangle = \{F\,t_1{:}_{\{x,v\}}A(x,v), F\,t_2{:}_{\{x,v\}}A(x,v),\ldots\}$ *where* $t_1, t_2, \ldots$ *are all the justification terms.*

5. $\langle\!\langle F\,(\forall x)\Box_1 A(x,v)\rangle\!\rangle = \{F\,(\forall x)D_1, F\,(\forall x)D_2,\ldots\}$, *where* D_1, D_2, $\ldots$ *are all the disjunctions of formulas of the form* $t{:}_{\{x,v\}}A(x,v)$ *for any justification term* t.

6. $\langle\!\langle F\,\Box_2\bot \supset (\forall x)\Box_1 A(x,v)\rangle\!\rangle = \{F\,p_2{:}_\emptyset\bot \supset (\forall x)D_1, F\,p_2{:}_\emptyset\bot \supset (\forall x)D_2,\ldots\}$, *where* D_1, D_2, $\ldots$ *are as in 5*

Definition 8.4 *Let* A *be a formula in the annotated language* L^*_{AM}. *A* quasi-realization *of* A *is a formula* $U_1 \vee \ldots \vee U_n$ *in the language* L^*_{FOLP} *where* $F\,U_1,\ldots,F\,U_n \in \langle\!\langle F\,A\rangle\!\rangle$. *For a formula in the language* L^*_M, *without annotations, a quasi-realization for it is any quasi-realization for* A', *where* A' *is an annotated version of* A.

Example 8.5 *Continuing Example 8.3, If* s, t, *and* u *are justification terms, then*

$$[p_2{:}_\emptyset\bot \supset (\forall x)[s{:}_{\{x,v\}}A(x,v) \vee t{:}_{\{x,v\}}A(x,v)]] \vee [p_2{:}_\emptyset\bot \supset (\forall x)u{:}_{\{x,v\}}A(x,v)]$$

is one quasi-realization of $\Box_2\bot \supset (\forall x)\Box_1 A(x,v)$ *and hence also of* $\Box\bot \supset (\forall x)\Box A(x,v)$.

It will be shown in the next section that every *provable* closed formula of first-order S4 has a quasi-realization that is *provable* in FOLP.

9 A Quasi-Realization Theorem

In this section we define an annotated S4 consistency property; call it $\mathcal{C}_q$ (the subscript is to suggest quasi-realization). The definition is very similar to that for the consistency property $\mathcal{C}_{ax}$ used as an example in Section 5, except that there is a detour through FOLP*. Call a set S of signed p-closed formulas of L^*_{AM} inconsistent if there are finite sets Γ and Δ of L^*_{FOLP} formulas such that $\bigwedge\Gamma \supset \bigvee\Delta$ is provable in FOLP*, and $T\,\Gamma \cup F\,\Delta \subseteq \langle\!\langle S\rangle\!\rangle$; call S consistent if it is not inconsistent. Let $\mathcal{C}_q$ be the collection of sets that are: 1) consistent in this sense and 2) omit infinitely many paramters. Note that members of $\mathcal{C}_q$ are sets of *modal* formulas in L^*_{AM}, while provability questions are shifted to FOLP*.

Theorem 9.1 *$\mathcal{C}_q$ is an annotated S4 consistency property.*

Proof For the entire of what follows, assume S is a set of signed p-closed formulas of L^*_{AM} that omits infinitely many parameters. We show that each part of Definition 4.1 (as modified in Section 7) is true for S by showing that assuming the contrary of any of items 0–6 implies that $S \notin \mathcal{C}_q$.

0. Suppose A is atomic, and both $T\,A$ and $F\,A$ are in S. Let $\Gamma = \Delta = \{A\}$. Then $T\,\Gamma = \{T\,A\} = \langle\!\langle T\,A \rangle\!\rangle \subseteq \langle\!\langle S \rangle\!\rangle$ and $F\,\Delta = \{F\,A\} = \langle\!\langle F\,A \rangle\!\rangle \subseteq \langle\!\langle S \rangle\!\rangle$. Since $A \supset A$ is provable in FOLP^*, that is, $\bigwedge \Gamma \supset \bigvee \Delta$, this implies that S is not consistent, and so $S \notin \mathcal{C}_q$. For similar reasons, $T\,\bot \notin S$.

1. Suppose $T\,A \supset B \in S$ but also $S \cup \{F\,A\} \notin \mathcal{C}_q$ and $S \cup \{T\,B\} \notin \mathcal{C}_q$. Since S omits infinitely many parameters, so does $S \cup \{F\,A\}$, so since $S \cup \{F\,A\} \notin \mathcal{C}_q$, it must be inconsistent in the present sense. Then there are finite sets Γ_A and Δ_A of L^*_{FOLP} formulas with $T\,\Gamma_A \cup F\,\Delta_A \subseteq \langle\!\langle S \rangle\!\rangle$, and there are L^*_{FOLP} formulas $A_1, \ldots, A_m$ with $F\,\{A_1, \ldots, A_m\} \subseteq \langle\!\langle F\,A \rangle\!\rangle$ so that we have provability in FOLP^* of the following.

$$\bigwedge \Gamma_A \supset \left[\bigvee \Delta_A \vee \bigvee \{A_1, \ldots, A_m\}\right] \tag{1}$$

   Similarly since $S \cup \{T\,B\} \notin \mathcal{C}_q$ there are finite sets Γ_B and Δ_B with $T\,\Gamma_B \cup F\,\Delta_B \subseteq \langle\!\langle S \rangle\!\rangle$, and there are formulas $B_1, \ldots, B_n$ with $T\,\{B_1, \ldots, B_n\} \subseteq \langle\!\langle T\,B \rangle\!\rangle$ so that we have FOLP^* provability of the following.

$$\left[\bigwedge \Gamma_B \wedge \bigwedge \{B_1, \ldots, B_n\}\right] \supset \bigvee \Delta_B \tag{2}$$

   Let $\Gamma = \Gamma_A \cup \Gamma_B$ and $\Delta = \Delta_A \cup \Delta_B$. Then the provability in FOLP^* of (1) and (2) gives us provability in FOLP^* of the following.

$$\bigwedge \Gamma \supset \left[\bigvee \Delta \vee \bigvee \{A_1, \ldots, A_m\}\right] \tag{3}$$

$$\left[\bigwedge \Gamma \wedge \bigwedge \{B_1, \ldots, B_n\}\right] \supset \bigvee \Delta \tag{4}$$

   By classical logic, formulas (3) and (4) together imply the following, where i ranges over $1, 2, \ldots, m$ and j ranges over $1, 2, \ldots, n$.

$$\left[\bigwedge \Gamma \wedge \bigwedge_{i,j} (A_i \supset B_j)\right] \supset \bigvee \Delta \tag{5}$$

   Since each $T\,A_i \supset B_j$ is in $\langle\!\langle T\,A \supset B \rangle\!\rangle$ and hence in $\langle\!\langle S \rangle\!\rangle$, and $T\,\Gamma \cup F\,\Delta \subseteq \langle\!\langle S \rangle\!\rangle$, provability of (5) in FOLP^* gives us the inconsistency of S, and hence $S \notin \mathcal{C}_q$.

2. Suppose $F\,A \supset B \in S$ but $S \cup \{T\,A, F\,B\} \notin \mathcal{C}_q$. S omits infinitely many parameters, hence so does $S \cup \{T\,A, F\,B\}$. Then since $S \cup \{T\,A, F\,B\} \notin \mathcal{C}_q$ there must be finite sets Γ and Δ of L^*_{FOLP} formulas with $T\,\Gamma \cup F\,\Delta \subseteq$

$\langle\!\langle S \rangle\!\rangle$, and there are L^*_{FOLP} formulas $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ with $T\{A_1, \ldots, A_m\} \subseteq \langle\!\langle T\,A \rangle\!\rangle$ and $F\{B_1, \ldots, B_n\} \subseteq \langle\!\langle F\,B \rangle\!\rangle$ so that we have FOLP^* provability of the following.

$$\left[\bigwedge \Gamma \wedge \bigwedge \{A_1, \ldots, A_m\}\right] \supset \left[\bigvee \Delta \vee \bigvee \{B_1, \ldots, B_n\}\right]$$

It follows by classical logic that we also have provability of the following, i ranging over $1, \ldots, m$ and j ranging over $1, \ldots, n$.

$$\bigwedge \Gamma \supset \left[\bigvee \Delta \vee \bigvee_{i,j} (A_i \supset B_j)\right] \tag{6}$$

Since each $F\,A_i \supset B_j \in \langle\!\langle F\,A \supset B \rangle\!\rangle$, this says that S is inconsistent, and hence $S \notin \mathcal{C}_q$.

Remark. The disjunction in formula (6) can be replaced by a simpler one. Instead of $\bigvee_{i,j}(A_i \supset B_j)$ with i ranging over $1, \ldots, m$ and j ranging over $1, \ldots, n$, we could use the following, with the same ranges for i and j as before: $\bigvee_i (A_i \supset B_1) \vee \bigvee_j (A_1 \supset B_j)$ (the use of 1 is arbitrary).

3. Suppose $T\,(\forall x)\varphi(x) \in S$ but $S \cup \{T\,\varphi(v)\} \notin \mathcal{C}_q$, for some parameter v. $S \cup \{T\,\varphi(v)\}$ omits infinitely many parameters, since S does, so there must be finite sets Γ and Δ of L^*_{FOLP} with $T\,\Gamma \cup F\,\Delta \subseteq \langle\!\langle S \rangle\!\rangle$, and L^*_{FOLP} formulas $\varphi_1(v)$, ..., $\varphi_n(v)$ with $T\{\varphi_1(v), \ldots, \varphi_n(v)\} \subseteq \langle\!\langle T\,\varphi(v) \rangle\!\rangle$ so that we have FOLP^* provability of the following.

$$\left[\bigwedge \Gamma \wedge \bigwedge \{\varphi_1(v), \ldots, \varphi_n(v)\}\right] \supset \bigvee \Delta$$

A small complication now arises. The formula $\varphi(x)$ may already contain occurrences of v, so while $\varphi(v) = \varphi(x)(x/v)$ it does not follow that $\varphi(x) = \varphi(v)(v/x)$, and similarly for each φ_i. This is taken care of by Lemma 8.2. Since $\varphi(v) = \varphi(x)(x/v)$, for $i = 1, \ldots, n$ there is some $\varphi_i'(x)$ so that $T\,\varphi_i'(x) \in \langle\!\langle T\,\varphi(x) \rangle\!\rangle$ and $\varphi_i(v) = \varphi_i'(x)(x/v) = \varphi_i'(v)$. Then using universal instantiation, $(\forall x)\varphi_i'(x) \supset \varphi_i(v)$ for each i, and so we have provability of the following,

$$\left[\bigwedge \Gamma \wedge \bigwedge \{(\forall x)\varphi_1'(x), \ldots, (\forall x)\varphi_n'(x)\}\right] \supset \bigvee \Delta \tag{7}$$

but also, for each i, $T\,(\forall x)\varphi_i'(x) \in \langle\!\langle T\,(\forall x)\varphi(x) \rangle\!\rangle$, so the provability in FOLP^* of (7) says S is inconsistent, hence $S \notin \mathcal{C}_q$.

4. Suppose $F\,(\forall x)\varphi(x) \in S$ but $S \cup \{F\,\varphi(v)\} \notin \mathcal{C}_q$ for every parameter v. Let r be a parameter that does not occur in S (recall that S omits infinitely many parameters). Then $S \cup \{F\,\varphi(r)\} \notin \mathcal{C}_q$, so there are finite sets Γ and Δ of L^*_{FOLP} formulas with $T\,\Gamma \cup F\,\Delta \subseteq \langle\!\langle S \rangle\!\rangle$, and there are L^*_{FOLP} formulas $\varphi_1(r), \ldots, \varphi_n(r)$ with $F\{\varphi_1(r), \ldots, \varphi_n(r)\} \subseteq \langle\!\langle F\,\varphi(r) \rangle\!\rangle$ such that the following is provable in FOLP^*.

$$\bigwedge \Gamma \supset \left[\bigvee \Delta \vee \bigvee \{\varphi_1(r), \ldots, \varphi_n(r)\}\right] \tag{8}$$

Just because $F\,\varphi_i(r) \in \langle\!\langle F\,\varphi(r)\rangle\!\rangle$, it does not follow that $F\,\varphi_i(x) \in \langle\!\langle F\,\varphi(x)\rangle\!\rangle$, but we can appeal to Lemma 8.2 as we did in the previous case. For each i there is some $\varphi'_i(x)$ so that $F\,\varphi'_i(x) \in \langle\!\langle F\,\varphi(x)\rangle\!\rangle$, and $\varphi_i(r) = \varphi'_i(x)(x/r) = \varphi'_i(r)$. Then from (8) we have provability of the following.

$$\bigwedge \Gamma \supset \left[\bigvee \Delta \vee \bigvee\{\varphi'_1(r), \ldots, \varphi'_n(r)\}\right] \tag{9}$$

Recall that for each signed formula Z, members of the set $\langle\!\langle Z\rangle\!\rangle$ and Z itself all have the same parameters. Since r does not occur in S, it does not occur in Γ or Δ. Then, using Corollary 6.3 and classical logic, the provability in FOLP^* of (9) gives us the FOLP^* provability of the following.

$$\bigwedge \Gamma \supset \left[\bigvee \Delta \vee (\forall x)\,[\varphi'_1(x) \vee \ldots \vee \varphi'_n(x)]\right] \tag{10}$$

But $F\,(\forall x)\,[\varphi'_1(x) \vee \ldots \vee \varphi'_n(x)] \in \langle\!\langle F\,(\forall x)\varphi(x)\rangle\!\rangle \subseteq \langle\!\langle S\rangle\!\rangle$, so the provability in FOLP^* of (10) gives us the inconsistency of S, and hence $S \notin \mathcal{C}_q$.

5. Suppose $T\,\Box_n A \in S$ but $S \cup \{T\,A\} \notin \mathcal{C}_q$. $S \cup \{T\,A\}$ omits infinitely many parameters. Then since $S \cup \{T\,A\} \notin \mathcal{C}_q$ there are finite sets Γ, Δ, and $\{A_1, \ldots, A_k\}$ consisting of L^*_{FOLP} formulas with $T\,\Gamma \cup F\,\Delta \subseteq \langle\!\langle S\rangle\!\rangle$ and $T\,\{A_1, \ldots, A_k\} \subseteq \langle\!\langle T\,A\rangle\!\rangle$ so that the following is provable in FOLP^*.

$$\left[\bigwedge \Gamma \wedge \bigwedge\{A_1, \ldots, A_k\}\right] \supset \bigvee \Delta \tag{11}$$

Let X be the set of parameters of A, which is the same as the set of parameters of A_i for each i. It follows from the provability in FOLP^* of (11) and Axiom **B1** that we have FOLP^* provability of the following.

$$\left[\bigwedge \Gamma \wedge \bigwedge\{p_n{:}_X A_1, \ldots, p_n{:}_X A_k\}\right] \supset \bigvee \Delta \tag{12}$$

Since $T\,p_n{:}_X A_i \in \langle\!\langle T\,\Box_n A\rangle\!\rangle$ for $i = 1, \ldots, k$, the provability in FOLP^* of (12) implies S is inconsistent, and hence $S \notin \mathcal{C}_q$.

6. Suppose $F\,\Box_n A \in S$ but $S^\sharp \cup \{F\,A\} \notin \mathcal{C}_q$. $S^\sharp \cup \{F\,A\}$ omits infinitely many parameters, since S does. Then there must be finite sets Γ and $\{A_1, \ldots, A_k\}$ consisting of L^*_{FOLP} formulas with $T\,\Gamma \subseteq \langle\!\langle S^\sharp\rangle\!\rangle$ and $F\,\{A_1, \ldots, A_k\} \subseteq \langle\!\langle F\,A\rangle\!\rangle$ so that the following is provable in FOLP^*.

$$\bigwedge \Gamma \supset \bigvee\{A_1, \ldots, A_k\}$$

All members of $S^\sharp$ are of the form $T\,\Box_{n_i} H_i$, and so members of $\langle\!\langle S^\sharp\rangle\!\rangle$ are of the form $T\,p_{n_i}{:}_{X_i} G_i$ where X_i is the set of parameters of G_i. Let us say $\Gamma = \{p_{n_1}{:}_{X_1} G_1, \ldots, p_{n_m}{:}_{X_m} G_m\}$, so we have provability of the following.

$$(p_{n_1}{:}_{X_1} G_1 \wedge \ldots \wedge p_{n_m}{:}_{X_m} G_m) \supset (A_1 \vee \ldots \vee A_k) \tag{13}$$

Then by Corollary 6.5 and the provability in FOLP^* of (13), the following is provable, for some justification term t, where X is the set of parameters of $A_1 \vee \ldots \vee A_k$, which is the same as the set of parameters of A.

$$(p_{n_1}{:}_{X_1} G_1 \wedge \ldots \wedge p_{n_m}{:}_{X_m} G_m) \supset t{:}_X (A_1 \vee \ldots \vee A_k) \tag{14}$$

Now $T\, p_{n_i}{:}_{X_i} G_i \in \langle\!\langle S^\sharp \rangle\!\rangle \subseteq \langle\!\langle S \rangle\!\rangle$, for each $i = 1, \ldots, m$. Also $F\, A_i \in \langle\!\langle F\, A \rangle\!\rangle$ for $i = 1, \ldots, k$, so $F\, t{:}_X(A_1 \vee \ldots \vee A_k) \in \langle\!\langle F\, \Box_n A \rangle\!\rangle \subseteq \langle\!\langle S \rangle\!\rangle$. Then the provability in FOLP^* of (14) implies that S is inconsistent, and hence $S \notin \mathcal{C}_q$.

■

Corollary 9.2 *Every closed formula of L_M that is provable in first-order* S4 *has a quasi-realization that is provable in* FOLP.

Proof Assume A is a closed formula of L_M that is provable in first-order S4. Annotate the occurrences of $\Box$ in A to produce a closed annotated formula A'. Since A is provable, A is valid in the family of Kripke first-order S4 models, and it follows that $\{F\, A'\}$ is not satisfiable (recall that semantically the annotations play no role). But since A' is a closed formula of L_{AM} it is trivially a p-closed formula of L^*_{AM}. By Theorem 9.1, $\mathcal{C}_q$ is an annotated consistency property, so by the Model Existence Theorem its members are satisfiable. Then $\{F\, A'\} \notin \mathcal{C}_q$, and thus the set is inconsistent in the sense used to define $\mathcal{C}_q$. Then there are $U_1, \ldots, U_n$ with $F\, U_1, \ldots, F\, U_n \in \langle\!\langle F\, A' \rangle\!\rangle$ so that $U_1 \vee \ldots \vee U_n$ is provable in FOLP^*. Since A', like A, contains no parameters, neither does any U_i, and thus each U_i is in the language L_{FOLP}. Then $U_1 \vee \ldots \vee U_n$ is provable in FOLP by Corollary 6.2, and hence is our provable quasi-realization of A. ■

Corollary 9.3 *Every formula of L_M, closed or not, that is provable in first-order* S4 *has a quasi-realization that is provable in* FOLP.

Proof Suppose A is a formula of L_M that is S4 provable, but which may contain free variables (not parameters, of course). Let $\forall A$ be the universal closure of A, which is also provable. By the previous corollary this has a quasi-realization that is provable in FOLP. This quasi-realization is of the form $U_1 \vee \ldots \vee U_n$ where for each i, $F\, U_i \in \langle\!\langle F\, \forall A' \rangle\!\rangle$ (where A' is an annotated version of A). By Definition 8.1, each U_i must be of the form $\forall D_i$, where D_i is a disjunction of members of $\langle\!\langle F\, A' \rangle\!\rangle$, with signs removed. By classical logic, $\forall D_i \supset D_i$, hence $(U_1 \vee \ldots \vee U_n) \supset (D_1 \vee \ldots \vee D_n)$ is provable in FOLP, and it follows that $D_1 \vee \ldots \vee D_n$ is also provable. But this is, itself, a disjunction (ignoring associativity issues) of formulas V where $F\, V \in \langle\!\langle F\, A' \rangle\!\rangle$, and hence is a quasi-realization of A. ■

10 Realizations

The machinery introduced below for our definition of realizations is rather different than usual, but the outcome is the same and the machinery is more appropriate for our purposes. We parallel the definition of quasi-realization from Section 8, but now disjunctions do not appear where they previously did in cases 3 and 4. We still assume that $p_1, p_2, \ldots$ is an enumeration of all justification variables of FOLP, with no justification variable repeated. It is important to note that parameters now play no role.

Definition 10.1 *The mapping $[\![\cdot]\!]$ associates with each signed formula of L_{AM} a set of signed formulas in the language L_{FOLP}. It is defined recursively, as follows.*

1. *If A is atomic, $[\![T\,A]\!] = \{T\,A\}$ and $[\![F\,A]\!] = \{F\,A\}$.*
2. $[\![T\,A \supset B]\!] = \{T\,U \supset V \mid F\,U \in [\![F\,A]\!], T\,V \in [\![T\,B]\!]\}$
   $[\![F\,A \supset B]\!] = \{F\,U \supset V \mid T\,U \in [\![T\,A]\!], F\,V \in [\![F\,B]\!]\}$.
3. $[\![T\,(\forall x)A]\!] = \{T\,(\forall x)U \mid T\,U \in [\![T\,A]\!]\}$
   $[\![F\,(\forall x)A]\!] = \{F\,(\forall x)U \mid F\,U \in [\![F\,A]\!]\}$.
4. *In the following, X is the set of free variables in A.*
   $[\![T\,\Box_n A]\!] = \{T\,p_n{:}_X U \mid T\,U \in [\![T\,A]\!]\}$
   $[\![F\,\Box_n A]\!] = \{F\,t{:}_X U \mid F\,U \in [\![F\,A]\!]$, *where t is any justification term*$\}$.
5. *The mapping extends to* sets *of signed annotated formulas by letting $[\![S]\!] = \cup\{[\![Z]\!] \mid Z \in S\}$.*

Definition 10.2 *Let A be a formula in the language L_{AM}. A* normal realization *of A is a formula U in the language L_{FOLP} where $F\,U \in [\![F\,A]\!]$. For a formula A in the language L_M, without annotations, a normal realization for A is any normal realization for A', where A' is an annotated version of A.*

The Comments that follow Definition 8.1 and are about the mapping $\langle\!\langle\cdot\rangle\!\rangle$ apply just as well to the mapping $[\![\cdot]\!]$, but are even simpler since there are no parameters now. The members of $[\![T\,A]\!]$ and of $[\![F\,A]\!]$ all have the same free variables as A.

11 About Substitution

In the next section we will be making considerable use of substitutions. The ones we need do not substitute for individual variables, instead they replace justification variables with justification terms. A substitution is a function $\sigma = \{p_{i_1}/t_1, \ldots, p_{i_n}/t_n\}$, that maps justification variable p_{i_k} to justification term t_k, and is the identity otherwise (it is assumed that each t_k is different from p_{i_k}). The *domain* of σ is $\{p_{i_1}, \ldots, p_{i_n}\}$. For a formula A of L_{FOLP}, the result of applying a substitution σ is denoted $A\sigma$, and likewise $t\sigma$ is the result of applying substitution σ to justification term t.

If A is a theorem of FOLP and σ is a substitution, $A\sigma$ will also be a theorem of FOLP. This is easy to see, because substitutions turn axioms into axioms (because axiomatization is by schemes), and turn rule applications into rule applications. But the role of constants must change with a substitution. We have not defined constant specifications—the rest of this paragraph is for those who know what they are, and it plays no role in the rest of this paper. Suppose $\mathcal{C}$ is a constant specification, A is an axiom, and $c{:}_\emptyset A$ is added to a proof using Axiom Necessitation, where this addition meets constant specification $\mathcal{C}$. Since $A\sigma$ is also an axiom, Axiom Necessitation allows us to add $c{:}_\emptyset A\sigma$ to a proof, but this may no longer meet specification $\mathcal{C}$. However a new constant specification, which we can call $\mathcal{C}\sigma$, can be computed from the original one—$c{:}_\emptyset A\sigma \in \mathcal{C}\sigma$ just in case $c{:}_\emptyset A \in \mathcal{C}$. Even if $\mathcal{C}$ was axiomatically appropriate, $\mathcal{C}\sigma$ need not be, but if $\mathcal{C}$ is axiomatically appropriate, $\mathcal{C} \cup \mathcal{C}\sigma$ will be. So, if A is provable using an axiomatically appropriate constant specification the same will be true for $A\sigma$. From now on we suppress such details.

Definition 11.1 *Let A be a formula of L_{AM}. A substitution σ* lives on *A provided, for every justification variable p_k in the domain of σ, $\Box_k$ occurs in A. Also σ* lives away from *A provided, for every justification variable p_k in the domain of σ, $\Box_k$ does not occur in A. We say σ meets the* no new variable condition *provided, for every p in the domain of σ, the justification term $p\sigma$ contains no variables other than p.*

There are a few simple results concerning these notions that will be useful in the next section.

Lemma 11.2 *Assume A is a formula of L_{AM}, σ_A is a substitution that lives on A, and σ_Z is a substitution that lives away from A.*

1. *If $T\,U \in [\![T\,A]\!]$ then $T\,U\sigma_Z \in [\![T\,A]\!]$; if $F\,U \in [\![F\,A]\!]$ then $F\,U\sigma_Z \in [\![F\,A]\!]$*
2. *If both σ_A and σ_Z meet the no new variable condition, then $\sigma_A\sigma_Z = \sigma_Z\sigma_A$.*

Proof Part 1: The proof is by induction on the complexity of A. The atomic case is trivial since no justification variables are present, and the $\supset$ and quantifier cases are straightforward. This leaves the modal cases. Suppose $A = \Box_n B$, and the result is known for simpler formulas.

First assume that $T\,p_n :_X U \in [\![T\,\Box_n B]\!]$. Since σ_Z lives away from A, $p_n\sigma_Z = p_n$. By the induction hypothesis $T\,U\sigma_Z \in [\![T\,B]\!]$. But then $T\,(p_n :_X U)\sigma_Z = T\,p_n{:}_X(U\sigma_Z) \in [\![T\,\Box_n B]\!]$.

Next assume $F\,t{:}_X U \in [\![F\,\Box_n B]\!]$. By the induction hypothesis, $F\,U\sigma_Z \in [\![F\,B]\!]$. Then $F\,(t{:}_X U)\sigma_Z = F\,t\sigma_Z{:}_X U\sigma_Z \in [\![F\,\Box_n B]\!]$

Part 2: Assume the hypothesis, and let p be a justification variable; we show $p\sigma_A\sigma_Z = p\sigma_Z\sigma_A$.

First, suppose $p = p_k$ where $\Box_k$ occurs in A. Since σ_A meets the no new variable condition, the only justification variable that can occur in $p\sigma_A$ is p. Since σ_Z lives away from A, $p\sigma_Z = p$, and so $p\sigma_A\sigma_Z = p\sigma_A$. But also, $p\sigma_Z\sigma_A = p\sigma_A$, hence $p\sigma_A\sigma_Z = p\sigma_Z\sigma_A$.

Second, suppose $p = p_k$ where $\Box_k$ does not occur in A. Since σ_A lives on A, $p\sigma_A = p$. And since σ_Z meets the no new variable condition, p is the only variable that can occur in $p\sigma_Z$. Then $p\sigma_Z\sigma_A = p\sigma_Z$, and $p\sigma_A\sigma_Z = p\sigma_Z$, so $p\sigma_A\sigma_Z = p\sigma_Z\sigma_A$ in this case too. ■

12 The Realization Theorem Itself

We have already shown a Quasi-Realization Theorem, in 9.2 and 9.3. We now use this to get a proper Realization Theorem. The key construction is embodied in the proof of Theorem 12.1—how it gives Realization is the simple content of Theorem 12.4. Since the work here is entirely constructive we give it as an algorithm, followed by a verification of its correctness. Parameters play no role in this section.

Theorem 12.1 *Let A be a formula of L_{AM}.*

1. *If $T\,A_1, \ldots, T\,A_k \in \langle\langle T\,A \rangle\rangle$ then there is a substitution σ that lives on A and meets the no new variable condition, and there is some $T\,A' \in [\![T\,A]\!]$ such that the following is provable in **FOLP**:*

$$A' \supset (A_1 \wedge \ldots \wedge A_k)\sigma$$

2. *If $F\,A_1, \ldots, F\,A_k \in \langle\langle F\,A \rangle\rangle$ then there is a substitution σ that lives on A and meets the no new variable condition, and there is some $F\,A' \in [\![F\,A]\!]$ such that the following is provable in **FOLP**:*

$$(A_1 \vee \ldots \vee A_k)\sigma \supset A'$$

Example 12.2 *We cannot directly continue with Examples 8.3 and 8.5, because they involved formulas containing a parameter, v. But suppose we replace v with a new individual variable y that is not a parameter—nothing essential changes in what we did. We will have that both $F\,p_2{:}_{\emptyset}\bot \supset (\forall x)[s{:}_{\{x,y\}}A(x,y) \vee t{:}_{\{x,y\}}A(x,y)]$ and $F\,p_2{:}_{\emptyset}\bot \supset (\forall x)u{:}_{\{x,y\}}A(x,y)$ are in $\langle\langle F\,\Box_2\bot \supset (\forall x)\Box_1 A(x,y)\rangle\rangle$, where s, t, and u are justification terms. In a similar way, $F\,p_2{:}_{\emptyset}\bot \supset (\forall x)(s+t+u){:}_{\{x,y\}}A(x,y)$ is in $[\![F\,\Box_2\bot \supset (\forall x)\Box_1 A(x,y)]\!]$. And in fact, the following is provable in **FOLP**, where σ is the empty substitution.*

$$\begin{aligned}\{[p_2{:}_{\emptyset}\bot \supset (\forall x)[s{:}_{\{x,y\}}A(x,y) \vee t{:}_{\{x,y\}}A(x,y)]] \vee [p_2{:}_{\emptyset}\bot \supset (\forall x)u{:}_{\{x,y\}}A(x,y)]\}\sigma \\ \supset \\ [p_2{:}_{\emptyset}\bot \supset (\forall x)(s+t+u){:}_{\{x,y\}}A(x,y)]\end{aligned}$$

Noting Example 8.5, we have that the quasi-realization of $\Box_2\bot \supset (\forall x)\Box_1 A(x,y)$

$$[p_2{:}_{\emptyset}\bot \supset (\forall x)[s{:}_{\{x,y\}}A(x,y) \vee t{:}_{\{x,y\}}A(x,y)]] \vee [p_2{:}_{\emptyset}\bot \supset (\forall x)u{:}_{\{x,y\}}A(x,y)]$$

*provably implies the normal realization in **FOLP***

$$p_2{:}_{\emptyset}\bot \supset (\forall x)(s+t+u){:}_{\{x,y\}}A(x,y)$$

though neither is a theorem separately.

The algorithm that follows shows how to construct an appropriate formula A' and a substitution σ, as used in Theorem 12.1. It proceeds by induction on the formula complexity of A—proof complexity plays no role here. The atomic case is simple, but for the other cases we make use of some notation to schematically present the construction for each case. The detailed meaning of the notation is fully spelled out in the correctness proof that follows, but here is a reading for the **F** $\supset$ case, as an example. Suppose formula A is $B \supset C$, each of

$F\,A_1, \ldots, F\,A_k$ is in $\langle\!\langle F\,A \rangle\!\rangle$, and each A_i is $B_i \supset C_i$ (and hence $T\,B_i \in \langle\!\langle T\,B \rangle\!\rangle$ and $F\,C_i \in \langle\!\langle F\,C \rangle\!\rangle$). Suppose also that it has already been established that $B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B$ and $(C_1 \vee \ldots \vee C_k)\sigma_C \supset C'$ are provable. Then the formula $(A_1 \vee \ldots \vee A_k)\sigma \supset A'$ will also be provable, where $\sigma = \sigma_B\sigma_C$ and $A' = B'\sigma_C \supset C'\sigma_B$.

Now here is the algorithm, with a correctness proof following. It is assumed that all formulas are in L_{FOLP}.

Algorithm 12.3 (Quasi-Realization to Realization Conversion)

Atomic *Trivial, since if A is atomic $\langle\!\langle A \rangle\!\rangle = [\![A]\!] = \{A\}$, and we can use the empty substitution.*

F $\supset$

$$\frac{A = B \supset C \quad F\,A_1, \ldots, F\,A_k \in \langle\!\langle F\,A \rangle\!\rangle \quad A_i = B_i \supset C_i \\ B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B \quad (C_1 \vee \ldots \vee C_k)\sigma_C \supset C'}{(A_1 \vee \ldots \vee A_k)\sigma \supset A' \text{ where } \sigma = \sigma_B\sigma_C \text{ and } A' = B'\sigma_C \supset C'\sigma_B}$$

T $\supset$

$$\frac{A = B \supset C \quad T\,A_1, \ldots, T\,A_k \in \langle\!\langle T\,A \rangle\!\rangle \quad A_i = B_i \supset C_i \\ (B_1 \vee \ldots \vee B_k)\sigma_B \supset B' \quad C' \supset (C_1 \wedge \ldots \wedge C_k)\sigma_C}{A' \supset (A_1 \wedge \ldots \wedge A_k)\sigma \text{ where } \sigma = \sigma_B\sigma_C \text{ and } A' = B'\sigma_C \supset C'\sigma_B}$$

F $\Box$

$$\frac{A = \Box_n B \quad F\,A_1, \ldots, F\,A_k \in \langle\!\langle F\,A \rangle\!\rangle \quad A_i = t_i{:}_X B_i \text{ where } B_i = \bigvee D_i \\ (B_1 \vee \ldots \vee B_k)\sigma_B \supset B'}{(A_1 \vee \ldots \vee A_k)\sigma \supset A' \text{ where } \sigma = \sigma_B \text{ and } A' = s{:}_X B' \\ s = (u_1 \cdot t_1\sigma_B) + \ldots + (u_k \cdot t_k\sigma_B) \\ u_i \text{ internalizes the proof of } B_i\sigma_B \supset B'}$$

T $\Box$

$$\frac{A = \Box_n B \quad T\,A_1, \ldots, T\,A_k \in \langle\!\langle T\,A \rangle\!\rangle \quad A_i = p_n{:}_X B_i \\ B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B}{A' \supset (A_1 \wedge \ldots \wedge A_k)\sigma \text{ where } \sigma = \sigma_B\{p_n/(s \cdot p_n)\} \text{ and } A' = p_n{:}_X B'\{p_n/(s \cdot p_n)\} \\ s = u_1 + \ldots + u_k \\ u_i \text{ internalizes the proof of } B' \supset B_i\sigma_B}$$

F $\forall$

$$\frac{A = (\forall x)B \quad F\,A_1, \ldots, F\,A_k \in \langle\!\langle F\,A \rangle\!\rangle \quad A_i = (\forall x)B_i \text{ where } B_i = \bigvee D_i \\ (B_1 \vee \ldots \vee B_k)\sigma_B \supset B'}{(A_1 \vee \ldots \vee A_k)\sigma \supset A' \text{ where } \sigma = \sigma_B \text{ and } A' = (\forall x)B'}$$

T $\forall$

$$\frac{A = (\forall x)B \quad T\,A_1, \ldots, T\,A_k \in \langle\!\langle T\,A \rangle\!\rangle \quad A_i = (\forall x)B_i \\ B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B}{A' \supset (A_1 \wedge \ldots \wedge A_k)\sigma \text{ where } \sigma = \sigma_B \text{ and } A' = (\forall x)B'}$$

Proof of correctness for Algorithm 12.3 We justify each non-atomic case of the algorithm, and along the way fully supply the meaning for the schematics used in the formulation of the algorithm as given above.

F $\supset$ Suppose A is $(B \supset C)$ and $F\,A_1, \ldots, F\,A_k \in \langle\!\langle F\,A \rangle\!\rangle$. For each i, say $A_i = (B_i \supset C_i)$; it follows that $T\,B_i \in \langle\!\langle T\,B \rangle\!\rangle$ and $F\,C_i \in \langle\!\langle F\,C \rangle\!\rangle$. Assume there are substitutions σ_B and σ_C, living on B and C respectively and meeting the no new variable condition, and there are B' and C' with $T\,B' \in [\![T\,B]\!]$ and $F\,C' \in [\![F\,C]\!]$ such that $B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B$ and $(C_1 \vee \ldots \vee C_k)\sigma_C \supset C'$ are both provable in FOLP. We will show $(A_1 \vee \ldots \vee A_k)\sigma \supset A'$ is provable where $\sigma = \sigma_B\sigma_C$ and $A' = (B'\sigma_C \supset C'\sigma_B)$. We will also show that both σ and A' meet the required conditions. (As noted above, all this is what is embodied in the **F** $\supset$ schematic of the algorithm as noted above.)

Since $A = (B \supset C)$ is a formula of L_{AM}, and in annotated formulas indexes can appear only once, then B and C have no indexes in common, and so σ_B and σ_C have disjoint domains. In particular, σ_B lives on B and so lives away from C, while σ_C lives on C and so lives away from B. Let σ be $\sigma_B\sigma_C = \sigma_C\sigma_B$ (these are equal by Lemma 11.2). It is easy to see that σ lives on $B \supset C$ and meets the no new variable condition.

Since $B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B$ is provable in FOLP, so is $[B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B]\sigma_C = [B'\sigma_C \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B\sigma_C]$ (though the constant specification may change). Slightly restated, $B'\sigma_C \supset (B_1 \wedge \ldots \wedge B_k)\sigma$ is provable. Similarly $(C_1 \vee \ldots \vee C_k)\sigma \supset C'\sigma_B$ is provable. Then by classical logic the following is FOLP provable: $[(B_1 \supset C_1) \vee \ldots \vee (B_k \supset C_k)]\sigma \supset (B'\sigma_C \supset C'\sigma_B)$, that is, $(A_1 \vee \ldots \vee A_k)\sigma \supset A'$ is provable.

By Lemma 11.2, $T\,B'\sigma_C \in [\![T\,B]\!]$ since $T\,B' \in [\![T\,B]\!]$ and σ_C lives away from B. Likewise $F\,C'\sigma_B \in [\![F\,C]\!]$. Then $F\,A' = F\,B'\sigma_C \supset C'\sigma_B \in [\![F\,B \supset C]\!]$, completing the case.

T $\supset$ Suppose A is $(B \supset C)$ and $T\,A_1, \ldots, T\,A_k \in \langle\!\langle T\,A \rangle\!\rangle$. Say $A_i = (B_i \supset C_i)$, where $F\,B_i \in \langle\!\langle F\,B \rangle\!\rangle$ and $T\,C_i \in \langle\!\langle T\,C \rangle\!\rangle$. Assume there are substitutions σ_B living on B, and σ_C living on C, both meeting the no new variable condition, and there are B' and C' with $F\,B' \in [\![F\,B]\!]$ and $T\,C' \in [\![T\,C]\!]$ such that both $(B_1 \vee \ldots \vee B_k)\sigma_B \supset B'$ and $C' \supset (C_1 \wedge \ldots \wedge C_k)\sigma_C$ are provable.

As in the previous case, if we set $\sigma = \sigma_B\sigma_C = \sigma_C\sigma_B$, the following is provable: $(B'\sigma_C \supset C'\sigma_B) \supset [(B_1 \supset C_1) \wedge \ldots \wedge (B_k \supset C_k)]\sigma$. Also $F\,B'\sigma_C \in [\![F\,B]\!]$ and $T\,C'\sigma_B \in [\![T\,C]\!]$, so $T\,B'\sigma_C \supset C'\sigma_B \in [\![T\,B \supset C]\!]$, and this is enough to establish this case.

F $\Box$ Suppose A is $\Box_n B$ and $F\,A_1, \ldots, F\,A_k \in \langle\!\langle F\,A \rangle\!\rangle$. In this case A_i is of the form $t_i{:}_X B_i$, where each t_i is some justification term, X is the set of free variables of B, $B_i = \bigvee D_i$, and $F\,D_i \subseteq \langle\!\langle F\,B \rangle\!\rangle$. Recall, the free variables of B are the same as the free variables of members of $\langle\!\langle F\,B \rangle\!\rangle$, which is the same as the free variables of the members of $[\![F\,B]\!]$. Let $D = D_1 \cup \ldots \cup D_k$; then $F\,D \subseteq \langle\!\langle F\,B \rangle\!\rangle$. Assume there is some substitution σ_B, living on B and meeting the no new variable condition, and there is some B' with $F\,B' \in [\![F\,B]\!]$ such that $\bigvee D\sigma_B \supset B'$ is provable. Equivalently, $(B_1 \vee \ldots \vee B_k)\sigma_B \supset B'$ is provable. Then for each i, $B_i\sigma_B \supset B'$ is provable and so, by the Lifting Lemma, Corollary 6.7, there is a justification term u_i (with no justification variables) such that $u_i{:}_X(B_i\sigma_B \supset B')$ is provable. But then $(t_i{:}_X B_i)\sigma_B \supset (u_i \cdot t_i\sigma_B){:}_X B'$ is also provable, using the fact that $(t_i : _X B_i)\sigma_B = (t_i\sigma_B) : _X (B_i\sigma_B)$. Let s be the justification term $(u_1 \cdot t_1\sigma_B) + \ldots + (u_k \cdot t_k\sigma_B)$. For each i we have the provability of $(t_i{:}_X B_i)\sigma_B \supset s{:}_X B'$, and hence provability of $(t_1{:}_X B_1 \vee \ldots \vee t_k{:}_X B_k)\sigma_B \supset s{:}_X B'$. Since $F\,s{:}_X B' \in [\![F\,\Box_n B]\!]$, this concludes the case.

T $\Box$ Suppose A is $\Box_n B$ and $T\,A_1, \ldots, T\,A_k \in \langle\!\langle T\,A \rangle\!\rangle$. In this case A_i is of the form $p_n{:}_X B_i$ where $T\,B_i \in \langle\!\langle T\,B \rangle\!\rangle$. Assume there is some substitution σ_B and some $T\,B' \in [\![T\,B]\!]$ such that $B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B$ is FOLP provable, where σ_B lives on B and meets the no new variable condition. We will show $A' \supset (A_1 \wedge \ldots \wedge A_k)\sigma$ is provable, where $\sigma = \sigma_B\{p_n/(s \cdot p_n)\}$ and $A' = p_n{:}_X B'\{p_n/(s \cdot p_n)\}$, for a particular s that is specified below. We will also show σ and A' meet the required conditions.

For each $i = 1, \ldots, k$, the formula $B' \supset B_i\sigma_B$ is provable, so by the Lifting Lemma, Corollary 6.7, there is a justification term u_i (with no justification variables) such that $u_i{:}_X(B' \supset B_i\sigma_B)$ is provable, where X is the set of free variables of B', equivalently of B, or of B_i. Let s be the justification term $u_1 + \ldots + u_k$. Then $s{:}_X(B' \supset B_i\sigma_B)$ is provable, for each i.

For notational convenience, let σ_0 be the substitution $\{p_n/(s \cdot p_n)\}$. For each $i = 1, \ldots, k$, $s{:}_X(B' \supset B_i\sigma_B)$ is provable, hence so is $[s{:}_X(B' \supset B_i\sigma_B)]\sigma_0$. Since s is a justification term with no justification variables, $s{:}_X(B'\sigma_0 \supset B_i\sigma_B\sigma_0)$ is provable. Then for each i, $p_n{:}_X B'\sigma_0 \supset (s \cdot p_n){:}_X B_i(\sigma_B\sigma_0)$ is provable. Since $\Box_n B$ is a formula of L_{AM}, index n cannot occur in B. Substitution σ_B lives on B, hence p_n is not in its domain, nor is it introduced by σ_B since σ_B meets the no new variable condition. It follows that $p_n(\sigma_B\sigma_0) = p_n\sigma_0 = (s \cdot p_n)$, and so $[p_n{:}_X B_i](\sigma_B\sigma_0) = (s \cdot p_n){:}_X B_i(\sigma_B\sigma_0)$. Then for each i, $p_n{:}_X B'\sigma_0 \supset [p_n{:}_X B_i](\sigma_B\sigma_0)$ is provable, and so $p_n{:}_X B'\sigma_0 \supset [p_n{:}_X B_1 \wedge \ldots \wedge p_n{:}_X B_k](\sigma_B\sigma_0)$ is provable; that is, $A' \supset (A_1 \wedge \ldots \wedge A_k)\sigma$ is provable.

The substitution σ_0 lives away from B so, since $T\,B' \in [\![T\,B]\!]$ then also $T\,B'\sigma_0 \in [\![T\,B]\!]$ by Lemma 11.2. Since A' is $p_n{:}_X B'\sigma_0$, we have that $T\,A' \in [\![T\,\Box_n B]\!] = [\![T\,A]\!]$. Also $\sigma = \sigma_B\sigma_0$, and it is easy to check that σ lives on A and meets the no new variable condition.

F $\forall$ Suppose A is $(\forall x)B$ and $F\,A_1, \ldots, F\,A_k \in \langle\!\langle F\,A \rangle\!\rangle$. This case is similar to the **F**$\Box$ case, but somewhat simpler. In the present instance A_i is of the form $(\forall x)B_i$, where $B_i = \bigvee D_i$ with $F\,D_i \subseteq \langle\!\langle F\,B \rangle\!\rangle$. Let $D = D_1 \cup \ldots \cup D_k$. Then $F\,D \subseteq \langle\!\langle F\,B \rangle\!\rangle$. Assume there is some substitution σ_B living on B and meeting the no new variable condition, and there is some B' with $F\,B' \in [\![F\,B]\!]$ such that $\bigvee D\sigma_B \supset B'$ is FOLP provable; equivalently $(B_1 \vee \ldots \vee B_k)\sigma_B \supset B'$ is provable. Then for each i, $B_i\sigma_B \supset$

B' is provable, hence so is $(\forall x)B_i\sigma_B \supset (\forall x)B'$, and then finally so is $[(\forall x)B_1 \vee \ldots \vee (\forall x)B_k]\sigma_B \supset (\forall x)B'$. Since $F\,(\forall x)B' \in [\![F\,(\forall x)B]\!]$ and $F\,(\forall x)B_i \in \langle\!\langle F\,(\forall x)B\rangle\!\rangle$, this concludes the case.

T $\forall$ Suppose A is $(\forall x)B$ and $T\,A_1, \ldots, T\,A_k \in \langle\!\langle T\,A\rangle\!\rangle$. Each A_i is of the form $(\forall x)B_i$ where $T\,B_i \in \langle\!\langle T\,B\rangle\!\rangle$. Assume there is a substitution σ_B living on B and meeting the no new variable condition, and there is B' with $T\,B' \in [\![T\,B]\!]$ so that $B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B$ is provable in FOLP. Then $(\forall x)B' \supset ((\forall x)B_1 \wedge \ldots \wedge (\forall x)B_k)\sigma_B$ is also FOLP provable, $T\,(\forall x)B' \in [\![T\,(\forall x)B]\!] = [\![T\,A]\!]$, and $\sigma = \sigma_B$ meets the necessary conditions.

■

The construction and proof above trace back to Proposition 7.8 in [5], with a modification and correction supplied in [7]. Of course the present version is for FOLP, while the earlier one was only for LP. Now realization is an easy consequence.

Theorem 12.4 (Realization) *Every formula of L_M that is provable in first-order S4 has a normal realization that is provable in FOLP.*

Proof Suppose B is a theorem of first-order S4. We can assume B is closed, since if it is not, a construction similar to that in the proof of Corollary 9.3 can be applied. Let A be an annotated version of B. Then from Corollary 9.2, there are $U_1, \ldots, U_k$ with $F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,A\rangle\!\rangle$ such that $U_1 \vee \ldots \vee U_k$ is a theorem of FOLP. By part 2 of Theorem 12.1 there is a substitution σ and a formula A' with $F\,A' \in [\![F\,A]\!]$ such that $(U_1 \vee \ldots \vee U_k)\sigma \supset A'$ is a theorem of FOLP. Since $(U_1 \vee \ldots \vee U_k)\sigma$ is also provable in FOLP, so is A', and this is a normal realization of A, and hence of B. ■

13 Future Work

The Realization Theorem proof given here for FOLP is non-constructive because of the use made of the Model Existence Theorem. In a subsequent paper we will explore the constructive content of the argument, and its extensions to other justification logics. Extensions to multi-agent justification logics may also be considered. The conversion from quasi-realizations to realizations, which is constructive, will also be explored further. It is probable that a modular, constructive method can be extracted from the present work, somewhat similar to [11], but of a simpler nature.

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