Modal Logics, Justification Logics, and Realization

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September 28, 2015

Abstract

Justification logics connect with modal logics, replacing unstructured modal operators with *justification terms* explicitly representing interdependence and flow of reasoning. The number of justification logics quickly grew from an initial single instance to a handful to about a dozen examples. In this paper we provide very general, though partly non-constructive, methods that cover all previous examples, and extend to an infinite family of modal logics. The full range of the phenomenon is not known. The extent to which constructive methods apply is also not known, but it is related to the availability of cut-free proof methods for modal logics.

1 Introduction

Justification logics are similar to modal logics, but with modal operators replaced by an infinite family of *justifications* that are intended to stand for explicit reasons. They are very fine-grained and, crucially, can internalize details of their own formal proofs. Justification logics are connected with modal logics via *realization* results. Roughly, realization replaces modal operators in a modal theorem with justification terms whose structure reflects the flow of reasoning involved in establishing the modal theorem. The first justification logic was LP, which was connected with the modal logic $\mathsf{S4}$. This is discussed in the next section. The number of modal logics known to have justification counterparts with connecting realization theorems has grown over the years. We now see we are dealing with a general phenomenon whose extent is not known. In this paper we give a general approach to justification logics and to realization. In particular we show that the family of modal logics with justification counterparts is infinite.

Since justification logics are not as familiar as modal logics, we begin with a brief history. In particular we discuss how they came about in the first place—their original motivation. Then we give a very general approach to justification logics and realization. We note that there are now versions of quantified justification logics, but in this paper we confine our discussion entirely to propositional cases.

2 The Origin Story

The family of justification logics grew from a single instance, *the logic of proofs*, LP. This was introduced by Sergei Artemov as an essential part of his program to create an arithmetic provability semantics for intuitionistic logic. Features of LP have had a significant influence on research into the developing group of justification logics, so it is appropriate that we begin with a brief discussion of LP history.

Gödel formulated, at least implicitly, a program to find an arithmetic semantics for intuitionistic logic. In a well-known note, [19], he introduced the modern axiomatization of the modal logic $\mathsf{S4}$, thinking of the $\Box$ operator as an informal provability operator. He also gave an embedding from intuitionistic logic to S4: put $\Box$ before every subformula. This amounts to thinking of intuitionistic truth as a kind of informal provability, with provability conditions reflected in conditions imposed on $\Box$. But he also noted that S4 does not embed into Peano arithmetic, translating $\Box$ as his formal provability operator. If it did, then the S4 theorem $\Box\bot \supset \bot$ would turn into a provable statement asserting consistency, something ruled out by his famous second incompleteness theorem. Since then it has been learned that the logic of formal arithmetic provability is GL, Gödel-Löb logic, but this does not relate in the desired way to intuitionistic logic.

In [20], Gödel introduced the idea that instead of thinking of the $\Box$ operator of S4 as *provability*, it could be thought of as an *explicit* proof representative. Each occurrence of $\Box$ could be translated in a different way. While $(\exists x)(x \ \mathsf{Proves} \ \bot) \supset \bot$ is not provable in Peano arithmetic, for each n that is the Gödel number of an arithmetic proof, $(n \ \mathsf{Proves} \ \bot) \supset \bot$ is provable. In a sense this moves the existential quantifier into the metalanguage. Using explicit proof representatives, an embedding of S4 into arithmetic should be possible, Gödel suggested. This proposal was not developed further by Gödel, and his observations were not published until many years later when his collected works appeared. By this time the idea had been rediscovered independently by Sergei Artemov. Artemov's formal treatment involved the introduction of a new logic, LP, standing for *logic of proofs*. This is a modal-like language, but with *proof terms* which one could think of as encapsulating explicit proofs. It was necessary to show that LP embedded into formal arithmetic, and this was done in Artemov's *Arithmetic Completeness Theorem* which we do not discuss here. But it was also necessary to show that S4 embedded into LP. This involved the formulation and proof of Artemov's *Realization Theorem*. A proper statement will be found in Section 4. The definitive presentation of all this is in [1].

The methods that connected S4 with LP could also make connections between the standard modal logics, K, K4, T, and some others and weaker versions of LP. The Artemov Realization Theorem extended to these logics as well, essentially by leaving cases out. There was also an arithmetic interpretation because these were sublogics of S4, but the connection with arithmetic was beginning to weaken. The term *justification logics* began being used because, while the connection with formal provability was fragmenting, proof terms (now called *justification terms*) still had the role of supplying explicit justifications for (epistemically) necessary statements. Two fairly comprehensive treatments of justification logics like these, and not just of the logic of proofs, can be found in [2] and [4].

The logic S5 extended the picture in a significant way. A justification logic counterpart was created in [26, 28, 29], and a realization theorem was proved. However, the resulting justification logic did not have a satisfactory arithmetical interpretation, and the proof of realization was not constructive. A non-constructive, semantic, proof of realization had been given in [9] for S4. It also applied to standard weaker logics without significant change. The extension to S5 required new ideas involving *strong* evidence functions. This will play a role here as well. The original Artemov proof of realization, connecting S4 and LP, was constructive. Indeed, as a key part of Artemov's program to provide an arithmetic semantics for intuitionistic logic, it was essential that it be constructive. S5 was the first example where a non-constructive proof was the initial version discovered. Eventually constructive proofs were found, but the door to a larger room was beginning to open.

In [9] I introduced a Kripke-style semantics for LP, building on an earlier semantics, [23], that did not use possible worlds. Using this semantics a non-constructive, semantic based, proof of realization was given. It is only now becoming clear that the non-constructive argument has a broad applicability, and there are many, many more justification logics out there than we once thought. In what follows we present the semantic, non-constructive, realization argument in a very broad setting. We also discuss aspects of making it constructive, though work on this is very much in progress among a number of investigators.

3 Basics

We set out the fundamental ideas, and the terminology, that we will need in our consideration of modal and justification logics.

3.1 Modal Logics

We are only interested in normal modal logics with a single necessity operator. Of course multi-modal logics are well studied, and justification analogs have been developed, but they do not concern us here. To the best of our knowledge, there are no justification counterparts for non-normal modal logics.

We have a modal operator, $\Box$. ($\Diamond$ plays little role so we mostly ignore it.) Modal formulas are built up from propositional letters, P, Q, …, in the usual way. We will be flexible about connectives. Often we will assume we just have $\supset$ and $\bot$, with other connectives defined. If convenient, we will assume a larger set of primitive connectives.

A *normal* modal logic is a *set* of (modal) formulas that contains all tautologies and all formulas of the form $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$, and is closed under *uniform substitution*, *modus ponens*, and *generalization* (if X is present, so is $\Box X$). We are primarily interested in normal modal logics for which a Hilbert system exists. We assume they are axiomatized using a finite set of axiom *schemes*. The smallest normal logic is K which is axiomatized in the familiar way. Axioms are all tautologies (or enough of them), and all formulas of the form $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$. Rules are *Modus Ponens* $X, X \supset Y \Rightarrow Y$ and *Necessitation* $X \Rightarrow \Box X$.

Many familiar normal logics can be axiomatized by adding a finite set of axiom schemes to the K axioms above. We assume everybody knows axiom systems like $\mathsf{K4}$, $\mathsf{S4}$, and so on. We will refer to such logics as *axiomatically formulated*. One can also characterize modal logics using frames. Given a set $\mathcal{F}$ of frames, the set of modal formulas valid in all modal models built on frames from $\mathcal{F}$ is a normal modal logic. We will refer to such logics as *frame based*. Relationships between normality, axiomatics, and frames can be complicated, but this does not concern us here.

To keep terminology simple, we call a formula X a *validity* of a modal logic, using the same word no matter what the formulation. If we have a normal modal logic thought of as a set of formulas, validity means being a member of the set. If we have an axiomatic formulation, validity means having an axiomatic proof. If we have a frame based modal logic validity means being valid in every model based on a frame in the given collection of frames.

3.2 Justification Logics

Justification logics, syntactically, are like modal logics except that *justification terms* take the place of $\Box$. We begin with the question of what is a justification term.

Definition 3.1 (Justification Term) Justification terms are built up as follows.

1. There is a set of *justification variables*, x, y, …. Every justification variable is a justification term.

2. There is a set of *justification constants*, a, b, …. Every justification constant is a justification term.

3. There are binary operation symbols, $+$ and $\cdot$. If u and v are justification terms, so are $(u+v)$ and $(u \cdot v)$.

4. There may be additional function symbols, f, g, $\ldots$, of various arities.. Which ones are present depends on the logic. If f is an n-place justification function symbol and $t_1, \ldots, t_n$ are justification terms, $f(t_1, \ldots, t_n)$ is a justification term.

Then *justification formulas* are built up from propositional letters using propositional connectives together with the formation rule: if t is a justification term and X is a justification formula, then $t{:}X$ is a justification formula. Informally a justification term represents a reason why something is so; $t{:}X$ asserts that X is so for reason t. If justification term t has a complex structure we will usually write $[t]{:}X$, using square brackets, as a visual aid. No formal meaning should be associated with this other than the usual behavior of parentheses.

Justification variables stand for arbitrary justification terms, and can be substituted for under certain circumstances. Justification constants stand for reasons that are not further analyzed—typically they are reasons for axioms. The $\cdot$ operation corresponds to *modus ponens*. If $X \supset Y$ is so for reason s and X is so for reason t, then Y is so for reason $s \cdot t$. (Note that reasons are not unique—Y may be true for other reasons too.) The $+$ operation is a kind of weakening. If X is so for either reason s or reason t, then $s + t$ is also a reason for X.

There is no simple justification analog of normality. For instance, we will not have (or want) closure under substitution of formulas for propositional letters. Justifications for formulas should become more complicated under such a replacement—see [10]. We assume the justification logics we are concerned with are always specified axiomatically, using axiom schemes. There can also be semantic specifications, or tableaus, or sequent calculi, but we will take axiomatics as basic.

The weakest justification logic is J_0, axiomatized as follows.

J_0 Axioms

- All tautologies (or enough of them)
- All formulas of the form $s{:}(X \supset Y) \supset (t{:}X \supset [s \cdot t]{:}Y)$
- All formulas of the forms $s{:}X \supset [s + t]{:}X$ and $t{:}X \supset [s + t]{:}X$

J_0 Rules

- *Modus Ponens* $X, X \supset Y \Rightarrow Y$

There is no necessitation rule, but one will come back shortly in a somewhat different form. Other justification logics are axiomatized by adding additional *axiom schemes* to J_0 (schemes, not just axioms). Even so, J_0 is not a direct analog of modal K, because of the lack of a necessitation rule.

Definition 3.2 (Justification Logics) A justification logic J is axiomatically characterized as J_0 extended with a finite set of axiom schemes, whose instances are called *axioms of* J. Axioms may involve function symbols besides $\cdot$ and $+$.

Axioms are simply assumed, and are not analyzed further. Constant symbols are used for their justification. But if A is an axiom and constant symbol c justifies it, $c{:}A$ itself has no deep analysis, and so a constant symbol, say d, can come in here too, as a justification for $c{:}A$, and thus would have $d{:}c{:}A$. Details don't matter much in what we do here, so how constants are used, and for what purposes, is turned into a kind of parameter called a *constant specification*.

Definition 3.3 (Constant Specification) A *constant specification* $\mathcal{CS}$ for a given justification logic J is a set of formulas meeting the following conditions.

1. Members of $\mathcal{CS}$ are of the form $c_n{:}c_{n-1}{:}\ldots c_1{:}A$ where $n > 0$, A is an axiom of J, and each c_i is a constant symbol.

2. $\mathcal{CS}$ contains all intermediate specifications in the sense that, whenever $c_n{:}c_{n-1}{:}\ldots c_1{:}A$ is in $\mathcal{CS}$, then $c_{n-1}{:}\ldots c_1{:}A$ is in $\mathcal{CS}$ too $(n > 1)$.

A constant specification $\mathcal{CS}$ for J is *axiomatically appropriate* provided, for every axiom instance A of J, $c_n{:}c_{n-1}{:}\ldots c_1{:}A \in \mathcal{CS}$ for some constant symbols $c_1, \ldots, c_n$, for every $n > 0$.

Other conditions are sometimes placed on constant specifications, but being axiomatically appropriate suffices for what we do here.

Definition 3.4 (Consequence) Suppose J is a justification logic, $\mathcal{CS}$ is a constant specification for J, and S is an arbitrary set of formulas (not schemes). We write $S \vdash_{\mathsf{J}(\mathcal{CS})} X$ if there is a finite sequence of formulas, ending with X, in which each formula is either an axiom of J, a member of $\mathcal{CS}$, a member of S, or follows from earlier formulas by *modus ponens*. As usual, if $\emptyset \vdash_{\mathsf{J}(\mathcal{CS})} X$ we just write $\vdash_{\mathsf{J}(\mathcal{CS})} X$.

Since there is no necessitation rule, the classical proof of the deduction theorem applies: $S, X \vdash_{\mathsf{J}(\mathcal{CS})} Y$ if and only if $S \vdash_{\mathsf{J}(\mathcal{CS})} X \supset Y$. Since formal proofs are finite we have compactness which, combined with the deduction theorem, tells us: $S \vdash_{\mathsf{J}(\mathcal{CS})} X$ if and only if $\vdash_{\mathsf{J}(\mathcal{CS})} Y_1 \supset (Y_2 \supset \ldots \supset (Y_n \supset X)\ldots)$ for some $Y_1, Y_2, \ldots, Y_n \in S$.

Instead of the usual necessitation rule justification logics have an explicit version, under the right circumstances.

Definition 3.5 (Internalization) Let J be a justification logic, and $\mathcal{CS}$ be a constant specification for it. We say J *has the internalization property relative to* $\mathcal{CS}$ provided, if $\vdash_{\mathsf{J}(\mathcal{CS})} X$ then for some justification term t, $\vdash_{\mathsf{J}(\mathcal{CS})} t{:}X$. Let us further say it has the *strong* interalization property if t contains no justification variables and no justification function symbols except $\cdot$.

Theorem 3.6 *If J is a justification logic and $\mathcal{CS}$ is an axiomatically appropriate constant specification for J then J has the strong internalization property relative to $\mathcal{CS}$.*

Proof By a simple induction on proof length. Suppose $\vdash_{\mathsf{J}(\mathcal{CS})} X$ and the result is known for formulas with shorter proofs. If X is an axiom of J or a member of $\mathcal{CS}$, there is a justification constant c such that $c{:}X$ is in $\mathcal{CS}$, and so $c{:}X$ is provable. If X follows by *modus ponens* from $Y \supset X$ and Y then, by the induction hypothesis, $\vdash_{\mathsf{J}(\mathcal{CS})} s{:}(Y \supset X)$ and $\vdash_{\mathsf{J}\mathcal{CS})} t{:}Y$ for some s, t containing no justification variables, and with $\cdot$ as the only function symbol. Using the J_0 Axiom for $\cdot$, $\vdash_{\mathsf{J}(\mathcal{CS})} [s \cdot t]{:}X$. ■

Here is a generalization of considerable use.

Theorem 3.7 (Lifting Lemma) *Suppose J is a justification logic and $\mathsf{J}(\mathcal{CS})$ has the internalization property. If $X_1, \ldots, X_n \vdash_{\mathsf{J}(\mathcal{CS})} Y$ then for any justification terms $t_1, \ldots, t_n$ there is a justification term u so that $t_1{:}X_1, \ldots, t_n{:}X_n \vdash_{\mathsf{J}(\mathcal{CS})} u{:}Y$.*

Proof Properly the proof is by induction on n, but it is quite straightforward. We illustrate it with the $n = 2$ case. Assume we have $X_1, X_2 \vdash_{\mathsf{J}(\mathcal{CS})} Y$. Then, using Internalization to introduce justification term c, we have the following.

$$\begin{aligned}
X_1, X_2 &\vdash_{\mathsf{J}(\mathcal{CS})} Y \\
&\vdash_{\mathsf{J}(\mathcal{CS})} X_1 \supset (X_2 \supset Y) \\
&\vdash_{\mathsf{J}(\mathcal{CS})} c{:}(X_1 \supset (X_2 \supset Y)) \\
&\vdash_{\mathsf{J}(\mathcal{CS})} t_1{:}X_1 \supset [c \cdot t_1]{:}(X_2 \supset Y) \\
&\vdash_{\mathsf{J}(\mathcal{CS})} t_1{:}X_1 \supset (t_2{:}X_2 \supset [c \cdot t_1 \cdot t_2]{:}Y) \\
t_1{:}X_1, t_2{:}X_2 &\vdash_{\mathsf{J}(\mathcal{CS})} [c \cdot t_1 \cdot t_2]{:}Y
\end{aligned}$$

■

We noted in Section 3.2 that substitution of formulas for propositional letters plays no role in our present examination of justification logic But we will be need substitutions that replace justification variables with justification terms.

Definition 3.8 (Substitution) A substitution is a function σ, often denoted $\{x_1/t_1, \ldots, x_n/t_n\}$, mapping each justification variable x_k to justification term t_k, and the identity otherwise (it is assumed that each t_k is different from x_k). The *domain* of σ is $\{x_1, \ldots, x_n\}$. For a justification formula X the result of applying a substitution σ is denoted $X\sigma$; likewise $t\sigma$ is the result of applying substitution σ to justification term t.

Substitutions map axioms of a justification logic into axioms (because axiomatization is by schemes), and they turn *modus ponens* applications into *modus ponens* applications. But one must be careful because the role of constants changes with a substitution. Suppose $\mathcal{CS}$ is a constant specification, A is an axiom, and $c{:}A$ is added to a proof where this addition is authorized by $\mathcal{CS}$. $A\sigma$ is also an axiom, but if we add $c{:}A\sigma$ to a proof this may no longer meet constant specification $\mathcal{CS}$. A new constant specification, call it $\mathcal{CS}\sigma$, can be computed from the original one—$c{:}A\sigma \in \mathcal{CS}\sigma$ just in case $c{:}A \in \mathcal{CS}$. If $\mathcal{CS}$ was axiomatically appropriate, $\mathcal{CS} \cup \mathcal{CS}\sigma$ will also be. So, if X is provable using an axiomatically appropriate constant specification $\mathcal{CS}$, the same will be true for $X\sigma$, but not using the original constant specification but rather $\mathcal{CS} \cup \mathcal{CS}\sigma$. But this is more detail than we need to care about. The following suffices for much of our purposes.

Theorem 3.9 (Substitution Closure) *Suppose J is a justification logic and X is provable in J using some (axiomatically appropriate) constant specification. Then for any substitution σ, $X\sigma$ is also provable in J using some (axiomatically appropriate) constant specification.*

We introduce some special notation that suppresses details of constant specifications when we don't need to care about these details. The notation will be particularly handy in stating our condensing algorithm in Section 11.2.

Definition 3.10 Let J be a justification logic. We write $\vdash_{\mathsf{J}} X$ as short for: there is some axiomatically appropriate constant specification $\mathcal{CS}$ so that $\vdash_{\mathsf{J}(\mathcal{CS})} X$.

Theorem 3.11 *Let J be a justification logic.*

1. *If $\vdash_{\mathsf{J}} X$ then $\vdash_{\mathsf{J}} X\sigma$ for any substitution σ.*

2. *If* $\vdash_{\mathsf{J}} X$ *and* $\vdash_{\mathsf{J}} X \supset Y$ *then* $\vdash_{\mathsf{J}} Y$.

Proof Item 1 is directly from Theorem 3.9. For item 2, suppose $\vdash_{\mathsf{J}} X$ and $\vdash_{\mathsf{J}} X \supset Y$. Then there are axiomatically appropriate constant specifications $\mathcal{CS}_1$ and $\mathcal{CS}_2$ so that $\vdash_{\mathsf{J}(\mathcal{CS}_1)} X$ and $\vdash_{\mathsf{J}(\mathcal{CS}_2)} X \supset Y$. Now $\mathcal{CS}_1 \cup \mathcal{CS}_2$ will also be an axiomatically appropriate constant specification and $\vdash_{\mathsf{J}(\mathcal{CS}_1 \cup \mathcal{CS}_2)} X$ and $\vdash_{\mathsf{J}(\mathcal{CS}_1 \cup \mathcal{CS}_2)} X \supset Y$. Then $\vdash_{\mathsf{J}(\mathcal{CS}_1 \cup \mathcal{CS}_2)} Y$ and hence $\vdash_{\mathsf{J}} Y$. ■

4 Counterparts

We have been talking about justification logics (such as LP) *corresponding to* modal logics (S4 in this case). Now we define precisely what this means.

There is a simple mapping from the language(s) of justification logic to modal language—it is called the *forgetful functor*. For each justification formula X, let X° be the result of replacing every subformula $t{:}Y$ with $\Box Y$. More formally, $P^\circ = P$ if P is atomic; $(X \supset Y)^\circ = (X^\circ \supset Y^\circ)$; and $[t{:}X]^\circ = \Box X^\circ$. We want to know the circumstances under which the set of theorems of a justification logic is mapped exactly to the set of theorems of a modal logic.

Definition 4.1 Suppose KL is a normal modal logic and JL is a justification logic. (That is, JL extends axiomatic J_0, from Section 3.2, with a finite number of additional axiom schemes.) We say JL is a *counterpart* of KL if the following holds.

1. If X is provable in JL using some constant specification then X° is a validity of KL.
2. If Y is a validity of KL then there is some justification formula X so that $X^\circ = Y$, where X is provable in JL using some (generally axiomatically appropriate) constant specification.

In other words, JL is a counterpart of KL if the forgetful functor is a mapping from the set of theorems of JL onto the set of theorems of KL (provided arbitrary constant specifications are allowed). Our definition is syntactic in nature. See Section 5.6 for further comments.

As the subject has developed so far, justification logics have generally been formulated so that item 1 of Definition 4.1 is simple to show. Item 2 is known as a *Realization* result—X is said to realize Y. This is not at all simple, and is the central topic of the present paper. Realization theorems have always been shown in a stronger form than just stated. Here is a proper formulation.

Definition 4.2 (Normal Realization) X of JL is a *normal* realization of theorem Y of KL if X meets condition 2 from Definition 4.1, and X results from the replacement of *negative* occurrences of $\Box$ in Y with distinct justification variables (and positive occurrences by justification terms that need not be variables).

That a particular justification logic is a counterpart of a particular modal logic has always been proved by showing *normal* realizations exist for modal theorems. For some modal logics there are algorithms for producing normal realizations, but there are cases where no known algorithms exist, though one still has normal realizations. The extent of algorithmic realization is unclear.

5 Examples

We have given a number of formal definitions now, along with a few theorems. We pause and look at some examples illustrating these.

5.1 S4 and LP

As we discussed in Section 2 the logic of proofs, LP, was the first of the justification logics. Besides $\cdot$ and $+$ it has a one-place function symbol, customarily written ! with no parentheses, with the following axiom schemes, displayed alongside their S4 analogs.

LP	S4
$t{:}X \supset X$	$\Box X \supset X$
$t{:}X \supset !t{:}t{:}X$	$\Box X \supset \Box\Box X$

LP is a counterpart of S4. Several algorithms have been developed to compute a normal realization for a modal formula X from a cut-free sequent calculus proof of X.

There are well-known sublogics of S4, namely K, K4, T, that are axiomatized by dropping schemes from the S4 list above. Dropping the corresponding axioms from the LP list yields justification counterparts using essentially the same realization algorithm that works for LP, but omitting parts that are no longer relevant.

A justification counterpart for a modal logic is generally not unique. If we replace the second LP axiom above with $t{:}X \supset f(t){:}g(t){:}X$ (where f and g are one-place function symbols) we get a different justification counterpart for S4. On the other hand, replacing $t{:}X \supset !t{:}t{:}X$ with $h(t,u){:}X \supset t{:}u{:}X$ almost certainly does not yield an S4 justification counterpart, though I have no proof of this. In an obvious sense using $t{:}X \supset f(t){:}g(t){:}X$ instead of $t{:}X \supset !t{:}t{:}X$ is more general since we get the latter by specializing $g(t)$ to be the identity function. LP was formulated as it was because of motivation coming from the project of providing an arithmetic semantics for intuitionistic logic.

5.2 S5, a very familiar modal logic

The modal logic S5 is perhaps the most commonly applied modal logic. It is axiomatized by adding a *negative introspection* scheme to S4, $\neg\Box X \supset \Box\neg\Box X$.

A justification counterpart appears in the literature, [26, 27], and goes under the name JT45. An additional one-place function symbol, ?, is added to the language (as with !, parentheses are dropped), and an axiom scheme $\neg t{:}X \supset ?t{:}(\neg t{:}X)$ is adopted. As with S4, this counterpart is not unique.

5.3 $\mathsf{K4}^3$, a somewhat obscure modal logic

The family $\mathsf{K4}^n$ is from [7]. For each n, $\mathsf{K4}^n$ extends K with the schema $\Box X \supset \Box^n X$. The logics are complete with respect to models based on frames $\langle \mathcal{G}, \mathcal{R}\rangle$ meeting a kind of extended transitivity condition: $\Gamma\mathcal{R}^n\Delta$ implies $\Gamma\mathcal{R}\Delta$, where $\Gamma\mathcal{R}^n\Delta$ means that for some $\Gamma_1, \ldots, \Gamma_{n-1}$, $\Gamma\mathcal{R}\Gamma_1\mathcal{R}\ldots\mathcal{R}\Gamma_{n-1}\mathcal{R}\Delta$.

We concentrate on the $n = 3$ case as illustrative of the family in general: $\mathsf{K4}^3$ has the schema $\Box X \supset \Box\Box\Box X$. As with S4, there is more than one justification counterpart for $\mathsf{K4}^3$, with the following being closest to LP. Let ! and !! be one-place function symbols. Then a justification counterpart for $\mathsf{K4}^3$ results by adding the axiom schema $t{:}X \supset !!t{:}!t{:}t{:}X$ to J_0. We call this justification logic $\mathsf{J4}^3$.

5.4 S4.2, a somewhat better known modal logic

Axiomatically S4.2 extends S4 with the axiom scheme $\Diamond\Box X \supset \Box\Diamond X$. This logic is complete with respect to the family of models based on frames that are reflexive, transitive, and *convergent*, meaning that whenever $\Gamma_1\mathcal{R}\Gamma_2$ and $\Gamma_1\mathcal{R}\Gamma_3$, there is some Γ_4 such that $\Gamma_2\mathcal{R}\Gamma_4$ and $\Gamma_3\mathcal{R}\Gamma_4$. We briefly present part of a soundness argument, because this will be instructive later on.

Suppose we have a modal model $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{V}\rangle$ whose frame is reflexive, transitive, and convergent. (Notation is defined in Section 6.1, but should be clear now.) Suppose that $\Gamma_1 \in \mathcal{G}$ and $\mathcal{M}, \Gamma_1 \Vdash \Diamond\Box X$ but $\mathcal{M}, \Gamma_1 \nVdash \Box\Diamond X$; we derive a contradiction. By the first item, for some $\Gamma_2 \in \mathcal{G}$ with $\Gamma_1\mathcal{R}\Gamma_2$, $\mathcal{M}, \Gamma_2 \Vdash \Box X$, and by the second item, for some $\Gamma_3 \in \mathcal{G}$ with $\Gamma_1\mathcal{R}\Gamma_3$, $\mathcal{M}, \Gamma_3 \nVdash \Diamond X$. Since the frame is convergent, there is some Γ_4 with $\Gamma_2\mathcal{R}\Gamma_4$ and $\Gamma_3\mathcal{R}\Gamma_4$. But then we have both $\mathcal{M}, \Gamma_4 \Vdash X$ and $\mathcal{M}, \Gamma_4 \nVdash X$, our contradiction.

For a justification counterpart we build on LP, adding two function symbols, f and g, each two-place, and adopting the following axiom scheme.

$$\neg f(t, u){:}\neg t{:}X \supset g(t, u){:}\neg u{:}\neg X \tag{1}$$

We call this justification logic J4.2.

A few words about intuitions concerning (1) might be in order. In LP, because of the axiom scheme $t{:}X \supset X$, we have provability of $(t{:}X \wedge u{:}\neg X) \supset \bot$, for any t and u, and thus provability of $\neg t{:}X \vee \neg u{:}\neg X$. In any context one of the disjuncts must hold. Axiom scheme (1) is equivalent to $f(t, u){:}\neg t{:}X \vee g(t, u){:}\neg u{:}\neg X$. Informally, this says that in any context we have means for computing a justification for the disjunct that holds. It is a strong assumption, but not implausible in some cases.

We have noted that justification counterparts may not be unique (in Section 5.1). We have another example of this here. A semantics for J4.2 will be given in Section 6.3.2. If condition 4 found there for evidence functions is modified in the obvious ways, it is easy to check that both $\neg f(u){:}\neg t{:}X \supset g(t){:}\neg u{:}\neg X$ and $\neg f(t){:}\neg t{:}X \supset g(u){:}\neg u{:}\neg X$ are sound and complete with respect to the semantics.

5.5 Generating Examples

We have seen several cases where there was more than one justification counterpart for a modal logic: S4, $\mathsf{K4}^3$ and S4.2 have this feature, and other examples could be mentioned. It appears that there is always a most general candidate for a justification counterpart, though it is not entirely clear what this means. We say briefly how it is produced, but this is something that needs further investigation.

Suppose we have a modal axiom scheme; say we use $(\Box P \vee \Box Q) \supset \Box\Box(P \vee Q)$ as a concrete example. Assign to each negative occurrence of $\Box$ a unique justification variable. In this case we have the occurrences before P and Q, and we assign x_1 and x_2. Replace these occurrences with the variables, as an intermediate step; for our example we get $(x_1{:}P \vee x_2{:}Q) \supset \Box\Box(P \vee Q)$. Next, assign to each positive occurrence of $\Box$ a unique function symbol with the same number of arguments as we have variables, and replace the positive occurrences with these function symbols applied to the variables. For our example we use f and g, each two-place, and we get $(x_1{:}P \vee x_2{:}Q) \supset f(x_1, x_2){:}g(x_1, x_2){:}(P \vee Q)$. All substitution instances of this (Definition 3.8) give us the justification scheme we would use as a counterpart of the original modal formula.

This does not always work. It does not work with Gödel-Löb logic, GL, for instance. When it does work it seems to be a most general counterpart, in a sense that has not yet been properly defined or investigated yet. It does apply to all the examples considered in this paper, including the infinite family of Geach logics investigated in Section 8. The example we just looked at is a Geach logic.

5.6 Final observations

We defined modal/justification counterparts in Section 4. For modal KL and justification JL to be counterparts, both items of Definition 4.1 must be satisfied. It is important to note that the two are quite independent of each other. We now have examples that illustrate this. In the following, our observations work for any constant specification, whether or not internalization holds.

First, in Definition 4.1, we can have condition 1 without condition 2. Consider the pair LP and S5. If X is a theorem of LP, X° will be a theorem of S5 because LP embeds into S4, and this is a sublogic of S5. But there are theorems of S5 that have no realization in LP; $\neg\Box X \supset \Box\neg\Box X$ is an obvious example. If $\neg\Box X \supset \Box\neg\Box X$ had a realization in LP, which is a counterpart of S4, the forgetful projection of this realization would be a theorem of S4, which means $\neg\Box X \supset \Box\neg\Box X$ would be an S4 theorem.

Next, in Definition 4.1 we can have condition 2 without condition 1. Consider the pair JT45 and S4. The forgetful functor does not map the set of theorems of JT45 into the set of theorems of S4; $\neg t{:}X \supset ?t{:}(\neg t{:}X)$ maps outside of S4. But it is the case that every theorem of S4 has a realization in JT45, because S4 and LP are counterparts, and LP is a sublogic of JT45.

6 Possible World Semantics

Several semantics have been introduced for justification logics: *arithmetic semantics*, due to Artemov, [1]; *Mkrtychev semantics*, [23]; a possible world semantics commonly known as *Fitting models*, [9]; and *modular models*, [3, 21]. Fitting models play a fundamental role in the non-constructive realization proof, a proof which originated in [9], but which is presented here with modifications coming from [13, 14, 15].

6.1 Fitting Models

Our justification semantics builds on top of standard Kripke modal semantics. As usual, a *frame* is a directed graph, $\langle\mathcal{G},\mathcal{R}\rangle$, with $\mathcal{G}$ being the nodes or possible worlds and $\mathcal{R}$ being the directed edges, or accessibility relation. One turns a frame into a modal model by specifying which atomic formulas are true at which possible worlds; more specifically there is a mapping $\mathcal{V}$ from atoms to subsets of $\mathcal{G}$ so that $\mathcal{V}(\bot) = \emptyset$. A propositional letter P is true at $\Gamma \in \mathcal{G}$ if $\Gamma \in \mathcal{V}(P)$.

For a modal model, truth of formulas is calculated at each possible world in the familiar way. If $\mathcal{M} = \langle\mathcal{G},\mathcal{R},\mathcal{V}\rangle$, we write $\mathcal{M},\Gamma \Vdash X$ to indicate that modal formula X is true at possible world Γ of $\mathcal{G}$. We list the familiar conditions for later reference.

1. if A is atomic, $\mathcal{M},\Gamma \Vdash A$ if $A \in \mathcal{V}(\Gamma)$
2. $\mathcal{M},\Gamma \Vdash X \supset Y$ if and only if $\mathcal{M},\Gamma \nVdash X$ or $\mathcal{M},\Gamma \Vdash Y$ (and similarly for other connectives)
3. $\mathcal{M},\Gamma \Vdash \Box X$ if and only if $\mathcal{M},\Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$

Now assume the language is not modal, but involves justification terms as in Section 3.2. Fitting models have an additional piece of machinery, syntactic in nature: an *evidence function*. Such a function, $\mathcal{E}$, maps justification terms and formulas to sets of possible worlds. Informally, $\Gamma \in \mathcal{E}(t,X)$ means that, at possible world Γ, t is relevant evidence for the truth of X. Relevant evidence need not be conclusive. Think of it as evidence that might be admitted in court, whose truth can then be discussed during the court proceedings. A Fitting model for a justification logic is a structure $\mathcal{M} = \langle\mathcal{G},\mathcal{R},\mathcal{V},\mathcal{E}\rangle$, where $\langle\mathcal{G},\mathcal{R},\mathcal{V}\rangle$ is as in Kripke models, and $\mathcal{E}$ is an evidence function. Truth conditions 1 and 2 are the same as above. Condition 3 is not pertinent since $\Box$ is not part of a justification language. It is replaced with the following.

4. $\mathcal{M}, \Gamma \Vdash t{:}X$ if and only if
   (a) $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$
   (b) $\Gamma \in \mathcal{E}(t, X)$

Thus $t{:}X$ is true at a possible world if X is true at all accessible worlds *and* t is relevant evidence for X at Γ.

With modal models various *frame conditions* are imposed, transitivity, symmetry, convergence, etc. This is done with Fitting models in exactly the same way. One also puts conditions on Fitting model evidence functions. We have assumed $+$ and $\cdot$ are always present axiomatically; correspondingly we always require the following semantically. There may be additional operations and conditions for particular justification logics.

5. $\mathcal{E}(s, X \supset Y) \cap \mathcal{E}(t, X) \subseteq \mathcal{E}(s \cdot t, Y)$
6. $\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$

A justification formula X is valid in a Fitting model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ provided, for each $\Gamma \in \mathcal{G}$, $\mathcal{M}, \Gamma \Vdash X$, and is valid in a class of Fitting models if it is valid in every member of the class.

A Fitting model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ *meets a constant specification* $\mathcal{CS}$ provided, for each $t{:}X \in \mathcal{CS}$, $\mathcal{E}(t, X) = \mathcal{G}$. It is easy to show that if all axioms of a particular axiomatic system are valid in $\mathcal{M}$, and if $\mathcal{M}$ meets constant specification $\mathcal{CS}$ for that axiom system, then all members of $\mathcal{CS}$ must be valid in $\mathcal{M}$.

If J is a justification logic and $\mathcal{CS}$ is a constant specification for it, we say a Fitting model $\mathcal{M}$ is a model for $\mathsf{J}(\mathcal{CS})$ if all axioms of J are valid in $\mathcal{M}$ and $\mathcal{M}$ meets $\mathcal{CS}$. It is easy to check that if $\mathcal{M}$ is a model for $\mathsf{J}(\mathcal{CS})$ then all theorems of $\mathsf{J}(\mathcal{CS})$ are valid in $\mathcal{M}$.

6.2 Special Kinds of Models

Special conditions are often imposed on Fitting models. The first we discuss is being *fully explanatory*, originating in [9]. Informally, a model is fully explanatory if anything that is necessary has a reason. More properly, a Fitting model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is fully explanatory provided that, whenever $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, there is some justification term t so that $\mathcal{M}, \Gamma \Vdash t{:}X$. This has a certain intuitive desirability, but in fact it will play no role in what we do here.

The second special condition is that of having a *strong evidence function*, something we believe originated in [28]. Condition 4 in Section 6.1 has two parts. It says $\mathcal{M}, \Gamma \Vdash t{:}X$ provided (a) $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, and (b) $\Gamma \in \mathcal{E}(t, X)$. A strong evidence function is one for which condition (b) implies condition (a), and is thus sufficient by itself. Informally this amounts to assuming that we are not dealing with mere *relevant* evidence, but with *conclusive* evidence. This condition will play an important role here.

Fully explanatory and strong evidence seem to behave very differently. Given a class $\mathcal{F}$ of Fitting models specified by conditions on accessibility and evidence, those that are fully explanatory generally form a proper subclass. There are no known examples of axiom systems whose soundness can be shown with respect to models in that subclass, but not for $\mathcal{F}$ itself. The completeness proofs considered here are justification versions of canonical model constructions, and it turns out that canonical justification models are always fully explanatory, provided an axiomatically appropriate constant specification is used. Thus being fully explanatory is a nice feature that is almost free.

The story with strong evidence functions is quite another thing, however. Once again, canonical justification models always have strong evidence functions. But there are examples where no natural class $\mathcal{F}$ of Fitting models exists with respect to which a particular justification logic is sound, without also imposing the requirement that we have a strong evidence function. A justification counterpart of S5 is such a case, [28], and so is J4.2 from Section 5.4. The bad news is that it can be quite difficult to exhibit particular models with strong evidence functions, though without the restriction it is much less of a problem. The good news is that this will not matter when we come to realization results.

6.3 Examples Again

Fitting models for LP and its sublogics are well-known, so we omit discussion of them here. Instead we give examples that involve the logics $\mathsf{K4}^3$ and $\mathsf{J4}^3$ from Section 5.3, and S4.2 and J4.2 from Section 5.4.

6.3.1 The Logic $\mathsf{J4}^3$

For $\mathsf{J4}^3$, introduced in Section 5.3, we consider the following class of Fitting models, which we simply call $\mathsf{J4}^3$ *models*. These are models $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ that meet the conditions below.

Following [7], for $\Gamma, \Delta \in \mathcal{G}$, let $\Gamma \mathcal{R}^3 \Delta$ mean that there are $\Delta_1, \Delta_2 \in \mathcal{G}$ so that $\Gamma \mathcal{R} \Delta_1 \mathcal{R} \Delta_2 \mathcal{R} \Delta$. A modal model for $\mathsf{K4}^3$ must meet the condition: if $\Gamma \mathcal{R}^3 \Delta$ then $\Gamma \mathcal{R} \Delta$, a generalization of transitivity. We require frames of $\mathsf{J4}^3$ models to meet the same condition.

Conditions for the evidence function generalize those for ! in a straightforward way.

$$\Gamma \in \mathcal{E}(t, X) \quad \Longrightarrow \quad \begin{cases} \Gamma \in \mathcal{E}(!!t, !t{:}t{:}X) & (a) \\ \Gamma \mathcal{R} \Delta \Longrightarrow \Delta \in \mathcal{E}(!t, t{:}X) & (b) \\ \Gamma \mathcal{R}^2 \Delta \Longrightarrow \Delta \in \mathcal{E}(t, X) & (c) \end{cases}$$

We verify that $t{:}X \supset !!t{:}!t{:}t{:}X$ is valid in all $\mathsf{J4}^3$ models. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ be such a model, $\Gamma \in \mathcal{G}$, and suppose $\mathcal{M}, \Gamma \Vdash t{:}X$ but $\mathcal{M}, \Gamma \not\Vdash !!t{:}!t{:}t{:}X$; we derive a contradiction

Since $\mathcal{M}, \Gamma \Vdash t{:}X$, we must have $\Gamma \in \mathcal{E}(t, X)$. Because we have this and (a), but $\mathcal{M}, \Gamma \not\Vdash !!t{:}!t{:}t{:}X$, there must be some $\Delta_1 \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta_1$, and $\mathcal{M}, \Delta_1 \not\Vdash !t{:}t{:}X$.

Since $\mathcal{M}, \Gamma \Vdash t{:}X$ and $\Gamma \mathcal{R} \Delta_1$, by (b) we must have $\Delta_1 \in \mathcal{E}(!t, t{:}X)$. By this, and $\mathcal{M}, \Delta_1 \not\Vdash !t{:}t{:}X$, there must be some $\Delta_2 \in \mathcal{G}$ with $\Delta_1 \mathcal{R} \Delta_2$, and $\mathcal{M}, \Delta_2 \not\Vdash t{:}X$.

Again, since $\mathcal{M}, \Gamma \Vdash t{:}X$ and $\Gamma \mathcal{R}^2 \Delta_2$, by (c) we must have $\Delta_2 \in \mathcal{E}(t, X)$. Then since $\mathcal{M}, \Delta_2 \not\Vdash t{:}X$, there must be some $\Delta \in \mathcal{G}$ with $\Delta_2 \mathcal{R} \Delta$ and $\mathcal{M}, \Delta \not\Vdash X$.

Now by assumption, $\mathcal{M}, \Gamma \Vdash t{:}X$. Also $\Gamma \mathcal{R}^3 \Delta$ so by the frame condition, $\Gamma \mathcal{R} \Delta$. It follows that $\mathcal{M}, \Delta \Vdash X$, and we have a contradiction.

6.3.2 The logic J4.2

J4.2 was introduced axiomatically in Section 5.4. The key axiom is $\neg f(t, u){:}\neg t{:}X \supset g(t, u){:}\neg u{:}\neg X$. Semantically, $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is a J4.2 Fitting model if it meets the following special conditions.

1. The frame $\langle \mathcal{G}, \mathcal{R} \rangle$ is reflexive, transitive, and convergent, as with S4.2 in Section 5.4.
2. $\mathcal{E}(t, X) \subseteq \mathcal{E}(!t, t{:}X)$
3. $\Gamma \in \mathcal{E}(t, X)$ and $\Gamma \mathcal{R} \Delta$ implies $\Delta \in \mathcal{E}(t, X)$.
4. $\mathcal{E}(f(t, u), \neg t{:}X) \cup \mathcal{E}(g(t, u), \neg u{:}\neg X) = \mathcal{G}$

5. $\mathcal{E}$ is a *strong* evidence function, as discussed in Section 6.2.

Items 1, 2 and 3 together give soundness of the usual LP axiom schemas $t{:}X \supset X$ and $t{:}X \supset !t{:}t{:}X$. We only verify soundness for (1), $\neg f(t,u){:}\neg t{:}X \supset g(t,u){:}\neg u{:}\neg X$. Suppose the formula fails at Γ in model $\mathcal{M}$; we derive a contradiction. If it fails at Γ then $\mathcal{M},\Gamma \not\Vdash f(t,u){:}\neg t{:}X$ and $\mathcal{M},\Gamma \not\Vdash g(t,u){:}\neg u{:}\neg X$. Because $\mathcal{E}$ is a strong evidence function, it must be that $\Gamma \notin \mathcal{E}(f(t,u), \neg t{:}X)$ and $\Gamma \notin \mathcal{E}(g(t,u), \neg u{:}\neg X)$. But this contradicts condition 4. Note that this argument is extremely trivial when compared to the argument in Section 5.4. The assumption of a strong evidence function is a strong assumption indeed.

This material will be revisited in a much broader setting, in Section 8. A separate section is required because of both length and significance.

6.4 A Remark About Strong Evidence Functions

Imposing a strong evidence assumption can sometimes make other requirements on a model redundant. For instance, if we restrict the semantics for $\mathsf{J4}^3$ in Section 6.3.1 to models with strong evidence functions, two of the three conditions on evidence functions are unnecessary. Assume, for this discussion, that $\Gamma \in \mathcal{E}(t,X)$. We retain consequence (1) that $\Gamma \in \mathcal{E}(!!t, !t{:}t{:}X)$, but we can drop (2) and (3) by the following reasoning.

Suppose also that $\Gamma\mathcal{R}\Delta$. We are still assuming $\Gamma \in \mathcal{E}(!!t, !t{:}t{:}X)$. Since $\mathcal{E}$ is a strong evidence function, it follows that $\mathcal{M},\Gamma \Vdash !!t{:}!t{:}t{:}X$. Then since $\Gamma\mathcal{R}\Delta$, we must have that $\mathcal{M},\Delta \Vdash !t{:}t{:}X$, and hence $\Delta \in \mathcal{E}(!t, t{:}X)$. Redundancy of the third condition has a similar argument.

For J4.2 in Section 6.3.2, a similar argument shows that condition 3 is unnecessary.

These are not meant to be deep observations, but they may be of some significance.

7 Canonical Models and Completeness

Canonical models for modal logics have possible worlds that are maximally consistent sets of formulas, with accessibility defined in a standard way. Roughly speaking, logics for which canonical models suffice to establish completeness are canonical logics. Not all modal logics are canonical, but those considered in Section 5 are, as are most common modal logics. Canonical methodology carries over to justification logics and plays a very fundamental role, as we will see. So for reference, even though this should be very familiar, we quickly review the ideas in the modal setting before moving to justification logics.

7.1 Canonical Modal Logics

Let KL be a normal modal logic. Call a set S of formulas *KL-consistent* provided, for no $X_1,\ldots,X_n \in S$ is $(X_1 \wedge \ldots \wedge X_n) \supset \bot$ valid in KL. Using Lindenbaum's Lemma, every KL-consistent set can be extended to a maximally KL-consistent set.

Let $\mathcal{G}$ be the set of all maximally KL-consistent sets. For a set S of formulas, let $S^\sharp = \{X \mid \Box X \in S\}$. For $\Gamma,\Delta \in \mathcal{G}$, let $\Gamma\mathcal{R}\Delta$ provided $\Gamma^\sharp \subseteq \Delta$. This gives us a frame, $\langle\mathcal{G},\mathcal{R}\rangle$. Finally, for atomic P set $\Gamma \in \mathcal{V}(P)$ if $P \in \Gamma$ (so that P is true at $\Gamma \in \mathcal{G}$ if $P \in \Gamma$) . This completely specifies a model $\mathcal{M} = \langle\mathcal{G},\mathcal{R},\mathcal{V}\rangle$, called the *canonical* model for KL.

The key fact is the *Truth Lemma*: for any $\Gamma \in \mathcal{G}$ and for any formula X, $X \in \Gamma \Leftrightarrow \mathcal{M},\Gamma \Vdash X$. The proof is by induction on the degree of X, is standard, and is widely known. It follows that a canonical model is a universal counter-model. For, suppose X is not valid in KL. Then $\{\neg X\}$ is KL-consistent, and so can be extended to a maximally consistent set Γ. Then $\Gamma \in \mathcal{G}$ and, by the Truth Lemma, $\mathcal{M},\Gamma \not\Vdash X$.

It can happen that the canonical model is not actually a model for its logic, but if it is, completeness is immediate. Thus canonical modal logics have completeness proofs that are essentially uniform across a broad range of logics. All noted above, all logics in Section 5 are canonical.

7.2 Canonical Justification Models

Canonical models for justification logics were introduced in [9]. We repeat things here because canonical justification models play a central role in proving not just completeness but realizability for a broad range of logics.

Let JL be some axiomatically formulated justification logic. Then it extends the system J_0 from Section 3.2. Also let $\mathcal{CS}$ be a constant specification for JL. We say S is *$\mathsf{JL}(\mathcal{CS})$-inconsistent* if $S \vdash_{\mathsf{JL}(\mathcal{CS})} \bot$, and S is *$\mathsf{JL}(\mathcal{CS})$-consistent* if it is not $\mathsf{JL}(\mathcal{CS})$-inconsistent.

The *canonical model* for $\mathsf{JL}(\mathcal{CS})$, $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is defined as follows.

- $\mathcal{G}$ is the set of all maximally $\mathsf{JL}(\mathcal{CS})$-consistent sets of formulas.
- If $\Gamma \in \mathcal{G}$, let $\Gamma^\sharp = \{X \mid t{:}X \in \Gamma$ for some justification term $t\}$. For $\Gamma, \Delta \in \mathcal{G}$, $\Gamma \mathcal{R} \Delta$ if $\Gamma^\sharp \subseteq \Delta$.
- For atomic A, $\Gamma \in \mathcal{V}(A)$ if $A \in \Gamma$.
- $\Gamma \in \mathcal{E}(t, X)$ if $t{:}X \in \Gamma$.

This completes the definition of canonical model. To show it is a Fitting model we must show it meets conditions 5 and 6 from Section 6.1. These are simple. Condition 5 is that $\mathcal{E}(s, X \supset Y) \cap \mathcal{E}(t, X) \subseteq \mathcal{E}(s \cdot t, Y)$. Well, suppose $\Gamma \in \mathcal{E}(s, X \supset Y)$ and $\Gamma \in \mathcal{E}(t, X)$. By definition of $\mathcal{E}$, $s{:}(X \supset Y) \in \Gamma$ and $t{:}X \in \Gamma$. Since $\mathsf{JL}(\mathcal{CS})$ axiomatically extends J_0, $s{:}(X \supset Y) \supset (t{:}X \supset s \cdot t{:}Y)$ is an axiom. Since Γ is maximally $\mathsf{JL}(\mathcal{CS})$ consistent, it follows that $s \cdot t : Y \in \Gamma$, and hence $\Gamma \in \mathcal{E}(s \cdot t, Y)$. Condition 6 is $\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$, and is treated the same way using JL_0 axioms $s{:}X \supset s + t{:}X$ and $t{:}X \supset s + t{:}X$.

As with modal canonical models, the key item to show is a truth lemma. Curiously, for justification logics the proof is simpler than in the modal case.

Theorem 7.1 (Truth Lemma) *In the canonical justification model $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$, for any $\Gamma \in \mathcal{G}$ and any formula X,*

$$\mathcal{M}_{\mathsf{JL}(\mathcal{CS})}, \Gamma \Vdash X \iff X \in \Gamma \tag{2}$$

Proof The proof is by induction on the degree of X. The atomic case is by definition. Propositional connective cases are by the usual argument, making use of maximal consistency of Γ. This leaves the justification case. Assume (2) holds for formulas simpler than X.

Suppose $t{:}X \in \Gamma$. By definition of $\mathcal{E}$ we have $\Gamma \in \mathcal{E}(t, X)$. Let Δ be an arbitrary member of $\mathcal{G}$ with $\Gamma \mathcal{R} \Delta$. Then $\Gamma^\sharp \subseteq \Delta$, so $X \in \Delta$, and by the induction hypothesis, $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})}, \Delta \Vdash X$. Then $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})}, \Gamma \Vdash t{:}X$.

Suppose $t{:}X \notin \Gamma$. Then $\Gamma \notin \mathcal{E}(t, X)$, so $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})}, \Gamma \nVdash t{:}X$. ■

The evidence function, $\mathcal{E}$, in the canonical justification model, is a strong evidence function. For, suppose $\Gamma \in \mathcal{E}(t, \Gamma)$. Then, by definition of $\mathcal{E}$, $t{:}X \in \Gamma$ so by Theorem 7.1, $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})}, \Gamma \Vdash t{:}X$. And then $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$.

It is also the case that if $\mathcal{CS}$ is axiomatically appropriate, the canonical $\mathsf{JL}(\mathcal{CS})$ model must be fully explanatory. We will not need this here, and so omit the proof.

As usual, the canonical model is a universal counter model, and by the usual argument. Then the completeness question reduces to the following—a direct analog of what happens in modal logic. We may have some class $\mathcal{F}_{\mathsf{JL}(\mathcal{CS})}$ of Fitting models, and we want to prove completeness of $\mathsf{JL}(\mathcal{CS})$ relative to this class. If it turns out that the canonical model $\mathcal{M}_{\mathsf{JL}(\mathcal{CS})}$ is in $\mathcal{F}_{\mathsf{JL}(\mathcal{CS})}$, we have succeeded. Typically this comes down to showing the frame and the evidence function have appropriate properties.

In modal logic there are well-known examples of completeness proofs that do not take the canonical model route (and cannot). Gödel-Löb logic is a prominent such logic. Other techniques are available for cases like these. For justification logics investigated thus far, *all* axiomatic completeness results make use of canonical models directly. This phenomenon needs to be investigated further.

7.3 An Example, $\mathsf{J4}^3$

In Section 5.3 a logic $\mathsf{J4}^3$ was introduced axiomatically, and a semantics was given in Section 6.3.1. Here we check that the canonical justification model for this is of the right semantic type, and so completeness follows. We also discussed $\mathsf{J4.2}$ in Section 5.4 with a semantics given in Section 6.3.2. We postpone considering this logic further until Sections 8 and especially 8.3, where it will be treated as one of a very general class of logics.

There are two parts to showing the canonical model for $\mathsf{J4}^3$ meets appropriate conditions. First, the evidence function should have the right properties. Second, the underlying frame should meet appropriate frame conditions. These are separate arguments.

Evidence Function Conditions We saw in Section 6.4 that when dealing with a strong evidence function, as the canonical model has, only one of the three evidence function conditions needs to be established. We must show $\Gamma \in \mathcal{E}(t, X) \Longrightarrow \Gamma \in \mathcal{E}(!!t, !t{:}t{:}X)$. From the definition of evidence function in canonical models, this is equivalent to $t{:}X \in \Gamma \Longrightarrow !!t{:}!t{:}t{:}X \in \Gamma$. Since $t{:}X \supset !!t{:}!t{:}t{:}X$ is an axiom of $\mathsf{J4}^3$ and possible worlds of canonical models are maximally consistent, this is immediate.

Frame Conditions We must show that, in the canonical model for $\mathsf{J4}^3$, if $\Gamma\mathcal{R}^3\Delta$ then $\Gamma\mathcal{R}\Delta$. Well, suppose $\Gamma\mathcal{R}\Delta_1\mathcal{R}\Delta_2\mathcal{R}\Delta$. Then $\Gamma^\sharp \subseteq \Delta_1$, $\Delta_1^\sharp \subseteq \Delta_2$, and $\Delta_2^\sharp \subseteq \Delta$. We must show $\Gamma^\sharp \subseteq \Delta$. Suppose $X \in \Gamma^\sharp$. Then for some justification term t, $t{:}X \in \Gamma$. Since $t{:}X \supset !!t{:}!t{:}t{:}X$ is an axiom, by maximal consistency of Γ, $!!t{:}!t{:}t{:}X \in \Gamma$. But then $!t{:}t{:}X \in \Delta_1$, and so $t{:}X \in \Delta_2$, and thus $X \in \Delta$.

8 Geach Logics

The modal logic $\mathsf{S4.2}$ incorporates an axiom, $\Diamond\Box X \supset \Box\Diamond X$, introduced by Peter Geach. The axiom is commonly known as G. In [22] this axiom was incorporated into an infinite family of modal logics that included many common logics, and it was shown that all members of the family were canonical. This was a significant forerunner of Sahlquist's later work, but the Sahlquist family goes beyond our considerations here. We follow notation from [7], and present the family of what we call Geach logics (they are also called *Lemmon-Scott logics*). A significant fact about them from the present standpoint is that *they all have justification counterparts*, with realization theorems connecting them. In particular, this tells us that we are dealing with a phenomenon that is not rare—infinitely many modal logics have justification counterparts.

Definition 8.1 (Geach Modal Logics) Formulas of the form $\Diamond^k\Box^l X \supset \Box^m\Diamond^n X$ where $k, l, m, n \geq 0$ are $\mathsf{G}^{k,l,m,n}$ formulas. All such formulas are *Geach formulas*. *Geach modal logics* are those axiomatized over K using these axiom schemes.

Many standard modal logics are Geach logics, as the table of familiar axiom schemes below shows.

$$\begin{array}{rcll} \mathsf{D} & = & \mathsf{G}^{0,1,0,1} & \Box X \supset \Diamond X \\ \mathsf{T} & = & \mathsf{G}^{0,1,0,0} & \Box X \supset X \\ \mathsf{B} & = & \mathsf{G}^{0,0,1,1} & X \supset \Box\Diamond X \\ \mathsf{4} & = & \mathsf{G}^{0,1,2,0} & \Box X \supset \Box\Box X \\ \mathsf{5} & = & \mathsf{G}^{1,0,1,1} & \Diamond X \supset \Box\Diamond X \\ \mathsf{G} & = & \mathsf{G}^{1,1,1,1} & \Diamond\Box X \supset \Box\Diamond X \end{array}$$

Geach logics are canonical. Semantically the formula $\Diamond^k\Box^l X \supset \Box^m\Diamond^n X$ corresponds to a generalized convergence condition on frames: if $\Gamma_1\mathcal{R}^k\Gamma_2$ and $\Gamma_1\mathcal{R}^m\Gamma_3$ then for some Γ_4, $\Gamma_2\mathcal{R}^l\Gamma_4$ and $\Gamma_3\mathcal{R}^n\Gamma_4$. A proof of this can be found in [7]. Essentially, what we do here is transfer the modal argument from [7] to the justification setting, paying careful attention to details.

Much of the material that we need related to Geach logics consists of rather technical results, and we begin with this in Sections 8.1 and 8.2. In Section 8.3 we show completeness for all justification counterparts of Geach logics. For the special case of $\mathsf{J4.2}$ the parameters k, j, l and m can all be set to 1 in Section 8.3 and much of the material can be simply skipped. Doing this might help in understanding what is going on in the general case.

8.1 Notation

We used $\Box^n$ and $\mathcal{R}^n$ above. When working with justification terms things are more complicated since each box of $\Box^n A$ might be realized by a different justification term. We introduce the following vector-style notation to help formulate this.

Definition 8.2 Let $\mathcal{T}$ be the set of justification terms of a justification logic. We define $\mathcal{T}^h = \{\langle t_1, \ldots, t_h\rangle \mid t_1, \ldots, t_h \in \mathcal{T}\}$. We allow h to be 0; the only member of $\mathcal{T}^0$ is $\langle\,\rangle$, the 0-tuple. If $\vec{t} \in \mathcal{T}^h$, we define $\vec{t}{:}X$ as follows. First, $\langle\,\rangle{:}X = X$. Second, $\langle t_h, \ldots, t_1\rangle{:}X$ is $t_h{:}\ldots t_1{:}X$.

For a set of modal formulas Γ, we used the notation $\Gamma^\sharp$ in Section 7.1, $\Gamma^\sharp = \{X \mid \Box X \in \Gamma\}$. Likewise in justification setting of Section 7.2 we used $\Gamma^\sharp = \{X \mid t{:}X \in \Gamma$ for some justification term $t\}$. The following extends this to bring vector notation into the picture, and also defines a kind of backward version.

Definition 8.3 Let Γ be a set of justification formulas and $h \geq 0$.

1. $\Gamma^{\sharp h} = \{X \mid \vec{t}{:}X \in \Gamma$ for some $\vec{t} \in \mathcal{T}^h\}$
2. $\Gamma^{\flat h} = \{\neg\vec{t}{:}\neg X \mid X \in \Gamma$ and $\vec{t} \in \mathcal{T}^h\}$

Note that $\Gamma^{\sharp 0}$ is simply Γ. $\Gamma^{\sharp 1}$ coincides with $\Gamma^\sharp$ as used earlier, and will generally be written that way for simplicity.

8.2 Basic Results

For this section, assume J is an axiomatically formulated justification logic with a constant specification (which is suppressed in the notation to make things more readable). We assume J has the internalization property. Also we assume $\mathcal{T}$ is the set of justification terms of J. All mention of consistency is with respect to J.

Theorem 8.4 (Lifting Lemma Generalized) *Suppose $X_1, \ldots, X_n \vdash_{\mathsf{J}} Y$. Then for any members $\vec{t_1}, \ldots, \vec{t_h}$ of $\mathcal{T}^h$ there is some $\vec{u}$ in $\mathcal{T}^h$ so that $\vec{t_1}{:}X_1, \ldots, \vec{t_h}{:}X_h \vdash_{\mathsf{J}} \vec{u}{:}Y$.*

Proof This follows by repeated use of the Lifting Lemma, Theorem 3.7. ∎

Theorem 8.5 *Let Γ, Δ be maximal consistent sets. Then*

$$\Gamma^{\sharp h} \subseteq \Delta \Longleftrightarrow \Delta^{\flat h} \subseteq \Gamma.$$

Proof We follow the modal presentation of [7], Theorem 4.29.

Left to right. Assume $\Gamma^{\sharp h} \subseteq \Delta$. Suppose $\neg\vec{t}{:}\neg X \in \Delta^{\flat h}$. We show $\neg\vec{t}{:}\neg X \in \Gamma$. By our supposition, $X \in \Delta$. By consistency, $\neg X \notin \Delta$, hence $\neg X \notin \Gamma^{\sharp h}$. Then by definition, $\vec{t}{:}\neg X \notin \Gamma$, so by maximality, $\neg\vec{t}{:}\neg X \in \Gamma$.

Right to left. Assume $\Delta^{\flat h} \subseteq \Gamma$. Suppose $X \in \Gamma^{\sharp h}$, but also that $X \notin \Delta$. We derive a contradiction. By our supposition, $\vec{t}{:}X \in \Gamma$ for some $\vec{t} \in \mathcal{T}^h$. Since $X \vdash_{\mathsf{J}} \neg\neg X$, by the Lifting Lemma Generalized there is some $\vec{u}$ so that $\vec{t}{:}X \vdash_{\mathsf{J}} \vec{u}{:}\neg\neg X$. Then by maximal consistency of Γ, $\vec{u}{:}\neg\neg X \in \Gamma$. Also $X \notin \Delta$ so by maximality $\neg X \in \Delta$, so $\neg\vec{u}{:}\neg\neg X \in \Delta^{\flat h}$ and hence by our initial assumption, $\neg\vec{u}{:}\neg\neg X \in \Gamma$, contradicting consistency of Γ.

A remark. If $n = 0$ the theorem reduces to $\Gamma \subseteq \Delta$ iff $\Delta \subseteq \Gamma$, which is true because these are *maximal* consistent sets. ∎

Theorem 8.6 *Assume Γ is maximally consistent. Then $\Gamma^{\sharp h}$ is closed under consequence. In particular, $X_1, \ldots, X_k \in \Gamma^{\sharp h}$ if and only if $(X_1 \wedge \ldots \wedge X_k) \in \Gamma^{\sharp h}$.*

Proof It is enough to show closure under consequence, since the rest then follows. Suppose $X_1, \ldots, X_n \in \Gamma^{\sharp h}$, and $X_1, \ldots, X_n \vdash_{\mathsf{J}} Y$. Then there are $\vec{t_1}, \ldots, \vec{t_n} \in \mathcal{T}^h$ so that $\vec{t_1}{:}X_1, \ldots \vec{t_n}{:}X_n \in \Gamma$. By Theorem 8.4, for some $\vec{u} \in \mathcal{T}^h$, $\vec{t_1}{:}X_1, \ldots, \vec{t_n}{:}X_n \vdash_{\mathsf{J}} \vec{u}{:}Y$. Since Γ is maximally consistent, it is closed under consequence, so $\vec{u}{:}Y \in \Gamma$ and hence $Y \in \Gamma^{\sharp h}$. ∎

Theorem 8.7 *For each $\vec{t_1}, \ldots, \vec{t_k} \in \mathcal{T}^h$ there is some $\vec{u} \in \mathcal{T}^h$ so that $\vdash (\vec{t_1}{:}X_1 \vee \ldots \vee \vec{t_k}{:}X_k) \supset \vec{u}{:}(X_1 \vee \ldots \vee X_k)$.*

Proof To keep notation simple we assume $k = 2$. The proof is by induction on h.

Base case. If $h = 0$ the theorem says $\vdash (X_1 \vee X_2) \supset (X_1 \vee X_2)$, which is trivial.

Induction step. Suppose the result is known for h, and we now have members $\langle c_{h+1}, c_h, \ldots, c_1\rangle$ and $\langle d_{h+1}, d_h, \ldots, d_1\rangle$ in $\mathcal{T}^{h+1}$. Let us write $\vec{c}$ for $\langle c_h, \ldots, c_1\rangle$ and $\vec{d}$ for $\langle d_h, \ldots, d_1\rangle$. By the induction hypothesis, there is $\vec{e} = \langle e_h, \ldots, e_1\rangle \in \mathcal{T}^h$ so that $\vdash_{\mathsf{J}} (\vec{c}{:}X_1 \vee \vec{d}{:}X_2) \supset \vec{e}{:}(X_1 \vee X_2)$. We then have $\vec{c}{:}X_1 \vdash_{\mathsf{J}} \vec{e}{:}(X_1 \vee X_2)$, so using the Lifting Lemma there is some e^c_{h+1} so that $c_{h+1}{:}\vec{c}{:}X_1 \vdash_{\mathsf{J}} e^c_{h+1}{:}\vec{e}{:}(X_1 \vee X_2)$. Similarly there is some e^d_{h+1} so that $d_{h+1}{:}\vec{d}{:}X_2 \vdash_{\mathsf{J}} e^d_{h+1}{:}\vec{e}{:}(X_1 \vee X_2)$. Now, $e^c_{h+1}{:}\vec{e}{:}(X_1 \vee X_2) \supset (e^c_{h+1} + e^d_{h+1}){:}\vec{e}{:}(X_1 \vee X_2)$ and $e^d_{h+1}{:}\vec{e}{:}(X_1 \vee X_2) \supset (e^c_{h+1} + e^d_{h+1}){:}\vec{e}{:}(X_1 \vee X_2)$ hence, setting $e_{h+1} = e^c_{h+1} + e^d_{h+1}$, we have the following, establishing the $h+1$ case.

$$\begin{aligned}
(\langle c_{h+1}, c_h, \ldots, c_1\rangle{:}X_1 \vee \langle d_{h+1}, d_h, \ldots, d_1\rangle{:}X_2) &= (c_{h+1}{:}\vec{c}{:}X_1 \vee d_{h+1}{:}\vec{d}{:}X_2)\\
&\vdash_{\mathsf{J}} (e^c_{h+1}{:}\vec{e}{:}(X_1 \vee X_2) \vee e^d_{h+1}{:}\vec{e}{:}(X_1 \vee X_2))\\
&\vdash_{\mathsf{J}} (e^c_{h+1} + e^d_{h+1}){:}\vec{e}{:}(X_1 \vee X_2)\\
&= e_{h+1}{:}\vec{e}{:}(X_1 \vee X_2)\\
&= \langle e_{h+1}, e_h, \ldots, e_1\rangle{:}(X_1 \vee X_2)
\end{aligned}$$

Now apply the Deduction Theorem. ∎

Theorem 8.8 *Assume Γ, Δ are maximally consistent sets of justification formulas. Then*

$$\Gamma^{\sharp(h+1)} \subseteq \Delta \text{ iff for some maximally consistent set } \Omega, \Gamma^{\sharp} \subseteq \Omega \text{ and } \Omega^{\sharp h} \subseteq \Delta.$$

Proof We follow the modal version in [7], where it is given as Theorem 4.31.

The argument from right to left is easy. Assume $\Gamma^{\sharp} \subseteq \Omega$ and $\Omega^{\sharp h} \subseteq \Delta$, where Ω is maximally consistent. Suppose $X \in \Gamma^{\sharp(h+1)}$; we show $X \in \Delta$. By the supposition, for some $\vec{t} \in \mathcal{T}^{h+1}$, $\vec{t}{:}X \in \Gamma$. Let us say $\vec{t} = \langle t_{h+1}, t_h, \ldots, t_1\rangle$. Then $\langle t_h, \ldots, t_1\rangle{:}X \in \Gamma^{\sharp}$, and so $\langle t_h, \ldots, t_1\rangle{:}X \in \Omega$. Then $X \in \Omega^{\sharp h}$, and hence $X \in \Delta$.

The left to right argument is harder. Assume $\Gamma^{\sharp(h+1)} \subseteq \Delta$. The main thing is to show $\Gamma^{\sharp} \cup \Delta^{\flat h}$ is consistent. Once this has been established, we can extend $\Gamma^{\sharp} \cup \Delta^{\flat h}$ to a maximal consistent set Ω. Of course $\Gamma^{\sharp} \subseteq \Omega$. But also $\Delta^{\flat h} \subseteq \Omega$ and hence $\Omega^{\sharp h} \subseteq \Delta$ by Theorem 8.5.

We now show by contradiction that $\Gamma^{\sharp} \cup \Delta^{\flat h}$ is consistent. Suppose $\Gamma^{\sharp} \cup \Delta^{\flat h} \vdash_{\mathsf{J}} \bot$. Then there are $G_1, \ldots, G_k \in \Gamma^{\sharp}$ and $\neg\vec{t_1}{:}\neg D_1, \ldots, \neg\vec{t_m}{:}\neg D_m \in \Delta^{\flat h}$ so that $G_1, \ldots, G_k \vdash_{\mathsf{J}} (\vec{t_1}{:}\neg D_1 \vee \ldots \vee \vec{t_m}{:}\neg D_m)$. Using Theorem 8.7, for some $\vec{u} \in \mathcal{T}^h$ we have $G_1, \ldots, G_k \vdash_{\mathsf{J}} \vec{u}{:}(\neg D_1 \vee \ldots \vee \neg D_m)$. Since $(\neg D_1 \vee \ldots \vee \neg D_m) \vdash_{\mathsf{J}} \neg(D_1 \wedge \ldots \wedge D_m)$, using the Lifting Lemma Generalized we have that there is some $\vec{v} \in \mathcal{T}^h$ so that $\vec{u}{:}(\neg D_1 \vee \ldots \vee \neg D_m) \vdash_{\mathsf{J}} \vec{v}{:}\neg(D_1 \wedge \ldots \wedge D_m)$, and hence we have $G_1, \ldots, G_k \vdash_{\mathsf{J}} \vec{v}{:}\neg(D_1 \wedge \ldots \wedge D_m)$. Each G_i is in $\Gamma^{\sharp}$, and so $g_i{:}G_i \in \Gamma$ for some justification term g_i. Using the Lifting Lemma, for some justification term h, $g_1{:}G_1, \ldots, g_k{:}G_k \vdash_{\mathsf{J}} h{:}\vec{v}{:}\neg(D_1 \wedge \ldots \wedge D_m)$. Since Γ is maximally consistent, it follows that $h{:}\vec{v}{:}\neg(D_1 \wedge \ldots \wedge D_m) \in \Gamma$. But then $\neg(D_1 \wedge \ldots \wedge D_m)$ is in $\Gamma^{\sharp(h+1)}$, and hence in Δ. But also $\neg\vec{t_1}{:}\neg D_1, \ldots, \neg\vec{t_m}{:}\neg D_m \in \Delta^{\flat h}$, so $D_1, \ldots, D_m \in \Delta$, and this implies the inconsistency of Δ. We have our contradiction. ■

8.3 Justification Counterparts of Geach Logics, Axiomatics

We looked at S4.2 and J4.2 in Sections 5.4 and 6.3.2. We saw what a justification version of $\Diamond\Box X \supset \Box\Diamond X$ would be, where this is the paradigm Geach formula. But for the general case, $\mathsf{G}^{k,l,m,n}$, much more needs to be said to produce a justification analog $\mathsf{JG}^{k,l,m,n}$. We have the general remarks from Section 5.5 as a guide, but we have infinitely many cases to cover now, and we want to do it in the simplest way we can. Assume k, l, m, n are fixed for the rest of this section.

For $\mathsf{JG}^{k,l,m,n}$ we enlarge the language of justification logic with additional function symbols besides the $\cdot$ and $+$ of J_0. Specifically we add function symbols $f_1, \ldots, f_k$ and $g_1, \ldots, g_m$, all distinct, each taking $l+n$ arguments. Justification terms are built up using these function symbols, as well as $\cdot$ and $+$.

In order to compactly formulate justification counterparts for all the various Geach logics it is convenient to make use of the special notation introduced in Definition 8.2. Recall that $\mathcal{T}^h$ consists of all length h vectors of justification terms (terms that we assume are in the present language). Now suppose $\vec{t} = \langle t_1, \ldots, t_l\rangle$ is a member of $\mathcal{T}^l$ and $\vec{u} = \langle u_1, \ldots, u_n\rangle$ is a member of $\mathcal{T}^n$. We write $f_i(\vec{t}, \vec{u})$ as short for $f_i(t_1, \ldots, t_l, u_1, \ldots, u_n)$. If $l = 0$ we must have $\vec{t} = \langle\,\rangle$, and we take $f_i(\vec{t}, \vec{u}) = f_i(\vec{u}) = f_i(u_1, \ldots, u_n)$. Similarly if $n = 0$. If both $l = 0$ and $n = 0$, $f_i(\vec{t}, \vec{u}) = f_i$ which can be understood as a 0 argument function. Of course the same notational conventions apply to $g_i(\vec{t}, \vec{u})$.

It is handy to also make use of vector notation for function symbols. Suppose $\vec{f} = \langle f_1, \ldots, f_k\rangle$, $\vec{t} \in \mathcal{T}^l$ and $\vec{u} \in \mathcal{T}^n$. We write $\vec{f}(\vec{t}, \vec{u})$ as short for $\langle f_1(\vec{t}, \vec{u}), \ldots, f_k(\vec{t}, \vec{u})\rangle$. If $k = 0$ we understand $\vec{f}(\vec{t}, \vec{u})$ to be $\langle\,\rangle$. Of course similarly for $\vec{g} = \langle g_1, \ldots, g_m\rangle$ and $\vec{g}(\vec{t}, \vec{u})$.

Finally we note that Definition 8.2 already supplies a meaning for $\vec{f}(\vec{t}, \vec{u}){:}X$. In particular if $k = 0$ then $\vec{f}(\vec{t}, \vec{u}) = \langle\,\rangle$, in which case $\vec{f}(\vec{t}, \vec{u}){:}X$ is just X. Also if $\vec{f} = \langle f_1\rangle$ then $\vec{f}(\vec{t}, \vec{u}){:}X$ is $f_1(\vec{t}, \vec{u}){:}X$. Similar comments apply to $\vec{g}(\vec{t}, \vec{u})$ too, of course.

Now we give an axiomatization for the justification counterpart of $\mathsf{G}^{k,l,m,n}$, which we understand as $\neg\Box^k\neg\Box^l X \supset \Box^m\neg\Box^n\neg X$ since the possibility operator does not match well with justification logic machinery. Let $\vec{f} = \langle f_1, \ldots, f_k\rangle$ and $\vec{g} = \langle g_1, \ldots, g_m\rangle$. The axiom scheme we want is all formulas of the following form, for any formula X, and any $\vec{t} \in \mathcal{T}^l$ and $\vec{u} \in \mathcal{T}^n$. (This should be compared with (1).)

$$\neg\vec{f}(\vec{t},\vec{u}){:}\neg\vec{t}{:}X \supset \vec{g}(\vec{t},\vec{u}){:}\neg\vec{u}{:}\neg X \tag{3}$$

These formulas are to be added to J_0 with the resulting justification logic denoted $\mathsf{JG}^{k,l,m,n}$. According to our vector notation conventions, if $k = 0$ then (3) reduces to $\vec{t}{:}X \supset \vec{g}(\vec{t},\vec{u}){:}\neg\vec{u}{:}\neg X$; if $l = 0$ it reduces to $\neg\vec{f}(\vec{t},\vec{u}){:}\neg X \supset \vec{g}(\vec{t},\vec{u}){:}\neg\vec{u}{:}\neg X$; and if both $k = 0$ and $l = 0$ it becomes $X \supset \vec{g}(\vec{t},\vec{u}){:}\neg\vec{u}{:}\neg X$. Similarly for m and n.

We give three examples, one of no special interest, the other two very familiar.

Example 1 $\mathsf{G}^{2,1,1,0}$, $\Diamond\Diamond\Box X \supset \Box X$, or equivalently $\neg\Box\Box\neg\Box X \supset \Box X$. For $\mathsf{JG}^{2,1,1,0}$, $\vec{f} = \langle f_1, f_2\rangle$ and $\vec{g} = \langle g_1\rangle$, where all function symbols are $1+0 = 1$ place. These are fixed. We have $\vec{t} = \langle t_1\rangle$, where t_1 is arbitrary, and $\vec{u} = \langle\,\rangle$. Then (3) specializes to $\neg f_1(t_1){:}f_2(t_1){:}\neg t_1{:}X \supset g_1(t_1){:}X$.

Example 2 $\mathsf{G}^{0,1,2,0}$, $\Box X \supset \Box\Box X$, the 4 axiom scheme. Here $\vec{f} = \langle\,\rangle$ and $\vec{g} = \langle g_1, g_2\rangle$. $\vec{t} = \langle t_1\rangle$ while $\vec{u} = \langle\,\rangle$. (3) becomes $t_1{:}X \supset g_1(t_1){:}g_2(t_1){:}X$. We saw this before, in Section 5.1.

Example 3 $\mathsf{G}^{0,0,1,1}$, $X \supset \Box\Diamond X$, equivalently $X \supset \Box\neg\Box\neg X$, known as B. For this, $\vec{f} = \langle\,\rangle$, $\vec{t} = \langle\,\rangle$, $\vec{g} = \langle g_1\rangle$, and $\vec{u} = \langle u_1\rangle$. Then (3) becomes $X \supset g(u_1){:}\neg u_1{:}\neg X$ (after eliminating a double negation).

8.4 Justification Counterparts of Geach Logics, Semantics

We use Fitting models as defined in Section 6.1. Beyond the basics, we need special conditions on evidence functions and frames. The frame condition is the same as for modal $\mathsf{G}^{k,l,m,n}$. To state the evidence function conditions in a compact way we introduce yet another piece of notation. Suppose $\vec{t} = \langle t_1, \ldots, t_m\rangle \in \mathcal{T}^m$ where $m > 0$. Define $F(\vec{t}) = t_1$ and $B(\vec{t}) = \langle t_2, \ldots, t_m\rangle$. The operator names are meant to suggest *first* and *but-first*. The operators are undefined on $\langle\,\rangle$ so conditions for the Evidence Function fall into four cases, depending on whether or not $\vec{f}(\vec{t},\vec{u}) = \langle\,\rangle$ and $\vec{g}(\vec{t},\vec{u}) = \langle\,\rangle$.

Evidence Function Condition for $\mathsf{JG}^{k,l,m,n}$ We require that $\mathcal{E}$ be a *strong* evidence function, and that the following must hold.

1. If $k > 0$ and $m > 0$, $\mathcal{E}(F(\vec{f}(\vec{t},\vec{u})), B(\vec{f}(\vec{t},\vec{u})){:}\neg\vec{t}{:}X) \cup \mathcal{E}(F(\vec{g}(\vec{t},\vec{u})), B(\vec{g}(\vec{t},\vec{u})){:}\neg\vec{u}{:}\neg X) = \mathcal{G}$
2. If $k = 0$ and $m > 0$, $\{\Gamma \mid \Gamma \Vdash \vec{t}{:}X\} \subseteq \mathcal{E}(F(\vec{g}(\vec{t},\vec{u})), B(\vec{g}(\vec{t},\vec{u})){:}\neg\vec{u}{:}\neg X)$
3. If $k > 0$ and $m = 0$, $\{\Gamma \mid \Gamma \Vdash \vec{u}{:}\neg X\} \subseteq \mathcal{E}(F(\vec{f}(\vec{t},\vec{u})), B(\vec{f}(\vec{t},\vec{u})){:}\neg\vec{t}{:}X)$
4. If $k = 0$ and $m = 0$, no conditions.

Frame Condition for $\mathsf{JG}^{k,l,m,n}$ If $\Gamma_1\mathcal{R}^k\Gamma_2$ and $\Gamma_1\mathcal{R}^m\Gamma_3$ then for some Γ_4, $\Gamma_2\mathcal{R}^l\Gamma_4$ and $\Gamma_3\mathcal{R}^n\Gamma_4$.

We examine again the examples from Section 8.3, discussing Evidence Function Conditions. In all cases we want strong evidence functions.

Example 1 $\mathsf{G}^{2,1,1,0}$, $\neg\Box\Box\neg\Box X \supset \Box X$. This falls into case 1, and the evidence function should satisfy $\mathcal{E}(f_1(t_1), f_2(t_1){:}\neg t_1{:}X) \cup \mathcal{E}(g_1(t_1), X) = \mathcal{G}$.

Example 2 $\mathsf{G}^{0,1,2,0}$, $\Box X \supset \Box\Box X$. This is in case 2. The evidence function should satisfy the condition $\{\Gamma \mid \Gamma \Vdash t_1{:}X\} \subseteq \mathcal{E}(g_1(t_1), g_2(t_1){:}X)$. Note that since we have a strong evidence function, we could also express this as $\mathcal{E}(t_1, X) \subseteq \mathcal{E}(g_1(t_1), g_2(t_1){:}X)$.

Example 3 $\mathsf{G}^{0,0,1,1}$, $X \supset \Box\neg\Box\neg X$. This also falls into case 2, and we have the evidence condition $\{\Gamma \mid \Gamma \Vdash X\} \subseteq \mathcal{E}(g(u_1), \neg u_1{:}\neg X)$. Unlike the previous example, this cannot be rewritten using only $\mathcal{E}$.

Case 4 for Evidence Function Conditions is vacuous. This is actually a familiar case, $\Box X \supset X$ and $\Box X \supset \Diamond X$ fall into it.

8.5 Soundness and Completeness

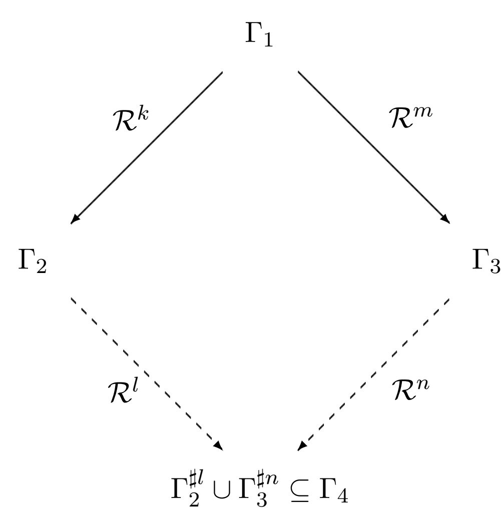
For case 1 of the Frame Condition for $\mathsf{JG}^{k,l,m,n}$, verifying soundness with respect to Fitting semantics is very similar to verifying soundness for $\mathsf{J4.2}$, which was presented in Section 6.3.2. We omit the details here. The other three cases are simpler and are left to the reader.

To show completeness we must show the canonical $\mathsf{JG}^{k,l,m,m}$ model is a model that meets the evidence and frame conditions specified in the previous section. That it satisfies the Evidence Function Condition is relatively straightforward. The main work comes in showing the frame of the canonical $\mathsf{JG}^{k,l,m,n}$ model meets the Frame Condition. *For this, internalization is essential and so an axiomatically appropriate constant specification is assumed throughout.* Recall that the frame of the canonical model for $\mathsf{JG}^{k,l,m,n}$ is $\langle \mathcal{G}, \mathcal{R} \rangle$ where: $\mathcal{G}$ is the collection of sets that are maximally consistent sets in axiomatic $\mathsf{JG}^{k,l,m,n}$ and, for $\Gamma, \Delta \in \mathcal{G}$, $\Gamma \mathcal{R} \Delta$ if $\Gamma^\sharp \subseteq \Delta$, with $\Gamma^\sharp$ as in Section 7.2.

There are a few useful observations before we get to the heart of the matter. Assume Γ and Δ are possible worlds in the canonical justification model. First, if $\Gamma \mathcal{R}^h \Delta$ then $\Gamma^{\sharp h} \subseteq \Delta$. This is by a straightforward induction on h, using monotonicity of the $\sharp$ operation. And second, if $\Gamma^{\sharp h} \subseteq \Delta$ then $\Gamma \mathcal{R}^h \Delta$. This is by repeated use of Theorem 8.8. We list this as

Important Fact $\Gamma \mathcal{R}^h \Delta$ if and only if $\Gamma^{\sharp h} \subseteq \Delta$ for possible worlds in the canonical justification model for $\mathsf{JG}^{k,l,m,n}$.

Now the details. From here on assume Γ_1, Γ_2, and Γ_3 are maximally consistent sets in $\mathsf{JG}^{k,l,m,n}$, and $\Gamma_1 \mathcal{R}^k \Gamma_2$ and $\Gamma_1 \mathcal{R}^m \Gamma_3$. We will produce an appropriate Γ_4, so that $\Gamma_2 \mathcal{R}^l \Gamma_4$ and $\Gamma_3 \mathcal{R}^n \Gamma_4$. Here is a picture to help keep things in mind.



We first show that $\Gamma_2^{\sharp l} \cup \Gamma_3^{\sharp n}$ is consistent, and we do this by contradiction. Suppose otherwise. Then there are $X_1, \ldots, X_i \in \Gamma_2^{\sharp l}$ and $Y_1, \ldots, Y_j \in \Gamma_3^{\sharp n}$ so that $X_1, \ldots, X_i, Y_1, \ldots, Y_j \vdash \perp$. Let $X = X_1 \wedge \ldots \wedge X_i$ and $Y = Y_1 \wedge \ldots \wedge Y_j$. Then using Theorem 8.6, $X \in \Gamma_2^{\sharp l}$ and $Y \in \Gamma_3^{\sharp n}$, and of course $X, Y \vdash \perp$, and hence $Y \vdash \neg X$. (The reason for introducing X and Y is simply to keep notational clutter down.)

Since $X \in \Gamma_2^{\sharp l}$, for some $\vec{t} \in \mathcal{T}^l$, $\vec{t}{:}X \in \Gamma_2$. Similarly, since $Y \in \Gamma_3^{\sharp n}$, $\vec{u}{:}Y \in \Gamma_3$ for some $\vec{u} \in \mathcal{T}^n$. And since $Y \vdash \neg X$, by the Lifting Lemma Generalized, for some $\vec{v} \in \mathcal{T}^n$, $\vec{u}{:}Y \vdash \vec{v}{:}\neg X$, and so $\vec{v}{:}\neg X \in \Gamma_3$. Since $\Gamma_1 \mathcal{R}^k \Gamma_2$ then $\Gamma_1^{\sharp k} \subseteq \Gamma_2$, so $\vec{f}(\vec{t}, \vec{v}){:}\neg\vec{t}{:}X \notin \Gamma_1$ since $\vec{t}{:}X \in \Gamma_2$ and Γ_2 is consistent. Then $\neg\vec{f}(\vec{t}, \vec{v}){:}\neg\vec{t}{:}X \in \Gamma_1$. Using the axiom $\vec{f}(\vec{t}, \vec{v}){:}\neg\vec{t}{:}X \vee \vec{g}(\vec{t}; \vec{v}){:}\neg\vec{v}{:}\neg X$ and maximal consistency of Γ_1, we have $\vec{g}(\vec{t}, \vec{v}){:}\neg\vec{v}{:}\neg X \in \Gamma_1$. Since $\Gamma_1 \mathcal{R}^m \Gamma_3$ then $\Gamma_1^{\sharp m} \subseteq \Gamma_3$, and so $\neg\vec{v}{:}\neg X \in \Gamma_3$. This contradicts the consistency of Γ_3.

We have now shown that that $\Gamma_2^{\sharp l} \cup \Gamma_3^{\sharp n}$ is consistent. Extend it to a maximally consistent set Γ_4, which will be a possible world in the canonical model. Since $\Gamma_2^{\sharp l} \subseteq \Gamma_4$ then $\Gamma_2 \mathcal{R}^l \Gamma_4$ by the Important Fact. Similarly $\Gamma_3 \mathcal{R}^n \Gamma_4$. Thus the canonical model meets the Frame Condition for $\mathsf{G}^{k,l,m,n}$.

9 Quasi-Ralization

Many of the proofs of realization do things in one pass. The first realization proof, which provides an algorithm suitable for LP and some closely related logics, can be found in [1]. It converts a sequent calculus S4 proof into an LP realization. Since then several different algorithmic proofs have been developed. In [11], for instance, an approach was introduced that does not work globally with a sequent calculus proof, but instead works through the sequent proof step by step. But here we follow a different line, originating in [9], where a non-constructive proof of realization for LP was presented. A key feature, whose significance I did not realize until much later, was that it was fundamentally a two-stage process. Eventually, in [15], the two stages were clearly separated, and the output of the first stage was given a name, *quasi-realization*. This was applied in [14] to provide not only an algorithm for LP, but an implementation of it. A direct forerunner of the present paper is [18] in which the two stages appear in a general, but non-constructive, setting. The present paper is the first full presentation of the ideas that have been developed.

Quasi-realizations are similar to realizations, but have a more complex form. They still provide an embedding from a modal logic into a justification logic, and can serve some of the functions that realizations were designed for. At a second stage, quasi-realizations are converted into realizations. There are features of both stages that are of significance.

The second stage, conversion from quasi-realizations to realizations, is algorithmic. It depends only on formula structure and not on the structure of a cut-free modal proof. And it is independent of the particular logic in question—that is, it is uniform across the family of justification logics.

The first stage may or may not be algorithmic. In this Section we present a non-algorithmic version that covers a very broad range of logics. Generally speaking, for quasi-realization to be algorithmic the modal logic must have some proof procedure which has the subformula property and in which formulas do not change polarity. This is generally described as cut-free. So far sequent calculi, tableau systems, hypersequents, nested sequents, and prefixed tableau systems have been used for this purpose. An abstract understanding is missing, but all the obvious candidates have worked.

9.1 Annotated and Signed Formulas

We will be mapping modal formulas to justification formulas. Since we want *normal* realizations, Definition 4.2, we must keep track of specific *occurrences* of $\Box$. In [11] we introduced *annotated formulas* for this purpose, and in [14, 15] a simpler version. It is the simpler version that we use now.

Definition 9.1 An *annotated modal formula* is like a standard modal formula except for the following.

1. Instead of a single modal operator $\Box$ there is an infinite family, $\Box_1$, $\Box_2$, ..., called *indexed* modal operators. Formulas are built up as usual, but using indexed modal operators instead of $\Box$. We assume that in an annotated formula, *no index occurs twice*.

2. If X is an annotated formula, and X' is the result of replacing all indexed modal operators, $\Box_n$, with $\Box$, thus forgetting the index, then X' is a conventional modal formula. We say X is an *annotated version* of X', and X' is an *unannotated version* of X.

Annotations, indexes, are purely for bookkeeping purposes. Semantically they are ignored. Thus $\Box_n$ and $\Box$ are understood to behave alike in models, so that a modal formula and an annotated version of it evaluate the same at each possible world.

It is also necessary, for normal realizations, that we keep track of positive and negative subformula occurrences. There are several more-or-less equivalent ways of doing this. We make use of signed formulas, familiar from tableau proof systems.

Definition 9.2 Let T and F be two symbols, not part of our modal or justification languages. A *signed* formula is $T\,X$ or $F\,X$, where X is a formula. We allow X to be a justification formula, a modal formula, or an annotated modal formula.

When working with tableaus one thinks of $T\,X$ as an assertion that X is true (under some circumstance), and $F\,X$ that X is false. All this plays a role in tableau theorem proving and in [14], but not here. In the present treatment, signs are simply for bookkeeping purposes. Still, we should note that for $T\,X \supset Y$ to be verified, informally, we must consider circumstances where either $F\,X$ or $T\,Y$ is so. Likewise, for $F\,X \supset Y$ to be verified, we must consider circumstances where $T\,X$ and $F\,Y$ are so. This observation may help motivate item 2 in Definition 9.3, and similarly for the other cases. Also, a tableau proof of X starts with $F\,X$, so we think of the 'top level' formula as having an F sign. This may help account for the details of Definition 9.4.

9.2 Quasi-Realizations

The definitions in this section have to do with languages, not logics. We define a mapping from signed annotated modal formulas to sets of signed justification formulas involving some set of function symbols including at least $+$ and $\cdot$. Justification terms are those appropriate for whatever justification language we are using, and details don't matter at this point.

Important Note From now on we assume that v_1, v_2, ... is an enumeration of all justification variables with no variable repeated. This enumeration is fixed once and for all.

Definition 9.3 The mapping $\langle\langle\cdot\rangle\rangle$ associates with each signed annotated modal formula a set of signed justification formulas. It is defined recursively, as follows.

1. If A is atomic, $\langle\!\langle T\,A \rangle\!\rangle = \{T\,A\}$ and $\langle\!\langle F\,A \rangle\!\rangle = \{F\,A\}$.

2. $\langle\!\langle T\,X \supset Y \rangle\!\rangle = \{T\,U \supset V \mid F\,U \in \langle\!\langle F\,X \rangle\!\rangle, T\,V \in \langle\!\langle T\,Y \rangle\!\rangle\}$
   $\langle\!\langle F\,X \supset Y \rangle\!\rangle = \{F\,U \supset V \mid T\,U \in \langle\!\langle T\,X \rangle\!\rangle, F\,V \in \langle\!\langle F\,Y \rangle\!\rangle\}$.

3. $\langle\!\langle T\,\Box_n X \rangle\!\rangle = \{T\,v_n{:}U \mid T\,U \in \langle\!\langle T\,X \rangle\!\rangle\}$.
   $\langle\!\langle F\,\Box_n X \rangle\!\rangle = \{F\,t{:}(U_1 \vee \ldots \vee U_k) \mid F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,X \rangle\!\rangle$ and t is any justification term$\}$.

4. The mapping is extended to *sets* of signed annotated formulas by letting $\langle\!\langle S \rangle\!\rangle = \cup\{\langle\!\langle Z \rangle\!\rangle \mid Z \in S\}$.

Definition 9.4 (Quasi-Realization) Let A be an annotated modal formula. If $F\,U_1, \ldots, F\,U_n \in \langle\!\langle F\,A \rangle\!\rangle$, we say the justification formula $U_1 \vee \ldots \vee U_n$ is a *quasi-realization* of A. (Disjunction may be primitive or defined depending on details of language.)

For a modal formula X without annotations, a quasi-realization for it is any quasi-realization for X', where X' is any annotated version of X.

Note that, in general, a modal formula, annotated or not, may have many quasi-realizations.

Example 9.5 Suppose t, u, and w are justification terms and P and Q are atomic formulas. Here are some quasi-realization calculations, leading up to $F\,\Box_1(\Box_2 P \supset \Box_3 Q)$. We do not produce *all* quasi-realizations; the set would be infinite since there are infinitely many justification terms.

1. $\{T\,P\} = \langle\!\langle T\,P \rangle\!\rangle$ and $\{F\,Q\} = \langle\!\langle F\,Q \rangle\!\rangle$

2. $\{T\,v_2{:}P\} = \langle\!\langle T\,\Box_2 P \rangle\!\rangle$

3. $\{F\,t{:}Q, F\,u{:}Q\} \subseteq \langle\!\langle F\,\Box_3 Q \rangle\!\rangle$

4. $\{F\,v_2{:}P \supset t{:}Q, F\,v_2{:}P \supset u{:}Q\} \subseteq \langle\!\langle F\,\Box_2 P \supset \Box_3 Q \rangle\!\rangle$

5. $\{F\,t{:}((v_2{:}P \supset t{:}Q) \vee (v_2{:}P \supset u{:}Q)), F\,w{:}(v_2{:}P \supset u{:}Q)\} \subseteq \langle\!\langle F\,\Box_1(\Box_2 P \supset \Box_3 Q) \rangle\!\rangle$

It follows that $t{:}((v_2{:}P \supset t{:}Q) \vee (v_2{:}P \supset u{:}Q)) \vee w{:}(v_2{:}P \supset u{:}Q)$ is a quasi-realization for $\Box(\Box P \supset \Box Q)$. There are many others.

The idea is that a quasi-realization, as defined above, is a *candidate* for what we want. We will determine circumstances under which a *provable* modal formula has a *provable* quasi-realization.

9.3 When Quasi-Realizations Exist

We now come to the central theorem of this Section. The result is not really new. It is implicit in [9]; we are now making it explicit.

Theorem 9.6 (Quasi-Realization Theorem) *Let KL be a normal modal logic characterized by a class of frames $\mathcal{FL}$. Let JL be an axiomatically formulated justification logic, $\mathcal{CS}$ be a constant specification for it, and assume $\mathsf{JL}(\mathcal{CS})$ has the internalization property. If the canonical Fitting model (Section 7.2) for $\mathsf{JL}(\mathcal{CS})$ is based on a frame in $\mathcal{FL}$, then every validity of KL has a provable quasi-realization in $\mathsf{JL}(\mathcal{CS})$.*

Once the following Lemma is shown, the Theorem follows easily. The Lemma and its proof benefit from some notational conventions. For a Fitting model $\mathcal{M}$ and an annotated formula X, $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,X \rangle\!\rangle$ means $\mathcal{M}, \Gamma \Vdash Y$ for every justification formula Y such that $T\,Y \in \langle\!\langle T\,X \rangle\!\rangle$, and $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,X \rangle\!\rangle$ means $\mathcal{M}, \Gamma \not\Vdash Y$ for every justification formula Y such that $F\,Y \in \langle\!\langle F\,X \rangle\!\rangle$. (It may be that neither $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,X \rangle\!\rangle$ nor $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,X \rangle\!\rangle$ holds.)

Lemma 9.7 *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ be the canonical Fitting model for the axiomatic justification logic $\mathsf{JL}(\mathcal{CS})$, a logic with the internalization property. Let $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ be the corresponding modal model that is formed by dropping the evidence function from $\mathcal{M}$. Then for every annotated modal formula X we have the following.*

1. $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,X \rangle\!\rangle \Longrightarrow \mathcal{N}, \Gamma \Vdash X$
2. $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,X \rangle\!\rangle \Longrightarrow \mathcal{N}, \Gamma \not\Vdash X$

Proof By induction on the degree of the annotated modal formula X.

Ground Case Suppose $X = A$, which is atomic. The only member of $\langle\!\langle T\,A \rangle\!\rangle$ is $T\,A$, so $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,A \rangle\!\rangle$ iff $\mathcal{M}, \Gamma \Vdash A$ iff $\Gamma \in \mathcal{V}(A)$ iff $\mathcal{N}, \Gamma \Vdash A$. This establishes item 1; item 2 is similar.

Implication Case Suppose $X = A \supset B$ and the results are known for A and for B.

1. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,A \supset B \rangle\!\rangle$. We divide the argument into two parts.
   Suppose first that $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,A \rangle\!\rangle$. By the induction hypothesis, $\mathcal{N}, \Gamma \not\Vdash A$, so $\mathcal{N}, \Gamma \Vdash A \supset B$.
   Suppose next that $\mathcal{M}, \Gamma \not\Vdash \langle\!\langle F\,A \rangle\!\rangle$. Then for some U with $F\,U \in \langle\!\langle F\,A \rangle\!\rangle$, $\mathcal{M}, \Gamma \Vdash U$. Let $T\,V$ be an arbitrary member of $\langle\!\langle T\,B \rangle\!\rangle$. Then $T\,U \supset V \in \langle\!\langle T\,A \supset B \rangle\!\rangle$ so by the assumption, $\mathcal{M}, \Gamma \Vdash U \supset V$, and hence $\mathcal{M}, \Gamma \Vdash V$. Since $T\,V$ was arbitrary, $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,B \rangle\!\rangle$. Then by the induction hypothesis, $\mathcal{N}, \Gamma \Vdash B$ and hence $\mathcal{N}, \Gamma \Vdash A \supset B$.
2. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,A \supset B \rangle\!\rangle$. Let $T\,U \in \langle\!\langle T\,A \rangle\!\rangle$ and $F\,V \in \langle\!\langle F\,B \rangle\!\rangle$ both be arbitrary. Then $F\,U \supset V \in \langle\!\langle F\,A \supset B \rangle\!\rangle$ so by the assumption, $\mathcal{M}, \Gamma \not\Vdash U \supset V$. Then $\mathcal{M}, \Gamma \Vdash U$ and $\mathcal{M}, \Gamma \not\Vdash V$. Since $T\,U$ and $F\,V$ were arbitrary, $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,A \rangle\!\rangle$ and $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,B \rangle\!\rangle$, so by the induction hypothesis, $\mathcal{N}, \Gamma \Vdash A$ and $\mathcal{N}, \Gamma \not\Vdash B$. Hence $\mathcal{N}, \Gamma \not\Vdash A \supset B$.

Modal Case Suppose $X = \Box_n A$ and the results are known for A.

1. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,\Box_n A \rangle\!\rangle$. Let $T\,U \in \langle\!\langle T\,A \rangle\!\rangle$ be arbitrary. By the assumption, $\mathcal{M}, \Gamma \Vdash v_n{:}U$. Let $\Delta \in \mathcal{G}$ be arbitrary, with $\Gamma \mathcal{R} \Delta$. Then $\mathcal{M}, \Delta \Vdash U$ and since $T\,U$ was arbitrary, $\mathcal{M}, \Delta \Vdash \langle\!\langle T\,A \rangle\!\rangle$. By the induction hypothesis, $\mathcal{N}, \Delta \Vdash A$ and, since Δ was arbitrary, $\mathcal{N}, \Gamma \Vdash \Box_n A$.
2. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,\Box_n A \rangle\!\rangle$. This case depends on the following Claim. We first show how the Claim is used, then we prove the Claim itself.

   Claim: We use notation from Section 7.2. The set $S = \Gamma^\sharp \cup \{\neg U \mid F\,U \in \langle\!\langle F\,A \rangle\!\rangle\}$ is consistent in the justification logic $\mathsf{JL}(\mathcal{CS})$.

   The Claim is used as follows. Since (assuming the claim) S is consistent in $\mathsf{JL}(\mathcal{CS})$, it can be extended to a maximally consistent set, Δ. Since $\mathcal{M}$ is a canonical model, $\Delta \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$ since $\Gamma^\sharp \subseteq \Delta$. Also $\{\neg U \mid F\,U \in \langle\!\langle F\,A \rangle\!\rangle\} \subseteq \Delta$ so by the Truth Lemma 7.1, $\mathcal{M}, \Delta \not\Vdash U$ for all U with $F\,U \in \langle\!\langle F\,A \rangle\!\rangle$, and so $\mathcal{M}, \Delta \Vdash \langle\!\langle F\,A \rangle\!\rangle$. Then by the induction hypothesis $\mathcal{N}, \Delta \not\Vdash A$, and hence $\mathcal{N}, \Gamma \not\Vdash \Box_n A$. Now to complete things, we establish the claim itself.

Proof of Claim: Suppose S is not consistent in $\mathsf{JL}(\mathcal{CS})$. Then there are formulas $G_1, \ldots, G_m \in \Gamma^\sharp$, and signed formulas $F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,A \rangle\!\rangle$ such that $\{G_1, \ldots, G_m, \neg U_1, \ldots, \neg U_k\}$ is not consistent, and so $G_1, G_2, \ldots, G_m \vdash_{\mathsf{JL}(\mathcal{CS})} U_1 \vee \ldots \vee U_k$. For each $1 \le i \le m$, $G_i \in \Gamma^\sharp$, and so there is some justification term g_i so that $g_i{:}G_i \in \Gamma$. Then by Theorem 3.7 there is some justification term u so that $g_1{:}G_1, g_2{:}G_2, \ldots, g_m{:}G_m \vdash_{\mathsf{JL}(\mathcal{CS})} u{:}(U_1 \vee \ldots \vee U_k)$. It follows from maximal consistency of Γ that $u{:}(U_1 \vee \ldots \vee U_k) \in \Gamma$, and hence by the Truth Lemma, $\mathcal{M}, \Gamma \Vdash u{:}(U_1 \vee \ldots \vee U_k)$. But this contradicts the assumption that $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,\Box_n A \rangle\!\rangle$.

■

With the proof of the fundamental Lemma 9.7 out of the way, we can now establish the main result of this Section.

Proof of Theorem 9.6 The proof is by contraposition. Suppose Y is a modal formula. Let X be an annotated version of Y and suppose that X has no provable quasi-realization in $\mathsf{JL}(\mathcal{CS})$. We show Y is not a validity of KL.

Since X has no provable quasi-realization, for every $U_1, \ldots, U_n$ with $F\,U_1, \ldots, F\,U_n \in \langle\!\langle F\,X \rangle\!\rangle$, $\not\vdash_{\mathsf{JL}(\mathcal{CS})} U_1 \vee \ldots \vee U_n$ (Definition 9.4). It follows that $\{\neg U \mid F\,U \in \langle\!\langle F\,X \rangle\!\rangle\}$ is consistent in $\mathsf{JL}(\mathcal{CS})$. (Because otherwise, $\{\neg U \mid F\,U \in \langle\!\langle F\,X \rangle\!\rangle\} \vdash_{\mathsf{JL}(\mathcal{CS})} \bot$, so $\neg U_1, \ldots, \neg U_n \vdash_{\mathsf{JL}(\mathcal{CS})} \bot$ for some $F\,U_1, \ldots, F\,U_n \in \langle\!\langle F\,X \rangle\!\rangle$, and hence $\vdash_{\mathsf{JL}(\mathcal{CS})} U_1 \vee \ldots \vee U_n$, contrary to what has been established.)

Since $\{\neg U \mid F\,U \in \langle\!\langle F\,X \rangle\!\rangle\}$ is consistent, it can be extended to a maximal consistent set Γ. In the canonical justification model $\mathcal{M}$, Γ is a possible world. Using the Truth Lemma, Theorem 7.1, $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,X \rangle\!\rangle$, and so by Lemma 9.7, $\mathcal{N}, \Gamma \not\Vdash X$. A hypothesis of Theorem 9.6 is that the canonical justification model is based on a frame in the class $\mathcal{FL}$, which determines the modal logic KL. We thus have a *modal* model $\mathcal{N}$ for KL in which X fails, and hence so does Y, the unannotated version of X. Then Y is not a validity of KL. ■

10 Quasi-Realizations to Realizations

In the previous section we gave conditions under which a provable quasi-realizer must exist. Now we give an algorithm that converts quasi-realizer sets to realizations, which are single justification formulas. This algorithm depends only on modal formula structure, and not on the particular logic involved. Details of the particular justification logic we work with don't matter either.

Some additional machinery must be brought in now—substitutions for justification variables. But before introducing this we give a new characterization of normal realization, analogous to the recursive definition of quasi-realization given earlier. This version is more useful for our purposes in this section. The two characterizations are equivalent, but we omit the proof.

11 Realizations

We gave the usual characterization of normal realization as Definition 4.2. This is not handy for our current work and we replace it with a recursive characterization that is analogous to our definition of quasi-realization given earlier. The two characterizations are equivalent, but we omit the proof. The only difference between quasi-realizations as in Definition 9.3 lies in the $F\,\Box$ part of case 3. Roughly speaking, the disjunction appearing in the definition of quasi-realizer will be folded into a single justification term by using the + operator. We still assume that v_1, v_2, ... is an enumeration of all justification variables, with no justification variable repeated.

Definition 11.1 The mapping $[\![\cdot]\!]$ is defined recursively on the set of signed annotated modal formulas.

1. If A is atomic, $[\![T\,A]\!] = \{T\,A\}$ and $[\![F\,A]\!] = \{F\,A\}$.
2. $[\![T\,X \supset Y]\!] = \{T\,U \supset V \mid F\,U \in [\![F\,X]\!], T\,V \in [\![T\,Y]\!]\}$
   $[\![F\,X \supset Y]\!] = \{F\,U \supset V \mid T\,U \in [\![T\,X]\!], F\,V \in [\![F\,Y]\!]\}$.
3. $[\![T\,\Box_n X]\!] = \{T\,v_n{:}U \mid T\,U \in [\![T\,X]\!]\}$.
   $[\![F\,\Box_n X]\!] = \{F\,t{:}U \mid F\,U \in [\![F\,X]\!]$ and t is any justification term$\}$.
4. The mapping is extended to *sets* of signed annotated formulas by letting $[\![S]\!] = \cup\{[\![Z]\!] \mid Z \in S\}$.

A *normal realization* of annotated modal X is any justification formula U where $F\,U \in [\![F\,X]\!]$. More generally, members of $[\![Z]\!]$ are called *realizers* of Z, where Z is a T signed or F signed, annotated modal formula. For a modal formula X without annotations, a normal realization for X is any normal realization for X', where X' is an annotated version of X

We are not requiring realizers to be provable. We considered using the term *potential realizer*, but found it too unwieldy. Of course what we are after is a *provable realizer* for each modal theorem.

11.1 Substitution

Substitution was discussed in Definition 3.8. Here are some syntactical results concerning substitution. They play a crucial role in verifying correctness of our algorithm.

Definition 11.2 Let σ be a substitution, and A be an annotated modal formula.

1. σ *lives on* A if, for every justification variable v_k in the domain of σ, $\Box_k$ occurs in A;
2. σ *lives away from* A if, for every justification variable v_k in the domain of σ, $\Box_k$ does not occur in A;
3. σ meets the *no new variable* condition if, for every v_k in the domain of σ, the justification term $v_k\sigma$ contains no variables other than v_k.

Theorem 11.3 *Assume A is an annotated modal formula, σ_A is a substitution that lives on A, and σ_Z is a substitution that lives away from A.*

1. *If $T\,U \in [\![T\,A]\!]$ then $T\,U\sigma_Z \in [\![T\,A]\!]$.*
   If $F\,U \in [\![F\,A]\!]$ then $F\,U\sigma_Z \in [\![F\,A]\!]$.
2. *If both σ_A and σ_Z meet the no new variable condition, then $\sigma_A\sigma_Z = \sigma_Z\sigma_A$.*

Proof Part 1: The proof is by induction on the complexity of A. The atomic case is trivial since no justification variables are present, and the propositional cases are straightforward. This leaves the modal cases. Suppose $A = \Box_n B$, and the result is known for simpler formulas.

Assume that $T\,v_n{:}U \in [\![T\,\Box_n B]\!]$. Since σ_Z lives away from A, $v_n\sigma_Z = v_n$. By the induction hypothesis $T\,U\sigma_Z \in [\![T\,B]\!]$. Then $T\,(v_n{:}U)\sigma_Z = T\,v_n{:}(U\sigma_Z) \in [\![T\,\Box_n B]\!]$.

Assume $F\, t{:}U \in [\![F\,\Box_n B]\!]$. By the induction hypothesis, $F\,U\sigma_Z \in [\![F\,B]\!]$. Then $F\,(t{:}U)\sigma_Z = F\,t\sigma_Z{:}U\sigma_Z \in [\![F\,\Box_n B]\!]$

Part 2: Assume the hypothesis, and let v_k be an arbitrary justification variable; we show $v_k\sigma_A\sigma_Z = v_k\sigma_Z\sigma_A$.

First, suppose $\Box_k$ occurs in A. Since σ_A meets the no new variable condition, the only justification variable that can occur in $v_k\sigma_A$ is v_k. Since σ_Z lives away from A, $v_k\sigma_Z = v_k$, and so $v_k\sigma_A\sigma_Z = v_k\sigma_A$. But also, $v_k\sigma_Z\sigma_A = v_k\sigma_A$, hence $v_k\sigma_A\sigma_Z = v_k\sigma_Z\sigma_A$.

Second, suppose $\Box_k$ does not occur in A. Since σ_A lives on A, $v_k\sigma_A = v_k$. And since σ_Z meets the no new variable condition, v_k is the only variable that can occur in $v_k\sigma_Z$. Then $v_k\sigma_Z\sigma_A = v_k\sigma_Z$, and $v_k\sigma_A\sigma_Z = v_k\sigma_Z$, so $v_k\sigma_A\sigma_Z = v_k\sigma_Z\sigma_A$. ∎

11.2 The Quasi-Realization to Realization Algorithm

We give an algorithm for *condensing* a quasi-realization set to a single realizer. The construction and verification trace back to Proposition 7.8 in [9], with a modification and correction supplied in [12]. Throughout the section J is some justification logic with an axiomatically appropriate constant specification. We make much use of of Definition 3.10.

We introduce some special notation to make the algorithm more easily presentable. $\mathcal{A} \xrightarrow{T\,A} (A', \sigma)$ can be read as saying that the set of quasi-realizers $\mathcal{A}$ for $T\,A$ *condenses* to the single realizer $T\,A'$ using substitution σ, and similarly for $\mathcal{A} \xrightarrow{F\,A} (A', \sigma)$.

Definition 11.4 (Condensing in J) Let A be an annotated modal formula, $\mathcal{A}$ be a finite set of justification formulas, A' be a single justification formula, and σ be a substitution that lives on A and meets the no new variable condition (Definition 11.2).

Notation	Meaning
$\mathcal{A} \xrightarrow{T\,A} (A', \sigma)$	$T\,\mathcal{A} \subseteq \langle\!\langle T\,A \rangle\!\rangle$
	$T\,A' \in [\![T\,A]\!]$
	$\vdash_{\mathsf{J}} A' \supset (\bigwedge \mathcal{A})\sigma$
$\mathcal{A} \xrightarrow{F\,A} (A', \sigma)$	$F\,\mathcal{A} \subseteq \langle\!\langle F\,A \rangle\!\rangle$
	$F\,A' \in [\![F\,A]\!]$
	$\vdash_{\mathsf{J}} (\bigvee \mathcal{A})\sigma \supset A'$

Algorithm 11.6 provides a constructive proof of the following.

Theorem 11.5 (Condensing Theorem) *Let A be an annotated modal formula. For each non-empty finite set $\mathcal{A}$ of justification formulas:*

1. *If $T\,\mathcal{A} \subseteq \langle\!\langle T\,A \rangle\!\rangle$ then there are A' and σ so that $\mathcal{A} \xrightarrow{T\,A} (A', \sigma)$.*

2. *If $F\,\mathcal{A} \subseteq \langle\!\langle F\,A \rangle\!\rangle$ then there are A' and σ so that $\mathcal{A} \xrightarrow{F\,A} (A', \sigma)$.*

Our algorithm proceeds by induction on the formula complexity of the annotated formula. The atomic case is simple. For the other cases we give a construction that makes use of the notation just introduced. In each case, if the schemes above the line are the case, so is the scheme below. We remind the reader that we are still assuming v_1, v_2, … is a list of all justification variables.

Algorithm 11.6 (Quasi-Realization to Realization Condensing)

Atomic Cases Trivial, since if P is atomic $\langle\!\langle P\rangle\!\rangle = [\![P]\!] = \{P\}$, and we can use the empty substitution, ϵ. So we have the following.

$$\{P\} \xrightarrow{T\,P} (P, \epsilon) \qquad\qquad \{P\} \xrightarrow{F\,P} (P, \epsilon)$$

T $\supset$ Case

$$\frac{\{A_1, \ldots, A_k\} \xrightarrow{F\,A} (A', \sigma_A) \qquad \{B_1, \ldots, B_k\} \xrightarrow{T\,B} (B', \sigma_B)}{\{A_1 \supset B_1, \ldots, A_k \supset B_k\} \xrightarrow{T\,A \supset B} (A'\sigma_B \supset B'\sigma_A, \sigma_A\sigma_B)}$$

F $\supset$ Case

$$\frac{\{A_1, \ldots, A_k\} \xrightarrow{T\,A} (A', \sigma_A) \qquad \{B_1, \ldots, B_k\} \xrightarrow{F\,B} (B', \sigma_B)}{\{A_1 \supset B_1, \ldots, A_k \supset B_k\} \xrightarrow{F\,A \supset B} (A'\sigma_B \supset B'\sigma_A, \sigma_A\sigma_B)}$$

T $\Box$ Case

$$\frac{\{A_1, \ldots, A_k\} \xrightarrow{T\,A} (A', \sigma_A)}{\{v_n{:}A_1, \ldots, v_n{:}A_k\} \xrightarrow{T\,\Box_n A} (v_n{:}A'\sigma, \sigma_A\sigma)}$$

where $\vdash_{\mathsf{J}} t_i{:}(A' \supset A_i\sigma_A)$
for $i = 1, \ldots, k$, and
$s = t_1 + \ldots + t_k$
$\sigma = \{v_n/(s \cdot v_n)\}$

F $\Box$ Case

$$\frac{\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k \xrightarrow{F\,A} (A', \sigma_A)}{\{t_1{:}\bigvee \mathcal{A}_1, \ldots, t_k{:}\bigvee \mathcal{A}_k\} \xrightarrow{F\,\Box_n A} (t\sigma_A{:}A', \sigma_A)}$$

where $\vdash_{\mathsf{J}} u_i{:}(\bigvee \mathcal{A}_i\sigma_A \supset A')$
for $i = 1, \ldots, k$, and
$t = u_1 \cdot t_1 + \ldots + u_k \cdot t_k$

Proof of correctness for Algorithm 11.6 We justify each case of the algorithm.

Atomic Cases These cases are immediate.

T $\supset$ Case Assume we have: 1. $\{A_1, \ldots, A_k\} \xrightarrow{F\,A} (A', \sigma_A)$ and 2. $\{B_1, \ldots, B_k\} \xrightarrow{T\,B} (B', \sigma_B)$. Suppose we also have 3. $\{T\,A_1 \supset B_1, \ldots, T\,A_k \supset B_k\} \subseteq \langle\!\langle T\,A \supset B\rangle\!\rangle$. By 3. we have $\{F\,A_1, \ldots, F\,A_k\} \subseteq \langle\!\langle F\,A\rangle\!\rangle$ and $\{T\,B_1, \ldots, T\,B_k\} \subseteq \langle\!\langle T\,B\rangle\!\rangle$. By 1. there are A' and σ_A so that σ_A lives on A, meets the no new variable condition, $F\,A' \in [\![F\,A]\!]$, and $\vdash_{\mathsf{JL}} (A_1 \vee \ldots \vee A_k)\sigma_A \supset A'$. Likewise by 2. there are B' and σ_B so that σ_B lives on B, meets the no new variable condition, $T\,B' \in [\![T\,B]\!]$, and $\vdash_{\mathsf{JL}} B' \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B$.

Since $A \supset B$ is an annotated modal formula, A and B have no indexes in common because indexes can appear only once in a formula. Then σ_A and σ_B have disjoint domains and in particular, σ_A lives on A and so lives away from B, while σ_B lives on B and so lives away from A. Then $\sigma_A\sigma_B = \sigma_B\sigma_A$ by Proposition 11.3. It is easy to see that $\sigma_A\sigma_B$ lives on $A \supset B$ and meets the no new variable condition.

Again by Proposition 11.3, $F\,A'\sigma_B \in [\![F\,A]\!]$ since $F\,A' \in [\![F\,A]\!]$ and σ_B lives away from A. Likewise $T\,B'\sigma_A \in [\![T\,B]\!]$. Then $T\,A'\sigma_B \supset B'\sigma_A \in [\![T\,A \supset B]\!]$.

Finally, since $\vdash_{\mathsf{J}} (A_1 \vee \ldots \vee A_k)\sigma_A \supset A'$ then also $\vdash_{\mathsf{J}} [(A_1 \vee \ldots \vee A_k)\sigma_A \supset A']\sigma_B$, that is, $\vdash_{\mathsf{J}} (A_1 \vee \ldots \vee A_k)\sigma_A\sigma_B \supset A'\sigma_B$. Similarly $\vdash_{\mathsf{J}} B'\sigma_A \supset (B_1 \wedge \ldots \wedge B_k)\sigma_B\sigma_A$, or equivalently, $\vdash_{\mathsf{J}} B'\sigma_A \supset (B_1 \wedge \ldots \wedge B_k)\sigma_A\sigma_B$. Then by classical logic,

$$\vdash_{\mathsf{J}} (A'\sigma_B \supset B'\sigma_A) \supset [(A_1 \supset B_1) \vee \ldots \vee (A_k \supset B_k)]\sigma_A\sigma_B$$

We have now established that $\{A_1 \supset B_1, \ldots, A_k \supset B_k\} \xrightarrow{T\,A \supset B} (A'\sigma_B \supset B'\sigma_A, \sigma_A\sigma_B)$.

F $\supset$ Case Assume we have 1. $\{A_1, \ldots, A_k\} \xrightarrow{T\,A} (A', \sigma_A)$ and 2. $\{B_1, \ldots, B_k\} \xrightarrow{F\,B} (B', \sigma_B)$. Also suppose 3. $\{F\,A_1 \supset B_1, \ldots, F\,A_k \supset B_k\} \subseteq \langle\!\langle F\,A \supset B \rangle\!\rangle$. As in the previous case, $\sigma_A\sigma_B = \sigma_B\sigma_A$, and also $T\,A'\sigma_B \in [\![T\,A]\!]$ and $F\,B'\sigma_A \in [\![F\,B]\!]$, so $F\,A'\sigma_B \supset B'\sigma_A \in [\![F\,A \supset B]\!]$. Likewise $\vdash_{\mathsf{J}} (A'\sigma_B \supset B'\sigma_A) \supset [(A_1 \supset B_1) \wedge \ldots \wedge (A_k \supset B_k)]\sigma_A\sigma_B$. All this establishes $\{A_1 \supset B_1, \ldots, A_k \supset B_k\} \xrightarrow{F\,A\supset B} (A'\sigma_B \supset B'\sigma_A, \sigma_A\sigma_B)$.

T $\Box$ Case Assume we have 1. $\{A_1, \ldots, A_k\} \xrightarrow{T\,A} (A', \sigma_A)$. And suppose we also have that 2. $\{T\,v_n{:}A_1, \ldots, T\,v_n{:}A_k\} \subseteq \langle\!\langle T\,\Box_n A \rangle\!\rangle$. From 2, $\{T\,A_1, \ldots, T\,A_k\} \subseteq \langle\!\langle T\,A \rangle\!\rangle$. From 1. there are σ_A and $T\,A' \in [\![T\,A]\!]$ such that $\vdash_{\mathsf{J}} A' \supset (A_1 \wedge \ldots \wedge A_k)\sigma_A$, where σ_A lives on A and meets the no new variable condition.

For each $i = 1, \ldots, k$, $\vdash_{\mathsf{J}} A' \supset A_i\sigma_A$, so by Theorem 3.6 there is a justification term t_i (with no justification variables) such that $\vdash_{\mathsf{J}} t_i{:}(A' \supset A_i\sigma_A)$. Let s be the justification term $t_1 + \ldots + t_k$. Then $\vdash_{\mathsf{J}} s{:}(A' \supset A_i\sigma_A)$, for each i.

Let σ be the substitution $\{v_n/(s \cdot v_n)\}$. For each $i = 1, \ldots, k$, $\vdash_{\mathsf{J}} s{:}(A' \supset A_i\sigma_A)$, hence also $\vdash_{\mathsf{J}} [s{:}(A' \supset A_i\sigma_A)]\sigma$. Since s is a justification term with no justification variables, $\vdash_{\mathsf{J}} s{:}(A'\sigma \supset A_i\sigma_A\sigma)$. Then for each i, $\vdash_{\mathsf{J}} v_n{:}A'\sigma \supset (s \cdot v_n){:}A_i(\sigma_A\sigma)$. Since $\Box_n A$ is an annotated modal formula indexes cannot occur more than once, hence index n cannot occur in A. Substitution σ_A lives on A, hence v_n is not in its domain. It follows that $v_n(\sigma_A\sigma) = v_n\sigma = (s \cdot v_n)$, and so $[v_n{:}A_i](\sigma_A\sigma) = (s \cdot v_n){:}A_i(\sigma_A\sigma)$. Then for each i, $\vdash_{\mathsf{J}} v_n{:}A'\sigma \supset [v_n{:}A_i](\sigma_A\sigma)$, and so $\vdash_{\mathsf{J}} v_n{:}A'\sigma \supset [v_n{:}A_1 \wedge \ldots \wedge v_n{:}A_k](\sigma_A\sigma)$.

The substitution σ lives away from A so, since $T\,A' \in [\![T\,A]\!]$ then also $T\,A'\sigma \in [\![T\,A]\!]$ by Proposition 11.3. Then $T\,v_n{:}A'\sigma \in [\![T\,\Box_n A]\!]$.

Finally, it is easy to check that $\sigma_A\sigma$ lives on $\Box_n A$ and meets the no new variable condition.

This is enough to establish $\{v_n{:}A_1, \ldots, v_n{:}A_k\} \xrightarrow{T\,\Box_n A} (v_n{:}A'\sigma, \sigma_A\sigma)$.

F $\Box$ Case Assume 1. $(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k) \xrightarrow{F\,A} (A', \sigma_A)$, and suppose also 2. $\{F\,t_1{:}\bigvee\mathcal{A}_1, \ldots, F\,t_k{:}\bigvee\mathcal{A}_k\} \subseteq \langle\!\langle F\,\Box_n A \rangle\!\rangle$.

From 2, $F\,\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k \subseteq \langle\!\langle F\,A \rangle\!\rangle$. Then by 1. there are σ_A and $F\,A' \in [\![F\,A]\!]$ such that $\vdash_{\mathsf{J}} \bigvee\{\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k\}\sigma_A \supset A'$, where σ_A lives on A and meets the no new variable condition.

For each i, $\vdash_{\mathsf{J}} \bigvee\mathcal{A}_i\sigma_A \supset A'$, so by Theorem 3.6 there is a justification term u_i (with no justification variables) such that $\vdash_{\mathsf{J}} u_i{:}(\bigvee\mathcal{A}_i\sigma_A \supset A')$. Then also $\vdash_{\mathsf{J}} (t_i\sigma_A){:}\bigvee\mathcal{A}_i\sigma_A \supset (u_i \cdot (t_i\sigma_A)){:}A'$, or equivalently, $\vdash_{\mathsf{J}} (t_i{:}\bigvee\mathcal{A}_i)\sigma_A \supset (u_i \cdot (t_i\sigma_A)){:}A'$. Since u_i contains no justification variables, this in turn is equivalent to $\vdash_{\mathsf{J}} (t_i{:}\bigvee\mathcal{A}_i)\sigma_A \supset ((u_i \cdot t_i)\sigma_A){:}A'$. Now let $t = u_1 \cdot t_1 + \ldots + u_k \cdot t_k$. Then for each i, $\vdash_{\mathsf{J}} (t_i{:}\bigvee\mathcal{A}_i)\sigma_A \supset (t\sigma_A){:}A'$. This gives us $\vdash_{\mathsf{J}} [t_1{:}\bigvee\mathcal{A}_1 \vee \ldots \vee t_k{:}\bigvee\mathcal{A}_k]\,\sigma_A \supset (t\sigma_A){:}A'$.

It is immediate that $F\,(t\sigma_A){:}A' \in [\![F\,\Box_n A]\!]$, and that σ_A lives on $\Box_n A$. We already know it meets the no new variable condition. We have thus verified $\{t_1{:}\bigvee\mathcal{A}_1, \ldots, t_k{:}\bigvee\mathcal{A}_k\} \xrightarrow{F\,\Box_n A} (t\sigma_A{:}A', \sigma_A)$.

■

12 Putting Things Together

Theorem 9.6 tells us, non-constructively, when provable quasi-realizers exist. Theorem 11.5 and its accompanying algorithm tells us how to convert a quasi-realizer to a realizer. Putting these together we get the following central result.

Theorem 12.1 (Realization, Non-Constructively) *Let* KL *be a normal modal logic that is characterized by a class of frames* $\mathcal{FL}$*. Let* JL *be a justification logic and let* $\mathcal{CS}$ *be an axiomatically appropriate constant specification for it. If the canonical Fitting model for* $\mathsf{JL}(\mathcal{CS})$ *is based on a frame in* $\mathcal{FL}$ *then every theorem of* KL *has a* JL *provable normal realizer.*

Proof Assume KL is a normal modal logic, characterized by the class of frames $\mathcal{FL}$, and that JL is a justification logic with an axiomatically appropriate constant specification $\mathcal{CS}$. Finally, assume the canonical Fitting model for $\mathsf{JL}(\mathcal{CS})$ is based on a frame from $\mathcal{FL}$.

Directly by Theorem 9.6, Y has a provable quasi-realization in $\mathsf{JL}(\mathcal{CS})$. More precisely there are $F\,Q_1, \ldots, F\,Q_k \in \langle\!\langle F\,Y \rangle\!\rangle$ so that $\vdash_{\mathsf{JL}(\mathcal{CS})} Q_1 \vee \ldots \vee Q_k$, and hence (making use of Definition 3.10) $\vdash_{\mathsf{JL}} Q_1 \vee \ldots \vee Q_k$.

By Theorem 11.5 there is some σ and some $F\,X \in [\![F\,Y]\!]$ so that $\{F\,Q_1, \ldots, F\,Q_k\} \xrightarrow{F\,Y} (X, \sigma)$. Then $\vdash_{\mathsf{JL}} (Q_1 \vee \ldots \vee Q_k)\sigma \supset X$ so by Theorem 3.11, $\vdash_{\mathsf{JL}} X$. We have our provable normal realization X. ■

12.1 Examples Already Discussed

In Section 5.3 we discussed a modal logic $\mathsf{K4}^3$ and a justification logic $\mathsf{J4}^3$. $\mathsf{K4}^3$ is complete with respect to frames meeting the condition $\Gamma\mathcal{R}^3\Delta \Longrightarrow \Gamma\mathcal{R}\Delta$. In Section 7.3 we showed the canonical model for $\mathsf{J4}^3$ had a frame meeting the $\mathsf{K4}^3$ condition. Then by Theorem 12.1, there is a realization result connecting $\mathsf{J4}^3$ and $\mathsf{K4}^3$.

$\mathsf{K4}^3$ is a Geach logic, Definition 8.1, along with many other common modal logics. In Section 8.3 we proposed candidates for justification counterparts of Geach logics. We also verified that the canonical models for these justification counterparts had frames meeting the appropriate conditions of their corresponding modal logics. Now we can appeal to Theorem 12.1 and say we have established the following general result. In particular it tells us that the modal/justification phenomenon has infinitely many examples.

General Justification Logic Existence Result All Geach logics have justification logic counterparts. In particular, these counterparts are connected with their Geach logics via realization.

12.2 A Few New Examples

All Geach logics have justification counterparts. We now give two simple examples to show that the phenomenon is not restricted to Geach logics. The book *Modal Logic* [6] contains an excellent treatment of Sahlqvist formulas. In Section 3.6 of that book Example 3.5.5 discusses some logics that are not of much intrinsic interest but which will do to illustrate our point.

The scheme $\Box(X \supset \Diamond X)$ is shown in [6] to correspond to frame condition $\Gamma\mathcal{R}\Delta \Longrightarrow \Delta\mathcal{R}\Delta$. Justification logics 'prefer' the $\Box$ operator over the $\Diamond$ operator, so we will use an equivalent modal axiom scheme: $\Box(\Box X \supset X)$. We propose the following scheme for a corresponding justification logic, where f is a one-place function symbol: $f(t){:}(t{:}X \supset X)$.

It is easy to see that the forgetful functor maps the justification logic into the modal logic in this case. To show realization we must show the canonical justification model is based on a frame meeting the given modal frame condition. Suppose $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is the canonical model for the justification logic extending J_0 with the scheme $f(t){:}(t{:}X \supset X)$ and an axiomatically appropriate constant specification. Assume that for some $\Gamma, \Delta \in \mathcal{G}$ we have $\Gamma\mathcal{R}\Delta$ but not $\Delta\mathcal{R}\Delta$. We derive a contradiction.

Since we do not have $\Delta\mathcal{R}\Delta$ then $\Delta^\sharp \not\subseteq \Delta$. Then for some justification formulas, $t{:}X \in \Delta$ but $X \notin \Delta$. Since Δ is maximally consistent, $t{:}X \supset X \notin \Delta$. Since Γ is maximally consistent, $f(t){:}(t{:}X \supset X) \in \Gamma$. And since $\Gamma\mathcal{R}\Delta$, $\Gamma^\sharp \subseteq \Delta$, so $t{:}X \supset X \in \Delta$. This is our contradiction, and realization is established using Theorem 12.1.

The axiom scheme $(X \wedge \Diamond\neg X) \supset \Diamond X$ corresponds to the frame condition $(\Gamma\mathcal{R}\Delta \wedge \Gamma \neq \Delta) \Longrightarrow \Gamma\mathcal{R}\Gamma$. Although it was not noted in [6], this reduces to the simpler frame condition $\Gamma\mathcal{R}\Delta \Longrightarrow \Gamma\mathcal{R}\Gamma$, and it is this condition that we will use. We rewrite the modal scheme to avoid $\Diamond$, obtaining $\Box X \supset (X \vee \Box\neg X)$. We use the following justification axiom scheme, where g is a one-place function symbol: $t{:}X \supset (X \vee g(t){:}\neg X)$.

Just as above, the forgetful functor maps the justification logic into the modal logic. Now suppose $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E}\rangle$ is the canonical justification model for the logic extending J_0 with the proposed scheme and an axiomatically appropriate constant specification. We show the frame $\langle \mathcal{G}, \mathcal{R}\rangle$ meets the appropriate condition.

Suppose $\Gamma\mathcal{R}\Delta$ but not $\Gamma\mathcal{R}\Gamma$; we derive a contradiction. We have $\Gamma^\sharp \subseteq \Delta$ and $\Gamma^\sharp \not\subseteq \Gamma$. By the second of these, $t{:}X \in \Gamma$ but $X \notin \Gamma$ for some $t{:}X$. Since Γ is maximally consistent, $t{:}X \supset (X \vee g(t){:}\neg X) \in \Gamma$, and hence $(X \vee g(t){:}\neg X) \in \Gamma$. And since $X \notin \Gamma$, $g(t){:}\neg X \in \Gamma$. Since $\Gamma^\sharp \subseteq \Delta$, both $X \in \Delta$ and $\neg X \in \Delta$, violating consistency.

Now realization follows.

12.3 A Very Concrete Example

We know modal logic S4.2 and justification logic J4.2 correspond. We now give a concrete instance of an S4.2 theorem and a corresponding realization. The S4.2 theorem is the following.

$$\vdash_{\mathsf{S4.2}} [\Diamond\Box X \wedge \Diamond\Box Y] \supset \Diamond\Box(X \wedge Y) \tag{4}$$

Our proof of Theorem 12.1 was not constructive, but for this example we are able to translate an S4.2 axiomatic proof step by step into a J4.2 proof, though we abbreviate considerably. More will be said about this at the end.

We recall Definition 3.10; if J is a justification logic then $\vdash_{\mathsf{J}} Z$ means there is some axiomatically appropriate constant specification $\mathcal{CS}$ so that $\vdash_{\mathsf{J}(\mathcal{CS})} Z$. We use $v_1, v_2 \ldots$ as justification variables. Finally, we actually realize $[\neg\Box\neg\Box X \wedge \neg\Box\neg\Box Y] \supset \neg\Box\neg\Box(X \wedge Y)$, which is equivalent to (4) but which does not contain $\Diamond$ operators. Nonetheless we will continue to use $\Diamond$ in our discussions because it makes reading easier.

Here is a sketch of a proof for (4). In it we assume several results provable in S4 and simpler logics.

$M1.$	$\vdash_{\mathsf{K}} [\Diamond\Box\Box X \wedge \Box\Diamond\Box Y] \supset \Diamond\Diamond\Box(X \wedge Y)$	
$M2.$	$\vdash_{\mathsf{K4}} \Diamond\Diamond\Box(X \wedge Y) \supset \Diamond\Box(X \wedge Y)$	since $\Diamond\Diamond Z \supset \Diamond Z$
$M3.$	$\vdash_{\mathsf{K4}} [\Diamond\Box\Box X \wedge \Box\Diamond\Box Y] \supset \Diamond\Box(X \wedge Y)$	by $M1.$ and $M2.$
$M4.$	$\vdash_{\mathsf{K4}} \Diamond\Box X \supset \Diamond\Box\Box X$	since $\Box Z \supset \Box\Box Z$
$M5.$	$\vdash_{\mathsf{K4}} [\Diamond\Box X \wedge \Box\Diamond\Box Y] \supset \Diamond\Box(X \wedge Y)$	by $M3.$ and $M4.$
$M6.$	$\vdash_{\mathsf{S4.2}} \Diamond\Box\Box Y \supset \Box\Diamond\Box Y$	since $\Diamond\Box Z \supset \Box\Diamond Z$
$M7.$	$\vdash_{\mathsf{S4.2}} [\Diamond\Box X \wedge \Diamond\Box\Box Y] \supset \Diamond\Box(X \wedge Y)$	by $M5.$ and $M6.$
$M8.$	$\vdash_{\mathsf{S4}} \Diamond\Box Y \supset \Diamond\Box\Box Y$	since $\Box Z \supset \Box\Box Z$
$M9.$	$\vdash_{\mathsf{S4.2}} [\Diamond\Box X \wedge \Diamond\Box Y] \supset \Diamond\Box(X \wedge Y)$	by $M7.$ and $M8.$

We begin with $[\Diamond\Box\Box X \wedge \Box\Diamond\Box Y] \supset \Diamond\Diamond\Box(X \wedge Y)$ which is a theorem of K, so we work in J_0, sketching a proof of a realization for it. In the following, j_1, j_2, and j_3 are justification terms whose existence is guaranteed by Theorem 3.6, abbreviated by 3.6. References to *dist* are to the distributivity axiom scheme, $s{:}(Z \supset W) \supset (t{:}Z \supset s \cdot t{:}W)$ We generally combine it with some simple propositional manipulation.

$J1.$	$\vdash_{\mathsf{J}_0} X \supset (Y \supset (X \wedge Y))$	tautology
$J2.$	$\vdash_{\mathsf{J}_0} j_1{:}\{X \supset (Y \supset (X \wedge Y))\}$	3.6 on $J1$
$J3.$	$\vdash_{\mathsf{J}_0} \neg[j_1 \cdot v_1 \cdot v_3]{:}(X \wedge Y) \supset (v_1{:}X \supset \neg v_3{:}Y)$	*dist* on $J2$
$J4.$	$\vdash_{\mathsf{J}_0} j_2{:}\{\neg[j_1 \cdot v_1 \cdot v_3]{:}(X \wedge Y) \supset (v_1{:}X \supset \neg v_3{:}Y)\}$	3.6 on $J3$
$J5.$	$\vdash_{\mathsf{J}_0} v_5{:}\neg[j_1 \cdot v_1 \cdot v_3](X \wedge Y) \supset \{\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}Y \supset \neg[v_2 \cdot v_1]{:}X\}$	*dist* on $J4$
$J6.$	$\vdash_{\mathsf{J}_0} j_3{:}\{v_5{:}\neg[j_1 \cdot v_1 \cdot v_3](X \wedge Y) \supset \{\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}Y \supset \neg[v_2 \cdot v_1]{:}X\}\}$	3.6 on $J5$
$J7.$	$\vdash_{\mathsf{J}_0} \{\neg[j_3 \cdot v_6 \cdot v_4]{:}\neg v_2{:}v_1{:}X \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}Y\}$ $\supset \neg v_6{:}v_5{:}\neg[j_1 \cdot v_1 \cdot v_3]{:}(X \wedge Y)$	*dist* on $J6$

$J7$ is a realizer for $M1$, and is provable. Thus we have the following provable realization for the K theorem $M1$ $[\Diamond\Box\Box X \wedge \Box\Diamond\Box Y] \supset \Diamond\Diamond\Box(X \wedge Y)$.

$$\vdash_{\mathsf{J}_0} \{\neg[j_3 \cdot v_6 \cdot v_4]{:}\neg v_2{:}v_1{:}X \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}Y\} \supset \neg v_6{:}v_5{:}\neg[j_1 \cdot v_1 \cdot v_3]{:}(X \wedge Y) \tag{5}$$

Next we realize the $\mathsf{K4}$ theorem $\Diamond\Diamond\Box(X \wedge Y) \supset \Diamond\Box(X \wedge Y)$. We have the LP axiom $v_7{:}\neg v_8{:}(X \wedge Y) \supset !v_7{:}v_7{:}\neg v_8{:}(X \wedge Y)$, so we immediately have the following.

$$\vdash_{\mathsf{LP}} \neg !v_7{:}v_7{:}\neg v_8{:}(X \wedge Y) \supset \neg v_7{:}\neg v_8{:}(X \wedge Y) \tag{6}$$

To match the antecedent of (6) with the consequent of (5) we set $v_6 = !v_7$, $v_7 = v_5$, and $v_8 = j_1 \cdot v_1 \cdot v_3$. (This is a unification problem.) When this is done antecedent and consequent match, so they can be 'cut out' yielding (7), which realizes $M3$.

$$\vdash_{\mathsf{LP}} \{\neg[j_3 \cdot !v_5 \cdot v_4]{:}\neg v_2{:}v_1{:}X \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}Y\} \supset \neg v_5{:}\neg[j_1 \cdot v_1 \cdot v_3]{:}(X \wedge Y) \tag{7}$$

The next step is to realize $M4$, $\Diamond\Box X \supset \Diamond\Box\Box X$. We have the LP axiom $v_9{:}X \supset !v_9{:}v_9{:}X$. This gives the theorem $\neg !v_9{:}v_9{:}X \supset \neg v_9{:}X$. Using Theorem 3.6, there is a justification term, call it j_4, provably justifying this. Then using distributivity, we get the desired realizer for $M4$.

$$\vdash_{\mathsf{LP}} \neg[j_4 \cdot v_{10}]{:}\neg v_9{:}X \supset \neg v_{10}{:}\neg !v_9{:}v_9{:}X \tag{8}$$

We match the consequent of (8) with the X part of the antecedent of (7) by setting $v_{10} = j_3 \cdot !v_5 \cdot v_4$, $v_2 = !v_9$, and $v_1 = v_9$. Then, again cutting out the common antecedent/consequent, we have (9), which realizes $M5$.

$$\vdash_{\mathsf{LP}} \{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot v_4]{:}\neg v_9{:}X \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot !v_9]{:}\neg v_3{:}Y\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(X \wedge Y) \tag{9}$$

Next we realize $M6$, $\Diamond\Box\Box Y \supset \Box\Diamond\Box Y$. This, at last, makes use of $\mathsf{J4.2}$ axiom scheme (1). By it we have $\neg f(v_{11}, v_{13}){:}\neg v_{11}{:}v_{12}{:}Y \supset g(v_{11}, v_{13})\neg v_{13}{:}\neg v_{12}{:}Y$. We match the consequent of this with the Y part of the antecedent of (9) by setting $v_4 = g(v_{11}, v_{13})$, $v_{13} = j_2 \cdot v_5 \cdot !v_9$, and $v_3 = v_{12}$. Then cutting out common parts, we get (10), which realizes $M7$.

$$\begin{aligned} \vdash_{\mathsf{J4.2}} \{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot g(v_{11}, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_9{:}X \wedge \\ \neg f(v_{11}, j_2 \cdot v_5 \cdot !v_9){:}\neg v_{11}{:}v_3{:}Y\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(X \wedge Y) \end{aligned} \tag{10}$$

Finally we realize $M8$, $\Diamond\Box Y \supset \Diamond\Box\Box Y$. Using Theorem 3.6, let us say j_5 justifies the LP theorem $\neg !v_3{:}v_3{:}Y \supset \neg v_3{:}Y$. Then using distributivity we can get $\neg[j_5 \cdot v_{14}]{:}\neg v_3{:}Y \supset \neg v_{14}{:}\neg[!v_3 \cdot v_3]{:}Y$. We match the consequent of this with the Y part of the antecedent of (10) by setting

$v_{14} = f(v_{11}, j_2 \cdot v_5 \cdot !v_9)$ and $v_{11} = !v_3$. Once again cutting out the common portion, we finally arrive at (11) which is a provable realization of $M9$.

$$\begin{aligned} \vdash_{\mathsf{J4.2}} \{ &\neg[j_4 \cdot j_3 \cdot !v_5 \cdot g(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_9{:}X \wedge \\ &\neg[j_5 \cdot f(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_3{:}Y\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(X \wedge Y) \end{aligned} \tag{11}$$

Some concluding remarks about this example. For S4, K4, and K, there are realization *algorithms*, and we used them behind the scenes above. We also used unification at various places. Not all modal proofs convert to justification logic proofs in this direct way. Ren-June Wang introduced the notion of a *non-circular* proof (for S4) in [30]. It is these that can be converted. It is something that warrants further study.

13 Conclusion

We have shown that justification logic counterparts with realization connections exist for a wide variety of modal logics. There are three topics that naturally come to mind for further investigation: 1. what is the range of modal logics to which this applies, 2. to what extent can the work be made constructive, and 3. can quantification be brought into the picture. We briefly discuss each of these open topics.

13.1 The Range of Realization

Theorem 12.1 is very general, and we have given many examples that it covers. We showed it applies to all Geach logics, an infinite family that includes all the modal logics that were previously known to have justification counterparts. But also, in Section 12.2, we gave two simple examples of modal logics outside the Geach family, but to which Theorem 12.1 applies. It is a natural question, to which logics does our methodology apply. We conjecture that the answer is the family of Sahlquist logics, or at least includes this family. At the moment, this is completely open.

13.2 Constructivity

The first proof of realization, connecting S4 and LP, was constructive, and almost all subsequent proofs have also been constructive. The first work on S5 used non-constructive methods, but eventually this was supplemented by several different constructive arguments. Constructive proofs of realization make use of cut-free proof systems, extracting a realizer from the proof of a formula and not from the formula itself. Essentially any cut-free approach seems to work: sequent calculus, tableau, prefixed tableau, nested sequent, hypersequent. Details differ, of course, but the connection between constructive realization and cut-free proofs seems clear.

Almost all constructive proofs have followed the same general path, producing a realization from a formal proof. In fact it is considerably easier to produce a *quasi*-realization since one needn't get involved with the behavior of substitution. Once a quasi-realization is obtained Algorithm 11.6 can be applied to produce a realization, and constructivity is preserved. We followed this route in [14] where we gave a constructive tableau based realization proof connecting S4 and LP that was detailed enough so that we could also give an implementation in Prolog.[1]

In [17] we gave new kinds of prefixed tableau and nested sequent cut-free proof systems specifically for the Geach logics. Also in [24] very general cut-free proof systems are given that cover

[1]The program as given in [14], a technical report, is missing a line. A corrected version is available from my web site, `http://www.melvinfitting.org/`.

even a wider range. It is not yet known if these can be adapted to provide constructive realization proofs, but if they can it would be very interesting.

All this raises the question of what is the connection between realizations and proof theory. Realizations seem to provide a kind of information flow analysis, though exactly what this means is rather vague. Nonetheless, there is the tantalizing suggestion that one might be able to extract some version of a cut-free proof directly from a realization. I'm not at all sure how to address such a project, but there is the feeling that there might be something of interest here.

13.3 Quantified Justification Logics

So far there is a single example of a quantified justification logic, a first-order version of LP. This was introduced axiomatically in [5] with a semantics appearing in [16]. A constructive realization theorem appears in [5], and a non-constructive version can be found in [15]. Investigation of quantified justification logics is in its early stages. For instance, semantics and completeness arguments that proceed along the lines of the present paper, using canonical justification models, are not always appropriate. When applied to quantified logics directly they produce varying domain, but monotonic, models. Thus such arguments are unsuitable for justification counterparts of quantified S5, for instance. This is work currently in progress. As always, modal quantification brings complexities, and the complexities become magnified in the justification setting.

13.4 Conclusion Concluded

The original motivation for justification logic was very clear—LP played an essential role in providing an arithmetic semantics for intuitionistic logic. The discovery of the broad range of justification logics was somewhat unexpected. It is not clear what they tell us about their corresponding modal logics. Our work is cut out for us.

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