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A MODAL LOGIC ANALOG OF SMULLYAN'S FUNDAMENTAL THEOREM

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§ 1. Introduction

In classical logic to some extent both HERBRAND's Theorem [3] and SMULLYAN's Fundamental Theorem [6] accomplish the same thing. Each substitutes for the problem of first-order provability of a given formula the problem of proving one of an infinite sequence of formulas in propositional logic, and they both do so constructively. They differ in the manner of associating the infinite sequence of formulas with the given one, but their overall effect is similar. We had entertained hopes of proving a natural analog of one or both of these theorems for the modal logic S4, partly because of its relationship to forcing [1]. Unfortunately, it seems impossible to do so. The parameters involved in the classical logic SMULLYAN's Theorem, for example, have characteristics of epsilon-terms, and there are reasons why analogs of such things can not be introduced into S4. See [2] for a discussion of this point. Fortunately, using a device of STALNAKER and THOMASON [7, 8] one can produce a logic, closely related to S4, in which the things needed can be introduced.

We begin this paper then with an informal discussion of the model theory of a logic clearly related to first order S4, a very natural logic to consider by any standards (we call it λS4). Having used model theory to make clear what we have in mind, we then formulate λS4 syntactically (indeed we do not use model theory in any of our proofs). Next we prove constructively that λS4 and more ordinary first-order S4 have the same constant-free theorems. Then we state and prove constructively an analog of SMULLYAN's Fundamental Theorem for λS4, reducing the problem of provability to that of provability of one of an infinite sequence of formulas in the propositional part of λS4.

Possibly the main value of this paper lies in the introduction of the logic λS4. It is a natural logic to study, as well as a fruitful one, as the Fundamental Theorem evidences. Indeed, we hope in a future paper to produce a natural analog of HERBRAND's Theorem for it. We believe there are many interesting things to be discovered concerning λS4 and we hope this paper stimulates work on them.

§ 2. The Logic λ S4

We assume the reader is familiar with the KRIPKE model theory for first-order S4 (without the BARCAN formula) [4, 5]. Starting from this we develop, in a highly informal manner, a model theory for a logic we call λS4. This model theory can, of course, be developed rigorously but we need it only for motivation.

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Associated with each of the possible worlds of a KRIPKE S4 model is a non-empty set of constants. These may be thought of as the "things" of that world. Thus $(\exists x)\, X(x)$ is true in a given possible world provided $X(c)$ is true there, where c is a constant associated with that world, that is, provided $X(x)$ is true of some "thing" in that world. Following STALNAKER and THOMASON [7, 8] we introduce a second kind of constant into the language of S4, a "name" constant. (We shall use the phrase *thing-constant* for the standard kind of constant above.) STALNAKER and THOMASON discuss "Miss America" and "The President of the United States" as informal examples, names naming different things in different possible worlds. Formally a *name-constant*, f, is a function defined on the collection of possible worlds of a KRIPKE S4 model (more precisely, on $\mathscr{R}$-closed subcollections) having the property that if Γ is a possible world, $f(\Gamma)$ is a thing constant associated with Γ. Informally we shall say f *names the thing* $f(\Gamma)$ *in the world* Γ.

Next we must specify when formulas involving these name constants are true in possible worlds. We would like to say, simply, $X(f)$ is true in the world Γ provided $X(x)$ is true of the thing which f names in Γ. Unfortunately, this immediately leads to an ambiguity in the case of $\Diamond X(f)$. To say $\Diamond X(f)$ is true in the world Γ could mean 1) $\Diamond X(f(\Gamma))$ is true in Γ, and so $X(f(\Gamma))$ is true in some world possible relative to Γ, or 2) $X(f)$ is true in some world possible relative to Γ, say Δ, and so $X(f(\Delta))$ is true in Δ. To eliminate this ambiguity, STALNAKER and THOMASON introduce an abstraction operator into the language, which we shall write using λ notation. Then, the convention followed is, $(\lambda x X)\,(f)$ is true in the possible world Γ provided $X(f(\Gamma))$ is true in Γ. The above ambiguous case now breaks into two distinct formulas, $(\lambda x \Diamond X)\,(f)$ and $\Diamond(\lambda x X)\,(f)$. For a fuller discussion of this point see their papers [7, 8]. Of course some convention must be adopted concerning atomic formulas, but this need not be gone into here.

We shall be interested in these new name-constants and not in the more usual thing-constants. What we want is a formulation of first-order S4 which does not mention thing-constants. If we require that our models have "enough" name-constants this can be done. When we say "enough" we have two conditions in mind. The first is, simply, that each thing have a name, that is, in each possible world and for each thing-constant of that world, some name-constant names that thing. The second condition is more complex and is directly related to the proof of the Fundamental Theorem which we will give. It is a requirement that there be certain "choice" names. Let Γ be a possible world and $X(x)$ a formula with only the variable x free. We require that if $X(x)$ is true of some thing in Γ and also of some thing in each world possible relative to Γ, then there is a name which, in each world possible relative to Γ, names a thing of which $X(x)$ is true. An equivalent way of stating this is: if $\Box(\exists x)\, X$ is true in Γ, then there is some name-constant, say f, such that $\Box(\lambda x X)\,(f)$ is true in Γ. f behaves like a "choice" name, choosing in each world something making $X(x)$ true. It is essentially an epsilon-term in HILBERT's sense. See [2] for a general treatment of S4 epsilon-terms.

Our first condition above allows us to state something akin to the usual rule of necessitation. Suppose, for example, that the only free variable of X is x, and suppose $(\lambda x \Box X)(f)$ is not valid. Then there is a possible world, Γ, in a KRIPKE model, in which this formula is not true. Let us say f names the thing c in the world Γ. Then we have that $\Box X(c)$ is not true in Γ. There must then be a world, say Δ, possible relative to Γ, in which $X(c)$ is not true. If we assume our model meets the first condition above, the thing c has a name in Δ, say g. Then $(\lambda x X)(g)$ is not true in Δ. Now, if h is a name not occurring in X, $(\lambda x X)(h)$ is not valid, for we can arrange matters so that h and g act alike in Δ, and $(\lambda x X)(g)$ is not true there. Turning this argument around we have: if $(\lambda x X)(h)$ is valid, where h does not occur in X, then $(\lambda x \Box X)(f)$ is also valid. This will be the basis for our rule of necessitation below.

The reader may verify that the second condition above allows us to conclude: if $\Box(\lambda x X)(f) \supset Y$ is valid, where f does not occur in X or Y, then $\Box(\exists x) X \supset Y$ is valid. This will be the basis for rule λR4 below.

The other rules and axioms we will assume are much more straightforward. For instance, if x is the only free variable of X and Y, and f is a name-constant, we would want

$$[\lambda x(X \wedge Y)](f) \equiv [(\lambda x X)(f) \wedge (\lambda x Y)(f)]$$

to be true. We will take a more general version of this as an axiom.

Now we have finished with our model theoretic motivation. We give, formally, the system λS4. The primitive symbols are those of ordinary first order S4 with the addition of λ. To be definite, we take $\wedge$, $\sim$, $\exists$, $\Box$ and λ as primitive and define the other connectives, quantifier and modal operator as usual. We use x, y, z, etc. for variables and f, g, h, etc. for (name) constants. The definition of formula is as usual, with the added clauses: 1) an atomic formula is an expression of the form $P(x_1, \ldots, x_n)$ where P is an n-place predicate letter and $x_1, \ldots, x_n$ are variables; 2) if X is a formula, x is a variable and f is a constant, then $(\lambda x X)(f)$ is a formula. We take x to be bound in $(\lambda x X)(f)$.

We often use the notation $X(x/f)$ to denote the result of substituting f for free occurrences of x in X. To simplify notation we use $(\lambda x_1 \ldots x_n X)(f_1, \ldots, f_n)$ as an abbreviation for $(\lambda x_1(\lambda x_2 \ldots (\lambda x_n X)(f_n) \ldots)(f_2))(f_1)$. Further, we often use $\boldsymbol{x}$ and $\boldsymbol{f}$ to denote sequences of variables and constants respectively, depending on context. Thus we may abbreviate $(\lambda x_1 \ldots x_n X)(f_1, \ldots, f_n)$ simply as $(\lambda \boldsymbol{x} X)(\boldsymbol{f})$.

If a formula has no occurrences of any free variables we will call it a *closed* formula. All our theorems are closed formulas. If $(\lambda \boldsymbol{x} X)(\boldsymbol{f})$ has no free variables we call it a *λ-closure* of the formula X. If X itself is closed we also consider it to be a λ-closure of itself.

The rules and axioms of λS4 are as follows.

Rules.

λR1: If X and Y are closed formulas then

$$\frac{X \quad X \supset Y}{Y}.$$

λR2: If $\boldsymbol{f}$ is a sequence of distinct constants, none of which occurs in X, $\boldsymbol{g}$ is a sequence of constants of the same length, and $(\lambda \boldsymbol{x} X)(\boldsymbol{f})$ is a closed formula, then

$$\frac{(\lambda \boldsymbol{x} X)(\boldsymbol{f})}{(\lambda \boldsymbol{x} \Box X)(\boldsymbol{g})}.$$

λR3: If $(\exists x) X$ and Y are closed formulas and f does not occur in X or Y, then

$$\frac{(\lambda x X)(f) \supset Y}{(\exists x) X \supset Y}.$$

λR4: If $(\exists x) X$ and Y are closed formulas and f does not occur in X or Y, then

$$\frac{\Box(\lambda x X)(f) \supset Y}{\Box(\exists x) X \supset Y}.$$

Axiom schemas.

λA1: If y is not free in X, but y is free for x in X, $(\lambda x X)(f) \equiv [\lambda y X(x/y)](f)$.

λA2: If x is not free in X, $(\lambda x X)(f) \equiv X$.

λA3: If $x \neq y$ and y is free for x in X, $(\lambda y x X)(f, f) \equiv [\lambda y X(x/y)](f)$.

λA4: If $x_1 \neq x_2$, $(\lambda x_1 x_2 X)(f_1, f_2) \equiv (\lambda x_2 x_1 X)(f_2, f_1)$.

λA5: $[\lambda \boldsymbol{x}(X \wedge Y)](\boldsymbol{f}) \equiv [(\lambda \boldsymbol{x} X)(\boldsymbol{f}) \wedge (\lambda \boldsymbol{x} Y)(\boldsymbol{f})]$.

λA6: $(\lambda \boldsymbol{x} \sim X)(\boldsymbol{f}) \equiv \sim(\lambda \boldsymbol{x} X)(\boldsymbol{f})$.

λA7: If y is not in the sequence $\boldsymbol{x}$, $[\lambda \boldsymbol{x}(\exists y) X](\boldsymbol{f}) \equiv (\exists y)[(\lambda \boldsymbol{x} X)](\boldsymbol{f})$.

λA8: X, if X is a classical tautology.

λA9: $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$.

λA10: $\Box X \supset X$.

λA11: $\Box X \supset \Box\Box X$.

λA12: $(\lambda x X)(f) \supset (\exists x) X$.

This completes the presentation of the system λS4.

§ 3. Statement of Results

Now that the system λS4 has been presented we can state precisely the principal results which will be established in the remaining sections of this paper. Since we want to establish a relationship between λS4 and a more conventional first-order S4, for definiteness sake we begin with a formulation of such a system, which we call FS4.

The language of FS4 differs from that of λS4 in not having λ as one of its primitive symbols, in not having f, g, h, etc. as symbols for name-constants, but in having a, b, c, etc. as symbols for thing-constants. The axioms and rules are as follows, where X and Y stand for any *closed* formulas.

Rules.

FR1: $\dfrac{X \quad X \supset Y}{Y}$.

FR2: $\dfrac{X}{\Box X}$.

FR3: If c does not occur in Y, then

$$\frac{X \supset Y}{(\exists x)\, X(c/x) \supset Y}.$$

Axiom schemas.

FA1: X, if X is a classical tautology.

FA2: $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$.

FA3: $\Box X \supset X$.

FA4: $\Box X \supset \Box\Box X$.

FA5: $X \supset (\exists x)\, X(c/x)$.

Then the first important result of this paper is

Theorem 1. *If X is a closed formula with no constants (of either kind) X is a theorem of* λS4 *if and only if X is a theorem of* FS4.

We will establish this constructively by showing how to translate proofs from each system to the other. The statement of the next theorem, the analog of SMULLYAN's Fundamental Theorem, requires several preliminary definitions.

By a *regular formula* we mean a closed formula of λS4 of one of the following three types:

(1) $(\lambda x X)(f) \supset (\exists x)\, X$,

(2) a λ-closure of $(\exists x) \Diamond X \supset \Diamond(\exists x)\, X$,

(3) $(\exists x)\, X \supset (\lambda x X)(f)$ where f does not occur in X.

In the type (3) formula above, we call f *chosen* by the formula. By a *regular sequence for Y* (a closed λS4 formula) we mean a finite sequence, $R_1, R_2, \ldots, R_n$, of regular formulas, such that if R_i is of type (3) the constant chosen by R_i does not occur in R_j for any $j > i$, or in Y. By a *regular set for Y* we mean a finite set R which can be arranged in a regular sequence for Y. We use R^c to denote any conjunction of all the elements of R. Finally, by the *propositional part of* λS4 we mean λS4 without axiom λA12 and rules λR3 and λR4. Then the analog of the classical Fundamental Theorem may be stated as follows.

Theorem 2. *Let X be a closed formula of* λS4. *X is a theorem of* λS4 *if and only if there is a regular set for X, call it R, such that $\Box R^c \supset X$ is a theorem of the propositional part of* λS4.

We have defined the propositional part of λS4 as we did because it is a logic with an intuitively appealing model theory, though we make no use of it in this paper. In fact, a more restricted logic will do, as our proof will show. If we call the *strict propositional part of* λS4 the propositional part of λS4 with rule λR2 replaced by the simpler

$$\lambda\text{R2*}: \quad \frac{X}{\Box X} \quad \text{if } X \text{ is a closed formula}$$

then we will actually show

Theorem 2*. *If X is a closed formula of λS4, X is a theorem of λS4 if and only if there is a regular set for X, call it R, such that $\Box R^c \supset X$ is a theorem of the strict propositional part of λS4.*

Our proof of this is entirely constructive. Since one can effectively generate the infinite sequence of sets regular for X, we have the form of the theorem referred to in the introduction. The decidability of either the propositional part of λS4 or of the strict propositional part is an interesting open question.

§ 4. Development of λS4

In this section we outline how λS4 may be developed. We give a condensed sample proof in the system (of the converse of the Barcan formula). Then we sketch proofs of certain metatheorems, and finally show half of our theorem 1. We note without proof that analogs of axioms λA5, λA6, λA7, λA12 and rules λR3 and λR4 hold for the other connectives and quantifiers.

Theorem. *If X has only x free, $(\exists x)\Diamond X \supset \Diamond(\exists x)X$ is a theorem.*

Proof. Choose a constant, f, not in X. By λA13, $(\lambda x X)(f) \supset (\exists x)X$. By λA2, $(\lambda x X)(f) \supset (\lambda x(\exists x)X)(f)$, so $(\lambda x[X \supset (\exists x)X])(f)$. Then by λR2, $(\lambda x \Box[X \supset (\exists x)X])(g)$ and we may choose g so that it does not occur in X. Adapting standard arguments we may conclude

$$(\lambda x[\Diamond X \supset \Diamond(\exists x)X])(g), \quad (\lambda x \Diamond X)(g) \supset (\lambda x \Diamond(\exists x)X)(g).$$

By λA2 again, $(\lambda x \Diamond X)(g) \supset \Diamond(\exists x)X$. Finally, by λR3, $(\exists x)\Diamond X \supset \Diamond(\exists x)X$.

We leave it to the reader as a good exercise to show the following generalization.

Theorem. *Any λ-closure of $(\exists x)\Diamond X \supset \Diamond(\exists x)X$ is provable.*

All the axiom schemas of λS4 are of the form: all λ-closures of X are provable. Let us introduce the notation $\vdash X$ to symbolize this, i.e. that all λ-closures of X are provable in λS4.

Theorem. $$\frac{\vdash X \quad \vdash X \supset Y}{\vdash Y}.$$

Proof. Suppose $\vdash X$ and $\vdash X \supset Y$. Let $(\lambda \boldsymbol{y} Y)(\boldsymbol{g})$ be a λ-closure of Y we wish to show is provable. Let $\boldsymbol{x}$ be a sequence consisting of all the free variables of

X other than those already in $\boldsymbol{y}$, and let $\boldsymbol{f}$ be a sequence of constants of the same length as $\boldsymbol{x}$. $\vdash X$ so $(\lambda \boldsymbol{xy} X)(\boldsymbol{f}, \boldsymbol{g})$ is a theorem. $\vdash X \supset Y$, so similarly, $(\lambda \boldsymbol{xy}(X \supset Y))(\boldsymbol{f}, \boldsymbol{g})$ is a theorem. But then,

$$(\lambda \boldsymbol{xy} X)(\boldsymbol{f}, \boldsymbol{g}) \supset (\lambda \boldsymbol{xy} Y)(\boldsymbol{f}, \boldsymbol{g})$$

is a theorem, so by λR1, $(\lambda \boldsymbol{xy} Y)(\boldsymbol{f}, \boldsymbol{g})$ is a theorem. Since the variables in $\boldsymbol{x}$ are not free in Y, use of λA2 produces $(\lambda \boldsymbol{y} Y)(\boldsymbol{g})$.

Theorem. $\dfrac{\vdash X}{\vdash \Box X}$.

Proof. By use of λR2.

Theorem. *Suppose x is not free in Y, then*

$$\frac{\vdash X \supset Y}{\vdash (\exists x)\, X \supset Y}.$$

Proof. Suppose $\vdash X \supset Y$. Let $(\lambda \boldsymbol{y}[(\exists x)\, X \supset Y])(\boldsymbol{f})$ be a λ-closure of $(\exists x)\, X \supset Y$ we wish to prove. Without loss of generality we may suppose x is not in the sequence $\boldsymbol{y}$ (x is not free in $(\exists x)\, X \supset Y$ and we have λA2). Let g be a constant not in $\boldsymbol{f}$ or in X or Y. Since $\vdash X \supset Y$, $(\lambda x(\lambda \boldsymbol{y}(X \supset Y))(\boldsymbol{f}))(g)$ is a theorem, hence so is $(\lambda x(\lambda \boldsymbol{y} X)(\boldsymbol{f}))(g) \supset (\lambda x(\lambda \boldsymbol{y} Y)(\boldsymbol{f}))(g)$. Using λA2, $(\lambda x(\lambda \boldsymbol{y} X)(\boldsymbol{f}))(g) \supset (\lambda \boldsymbol{y} Y)(\boldsymbol{f})$. Now by λR3, $(\exists x)(\lambda \boldsymbol{y} X)(\boldsymbol{f}) \supset (\lambda \boldsymbol{y} Y)(\boldsymbol{f})$. By λA7, $(\lambda \boldsymbol{y}(\exists x)\, X)(\boldsymbol{f}) \supset (\lambda \boldsymbol{y} Y)(\boldsymbol{f})$. So, finally, $(\lambda \boldsymbol{y}[(\exists x)\, X \supset Y])(\boldsymbol{f})$.

Theorem. *If y is not in X,* $\vdash X(x/y) \supset (\exists x)\, X$.

Proof. Using λA12 and λA1.

One may show a variant of the replacement theorem, as usual, by an induction on degree.

Theorem. *Let A, B, X and Y be formulas. Let Y be the result of replacing, in X, the formula A at some or all of its occurrences by B. Then*

$$\frac{\vdash A \equiv B}{\vdash X \equiv Y}.$$

This form is most convenient for λS4 as presented; it is closely connected with more conventional formulations, as the following shows.

Theorem. $\vdash X$ *if and only if the universal closure of X is a theorem.*

Finally, using the above we show

Theorem. *If X is a closed formula with no constant symbols which is a theorem of* FS4, *then X is a theorem of* λS4.

Proof. Let $X_1, X_2, \ldots, X_n = X$ be a proof of the constant-free formula X in FS4. Let $c_1, c_2, \ldots, c_k$ be all the thing-constants occurring in the proof, and let $x_1, x_2, \ldots, x_k$ be k distinct variables not occurring in any formula of the proof. Let $X_i^* = X_i(\boldsymbol{c}/\boldsymbol{x})$. We claim $\vdash X_i^*$ for $i = 1, 2, \ldots, n$.

But this is easy to see, for if X_i is one of the axioms FA1, FA2, FA3 or FA4, $\vdash X_i^*$ by λA8, λA9, λA10 or λA11 respectively. If X_i is an FA5 axiom, $\vdash X_i^*$ using one of the above theorems. Finally, if X_i comes from earlier lines of the proof using FR1, FR2 or FR3, $\vdash X_i^*$, again, using the above theorems. Thus $\vdash X_n^*$. Since X_n has no constants, $X_n^* = X_n = X$ and we are done.

Remark. The above does not make use of λR4.

§ 5. The Relation of λS4 and FS4

We devote this section to a proof of the converse of the last theorem of § 4, namely

Theorem. *If X is a closed formula with no constant symbols which is a theorem of λS4, then X is a theorem of* FS4.

We will rely chiefly on two lemmas which we give before discussing the rather complicated proof translating procedure

Lemma 1. *Let $A(x)$ be a formula of* FS4. *Then the universal closure of the following is an* FS4 *theorem:*

$$(\forall x)\{[(\exists x)\,A(x) \supset A(x)] \supset A(x)\} \equiv (\exists x)\,A(x).$$

Lemma 2. *Let $A(x)$ and B be formulas of* FS4 *with x not free in B. Then the universal closure of the following is an* FS4 *theorem;*

$$(\forall x)\{[(\exists x)\,A(x) \supset A(x)] \supset B\} \equiv B.$$

Now we begin discussing the translation process. Rule λR4 allows us to pass from $\Box(\lambda x X)(f) \supset Y$ to $\Box(\exists x)\,X \supset Y$ provided f does not occur in X or Y. Let us call these name-constants, like f here, which are thus used in λR4 applications *special* (with respect to a given proof) and the other name-constants *ordinary.*

Suppose $X_1, X_2, \ldots, X_n$ is a λS4 proof and that we have used λR4 to conclude $X_j = \Box(\exists x)\,X \supset Y$ from $X_i = \Box(\lambda x X)(f) \supset Y$ $(j > i)$. Let g be some name-constant not occurring in the proof and consider the following sequence:

$$X_1, X_2, \ldots, X_i, X_1(f/g), X_2(f/g), \ldots, X_i(f/g), X_{i+1}, \ldots, X_n.$$

This is still a λS4 proof of X_n, but we may now infer X_j from $X_i(f/g)$ instead of X_i. The gain is this: the special constant, g, involved in this new λR4 application does not occur in the proof anywhere after the hypothesis of this λR4 application (while f might). By repeated uses of this sort of trick we may produce a proof in λS4 of X_n having the following properties:

1) No special constant occurs in the proof after the hypothesis of the λR4 rule application in which it is involved (and hence different applications of λR4 involve different special constants).

2) If rule λR2 is used to conclude $(\lambda x \Box X)(\boldsymbol{g})$ from $(\lambda x X)(\boldsymbol{f})$, none of $\boldsymbol{f}$ are special, and none of $\boldsymbol{f}$ occur in the proof after $(\lambda x X)(\boldsymbol{f})$.

3) If rule λR3 is used to conclude $(\exists x)\,X \supset Y$ from $(\lambda x X)(f) \supset Y$, f is not special, and f does not occur in the proof after $(\lambda x X)(f) \supset Y$.

Let us call a λS4 proof satisfying these three conditions a *normal* proof. We have, then, if X has a λS4 proof, X has a normal proof. We show how to translate a normal proof from λS4 into a proof in FS4. The translation depends on the number of applications of λR4 which are involved. To make the notation simpler, let us work with a proof in which there are *three* such applications, though the method will be seen to be general. Furthermore we may suppose the conclusion of each λR4 application immediately follows the hypothesis, to further simplify notation. Thus, let us suppose X is a formula with no constants, and the following is a normal proof of X, wherein all λR4 applications are indicated.

$$\begin{array}{lll}
1) & X_1 & \\
2) & X_2 & \\
\vdots & \vdots & \\
n_3) & X_{n_3} & = \Box(\lambda x P_3)(f_3) \supset Y_3 \\
n_3+1) & X_{n_3+1} & = \Box(\exists x) P_3 \supset Y_3 \\
\vdots & \vdots & \\
n_2) & X_{n_2} & = \Box(\lambda x P_2)(f_2) \supset Y_2 \\
n_2+1) & X_{n_2+1} & = \Box(\exists x) P_2 \supset Y_2 \\
\vdots & \vdots & \\
n_1) & X_{n_1} & = \Box(\lambda x P_1)(f_1) \supset Y_1 \\
n_1+1) & X_{n_1+1} & = \Box(\exists x) P_1 \supset Y_1 \\
\vdots & \vdots & \\
n) & X_n & = X.
\end{array}$$

By the conditions of λR4, f_3 does not occur in P_3 or Y_3. Moreover, since the proof is normal, f_3 does not occur after the n_3th step. Similarly, f_2 does not occur in P_2, Y_2, or after the n_2th step, and f_1 does not occur in P_1, Y_1, or after the n_1th step.

The special constants of this proof are f_1, f_2 and f_3. Let $g_1, g_2, \ldots, g_k$ be the ordinary constants. Let x_1, x_2 and x_3 be three variables not used in the proof, and $c_1, c_2, \ldots, c_k$ be k different FS4 thing-constants.

Suppose $\Box W$ is a subformula of X_j. We say f_i is *attached to* $\Box W$ if W has a subformula of the form $(\lambda x Z)(f_i)$ which is not a subformula of $\Box R$, a subformula of W. That is, if by taking a subformula of a subformula of etc. of W we can reach a formula of the form $(\lambda x Z)(f_i)$ without first reaching one of the form $\Box R$.

Now we are ready to define our translation, or more precisely, a sequence of four translations.

First, let us define $T_0(Z)$ to be the result of replacing each subformula of Z of the form $(\lambda x W)(g_i)$ (g_i ordinary) by $W(x/c_i)$. Let us note that since $X = X_n$ has no constants, $T_0(X) = X$. Moreover, if Z has no special constants, $T_0(Z)$ is a formula of FS4. Thus $T_0(X_i)$ is an FS4 formula provided $i > n_1$.

Next, let S_1 be the formula $T_0((\exists x_1) P_1(x/x_1) \supset P_1(x/x_1))$ that is, $(\exists x_1) T_0(P_1)(x/x_1) \supset \supset T_0(P_1)(x/x_1)$. Let us define a translation T_1 as follows. Let Z be some λS4 formula. First, form $T_0(Z)$. In the resulting formula replace each subformula of the form $\Box W$ which has f_1 attached by $\Box(\forall x_1)(S_1 \supset W)$. Finally, in the resulting formula, replace each subformula of the form $(\lambda x\, Q)(f_1)$ by $Q(x/x_1)$. Call the result $T_1(Z)$.

We note that if Z has no occurrences of f_2 or f_3 then $T_1(Z)$ is a formula of FS4. Thus $T_1(X_i)$ is an FS4 formula provided $i > n_2$. Further, $T_1(X_i) = T_0(X_i)$ provided $i > n_1$.

Next, let S_2 be the formula $T_1((\exists x_2) P_2(x/x_2) \supset P_2(x/x_2))$ that is, $(\exists x_2) T_1(P_2)(x/x_2) \supset \supset T_1(P_2)(x/x_2)$. We define our next translation, T_2, as follows. Let Z be some λS4 formula. First, form $T_1(Z)$. In the resulting formula, replace each subformula of the form $\Box W$ which has f_2 attached by $\Box(\forall x_1)(\forall x_2)[(S_1 \wedge S_2) \supset W]$. Finally, in the resulting formula, replace each subformula of the form $(\lambda x\, Q)(f_2)$ by $Q(x/x_2)$. Call the result $T_2(Z)$.

As above, $T_2(X_i)$ is an FS4 formula provided $i > n_3$, and $T_2(X_i) = T_1(X_i)$ provided $i > n_2$.

Again, let S_3 be the formula $T_2((\exists x_3) P_3(x/x_3) \supset P_3(x/x_3))$ that is, $(\exists x_3) T_2(P_3)(x/x_3) \supset T_2(P_3)(x/x_3)$. We define our last translation, T_3, following the above pattern. Let Z be a λS4 formula. First, form $T_2(Z)$. In the resulting formula, replace each subformula of the form $\Box W$ which has f_3 attached by $\Box(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset W]$. Finally, in the resulting formula replace each subformula of the form $(\lambda x\, Q)(f_3)$ by $Q(x/x_3)$. Call the result $T_3(Z)$.

Now, $T_3(X_i)$ is an FS4 formula for *each* i, and $T_3(X_i) = T_2(X_i)$ if $i > n_3$. We assert the following is a sequence of FS4 theorems:

1) $(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset T_3(X_1)]$

2) $(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset T_3(X_2)]$

$\vdots$ $\quad$ $\vdots$

n_3) $(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset T_3(X_{n_3})]$

$n_3 + 1$) $(\forall x_1)(\forall x_2)[(S_1 \wedge S_2) \supset T_2(X_{n_3+1})]$

$\vdots$ $\quad$ $\vdots$

n_2) $(\forall x_1)(\forall x_2)[(S_1 \wedge S_2) \supset T_2(X_{n_2})]$

$n_2 + 1$) $(\forall x_1)[S_1 \supset T_1(X_{n_2+1})]$

$\vdots$ $\quad$ $\vdots$

n_1) $(\forall x_1)[S_1 \supset T_1(X_{n_1})]$

$n_1 + 1$) $T_0(X_{n_1+1})$

$\vdots$ $\quad$ $\vdots$

n) $T_0(X_n)$.

Then, since $T_0(X_n) = T_0(X) = X$, we will have finished a proof of theorem 1.

It is a simple but useful observation that we may replace

$$(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset W]$$

by $(\forall x_1)(\forall x_2)[(S_1 \wedge S_2) \supset W]$ wherever it occurs in a formula, provided x_3 is not free in W. This may be easily shown.

$$(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset W]$$

is equivalent to

$$(\forall x_1)(\forall x_2)(\forall x_3)\{S_3 \supset [(S_1 \wedge S_2) \supset W]\}.$$

But, S_3 is $(\exists x_3)\, T_2(P_3)(x/x_3) \supset T_2(P_3)(x/x_3)$ and by construction, x_3 is not free in S_1, S_2 or W. So by lemma 2, the above formula is equivalent to

$$(\forall x_1)(\forall x_2)[(S_1 \wedge S_2) \supset W].$$

There are similar useful replacements for the case that x_3 and x_2 are not free, and x_3, x_2 and x_1. One immediate consequence of this is that the proof translation as given above is equivalent to: replace each X_i by

$$(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset T_3(X_i)].$$

For instance, consider $n_2 \geqq i \geqq n_3$. Then

$$(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset T_3(X_i)]$$

is equivalent to

$$(\forall x_1)(\forall x_2)[(S_1 \wedge S_2) \supset T_3(X_i)]$$

since x_3 is not free in $T_3(X_i)$ by construction, and for $i > n_3$, $T_3(X_i) = T_2(X_i)$. If we call the formula $(\forall x_1)(\forall x_2)(\forall x_3)[(S_1 \wedge S_2 \wedge S_3) \supset T_3(Z)]$ the *T_3-translation* of Z (and similarly for T_2 and T_1) we have shown that, for each i, the translation of X_i as given above is equivalent to the T_3 translation of X_i. We will make much use of these remarks below.

Now we show the translation of each X_i is indeed a theorem of FS4.

If X_i is one of the axioms λA1 – λA8 or λA12 it is not difficult to see its translate is a theorem of FS4. Suppose X_i is an instance of λA9, let us show the T_3 translate of X_i is an FS4 theorem. To simplify notation we use S for $S_1 \wedge S_2 \wedge S_3$, and $(\forall \boldsymbol{x})$ for $(\forall x_1)(\forall x_2)(\forall x_3)$. Suppose X_i is some λ-closure of $\Box(Z \supset W) \supset (\Box Z \supset \Box W)$. The universal closure of

$$\Box(\forall \boldsymbol{x})[S \supset (T_3(Z) \supset T_3(W))] \supset [\Box(\forall \boldsymbol{x})(S \supset T_3(Z)) \supset \Box(\forall \boldsymbol{x})(S \supset T_3(W))]$$

is an FS4 theorem, and from this the T_3 translate of X_i follows, using the above remarks. Suppose X_i is an instance of λA10, say a λ-closure of $\Box Z \supset Z$. The following is an FS4 theorem:

$$(\forall \boldsymbol{x})\{S \supset [\Box(\forall \boldsymbol{x})(S \supset T_3(Z)) \supset T_3(Z)]\}$$

and again using the above remarks the T_3 translate of X_i follows. Finally, if $X.$ is an instance of axiom λA11, its translate is easily seen to be a theorem of FS4. Thus the translations of each λS4 axiom used in the proof are theorems of FS4.

Next, suppose X_i follows by λR1 from X_j and $X_j \supset X_i$, both occurring earlier in the proof than X_i, and suppose the T_3 translates of both are theorems of FS4.

Thus we have in FS4

$$(\forall \boldsymbol{x})\,[S \supset T_3(X_j)], \qquad (\forall \boldsymbol{x})\,[S \supset (T_3(X_j) \supset T_3(X_i))].$$

From this we easily deduce in FS4 $(\forall \boldsymbol{x})\,[S \supset T_3(X_i)]$ which is the T_3 translate of X_i.

Suppose X_i has been deduced from X_j, occurring earlier in the proof than X_i, using λR3, and suppose the translate of X_j is a theorem of FS4. To be definite, let us suppose

$$X_j = (\lambda x\, Z)\,(g_q) \supset Y, \qquad X_i = (\exists x)\, Z \supset Y$$

and $n_2 \geqq j > n_3$. The T_2 translate of X_j is an FS4 theorem. This is (recall, since the proof is normal, g_q is ordinary) $(\forall \boldsymbol{x})\,\{S \supset [T_2(Z)\,(x/c_q) \supset T_2(Y)]\}$ where we use S now for $S_1 \wedge S_2$ and $(\forall \boldsymbol{x})$ for $(\forall x_1)\,(\forall x_2)$. Equivalently, we have

$$(\forall \boldsymbol{x})\,\{T_2(Z)\,(x/c_q) \supset [S \supset T_2(Y)]\}. \qquad (*)$$

Since the proof is normal, g_q does not occur in the proof after the j^{th} line, hence g_q can not occur in P_2 or P_1. It follows that c_q does not occur in S_2 or S_1 and hence not in S. Furthermore, by the λR3 conditions, g_q does not occur in Z or Y. Thus the only occurrence of c_q in (*) is the one indicated. It follows then that

$$(\forall \boldsymbol{x})\,[(\exists x)\, T_2(Z) \supset (S \supset T_2(Y))], \qquad (\forall \boldsymbol{x})\,[S \supset ((\exists x)\, T_2(Z) \supset T_2(Y))],$$
$$(\forall \boldsymbol{x})\,[S \supset T_2((\exists x)\, Z \supset Y)]$$

and this is the T_2 translate of X_i.

We leave applications of λR2 to the reader. They have features in common with the above.

Finally, suppose X_i has been deduced from an earlier formula by an application of λR4. To be specific, let us suppose the translate of X_{n_2} is a theorem of FS4 and let us show the translate of X_{n_2+1} is also a theorem. Thus we suppose the following is provable in FS4:

$$(\forall x_1)\,(\forall x_2)\,[(S_1 \wedge S_2) \supset T_2(X_{n_2})].$$

Now, $T_2(X_{n_2})$ is $\Box(\forall x_1)\,(\forall x_2)\,[(S_1 \wedge S_2) \supset T_2(P_2)\,(x/x_2)] \supset T_2(Y_2)$ or equivalently, $\Box(\forall x_1)\,(\forall x_2)\,[S_1 \supset (S_2 \supset T_2(P_2)\,(x/x_2))] \supset T_2(Y_2)$. Now, by the λR4 conditions, f_2 is not in P_2 or in Y_2 (and neither is f_3 since the proof is normal). Hence

$$T_2(P_2) = T_1(P_2), \qquad T_2(Y_2) = T_1(Y_2).$$

So the above is $\Box(\forall x_1)\,(\forall x_2)\,[S_1 \supset (S_2 \supset T_1(P_2)\,(x/x_2))] \supset T_1(Y_2)$. Next, x_2 is not free in S_1, since f_2 is not in P_1. Thus the above is equivalent to

$$\Box(\forall x_1)\,\{S_1 \supset (\forall x_2)\,[S_2 \supset T_1(P_2)\,(x/x_2)]\} \supset T_1(Y_2).$$

Moreover, $(\forall x_2)\,[S_2 \supset T_1(P_2)\,(x/x_2)]$ is, written out,

$$(\forall x_2)\,\{[(\exists x_2)\, T_1(P_2)\,(x/x_2) \supset T_1(P_2)\,(x/x_2)] \supset T_1(P_2)\,(x/x_2)\}$$

and by lemma 1, we may replace this with $(\exists x_2)\, T_1(P_2)\,(x/x_2)$. Thus $T_2(X_{n_2})$ is equivalent to

$$\Box(\forall x_1)\,\{S_1 \supset (\exists x_2)\, T_1(P_2)\,(x/x_2)\} \supset T_1(Y_2).$$

This is equivalent to $T_1(\Box(\exists x) P_2 \supset Y_2)$ or $T_1(X_{n_2+1})$. Thus, if the T_2 translate of X_{n_2} is provable, so is the T_1 translate of X_{n_2+1}. The other two λR4 applications are treated similarly.

It follows now that the translate of each X_i is provable in FS4, hence so is that of X_n, that is, X itself is provable, and we are done.

§ 6. The System IS4

In this section we present an S4 type system much like the system εS4 of [2]. It is an epsilon-calculus formulation of S4 except that there are epsilon terms only for formulas with at most one free variable. Proving that IS4 is a conservative extension of λS4 will be seen to be equivalent to proving the Fundamental Theorem for λS4.

The language of IS4 is an extension of that of λS4, in which we associate new constants with formulas having single free variables. We do this in the following way. Let C_0 be the collection of name-constants of λS4 and let F_0 be the set of all λS4 formulas. To each formula, $X \in F_0$ having at most one free variable associate a distinct new constant, ε_X. Let C_1 be C_0 together with all these new constants, and let F_1 be the set of all formulas with constants from C_1. Similarly associate distinct new constants with those formulas of $F_1 - F_0$ having at most one free variable, let C_2 be C_1 together with these new constants, and let F_2 be the set of formulas with constants from C_2. And so on. Let $F = \bigcup_n F_n$ and $C = \bigcup_n C_n$. The set of formulas of IS4 is F. Thus, in IS4, to each formula X with at most one free variable there is associated a unique constant, ε_X.

The rules of IS4 are λR1 and

λR2*: if X is closed,

$$\frac{X}{\Box X}.$$

The axioms of IS4 are those of λS4 (with the domain of constants enlarged from C_0 to C), together with the following:

IA13: all λ-closures of $(\exists x) \Diamond X \supset \Diamond(\exists x) X$.

IA14: If $(\exists x) X$ is closed, $(\exists x) X \supset (\lambda x X)(\varepsilon_X)$.

This completes the presentation of IS4. Now we show that it is an extension of λS4. First we note

Lemma 1. *Let X be a closed formula of* IS4*, let $f \in C_0$ and $g \in C$. If X is a theorem of* IS4, *so is $X(f/g)$.*

Proof. If $X_1, X_2, \ldots, X_n = X$ is a proof in IS4 of X, $X_1(f/g), X_2(f/g), \ldots, X_n(f/g)$ is a proof of $X(f/g)$.

Now, all the axioms and one of the rules of λS4 are directly in the system IS4. If we show rules λR2, λR3 and λR4 are derivable in IS4 we are done.

Lemma 2. *Suppose* $(\lambda x X)(f)$ *is a closed formula of* IS4, *all of* f *are in* C_0, *distinct, and none of* f *occurs in* X. *Then if* $(\lambda x X)(f)$ *is a theorem of* IS4, *so is* $(\lambda x \Box X)(g)$ *where* g *is any sequence from* C.

Proof. Suppose $(\lambda x X)(f)$ is provable. By repeated use of lemma 1, $(\lambda x X)(\varepsilon)$ is also provable, where ε is a suitable sequence of IS4 epsilon terms which will enable us, by repeated use of IA14, to conclude $(\forall x) X$. Then by λR2*, $\Box(\forall x) X$. Next, by repeated applications of IA13, we may get $(\forall x) \Box X$. Lastly, by λA12 repeatedly, $(\lambda x \Box X)(g)$.

Lemma 3. *Let* $(\lambda x X)(f) \supset Y$ *be a closed formula of* IS4, *where* $f \in C_0$ *but* f *does not occur in* X *or* Y. *If this is a theorem of* IS4, *so is* $(\exists x) X \supset Y$.

Proof. $(\lambda x X)(f) \supset Y$ is a theorem of IS4 and $f \in C_0$, so by lemma 1, $(\lambda x X)(\varepsilon_X) \supset Y$ is provable in IS4. Then by IA14, $(\exists x) X \supset Y$ is a theorem.

Rule λR4 is treated similarly. Thus we indeed have

Theorem. IS4 *is an extension of* λS4.

We note for later use that we may show a deduction theorem for IS4 as follows. Call Y deducible from $X_1, \ldots, X_n$ provided that if $X_1, \ldots, X_n$ are added to IS4 as axioms, Y is provable.

Theorem. *Suppose* Y *is deducible from* $X_1, \ldots, X_n$ *in* IS4. *Then* $\Box(X_1 \wedge \cdots \wedge X_n) \supset Y$ *is a theorem of* IS4. (*Equivalently,* $(\Box X_1 \wedge \cdots \wedge \Box X_n) \supset Y$.)

The proof is as usual, by induction on the length of the IS4 deduction. The $\Box$ symbol before the $X_1 \wedge \cdots \wedge X_n$ arises from the presence of λR2*. λA11 is also needed here.

§ 7. The Fundamental Theorem

In the last section we showed IS4 was an extension of λS4. The primary result of this section is a proof that the extension is conservative. From it follows the main part of the Fundamental Theorem. First, however, we establish directly the easier part.

Lemma 1. *Let* X *be a formula of* λS4 *with at most* x *free. Then* $\Box(\exists x)[(\exists x) X \supset X]$ *is a theorem of* λS4.

Proof. Let f be a constant not in X.

$$\begin{aligned}(\lambda x X)(f) &\supset [(\exists x) X \supset (\lambda x X)(f)] \\ &\supset [(\lambda x (\exists x) X)(f) \supset (\lambda x X)(f)] \\ &\supset (\lambda x[(\exists x) X \supset X])(f) \\ &\supset (\exists x)[(\exists x) X \supset X].\end{aligned}$$

Then, using λR3, $(\exists x) X \supset (\exists x)[(\exists x) X \supset X]$. But also,

$$\sim(\exists x) X \supset [(\exists x) X \supset (\lambda x X)(f)]$$

so again

$$\sim(\exists x)\, X \supset (\lambda x[(\exists x)\, X \supset X])\,(f)$$
$$\supset (\exists x)\,[(\exists x)\, X \supset X].$$

Thus we have $(\exists x)\,[(\exists x)\, X \supset X]$ and now, by λR2 we are done.

Lemma 2. *Suppose* $\Box[(\exists x)\, X \supset (\lambda x X)\,(f)] \supset Y$ *is a theorem of* λS4, *where* f *does not occur in* X *or* Y. *Then* Y *is a theorem of* λS4.

Proof. $\Box[(\exists x)\, X \supset (\lambda x X)\,(f)] \supset Y$ so $\Box(\lambda x[(\exists x)\, X \supset X])\,(f) \supset Y$. Then by λR4, $\Box(\exists x)\,[(\exists x)\, X \supset X] \supset Y$. Now, by lemma 1 we are finished.

Remark. This is the only place in the proof of the Fundamental Theorem that λR4 is needed.

Theorem. *Let* R *be regular for* Y *and suppose* $\Box R^c \supset Y$ *is a* λS4 *theorem. Then so is* Y.

Proof. Let $R = \{R_1, R_2, \ldots, R_n\}$. Then $\Box R^c \supset Y$ is equivalent to

$$\Box R_1 \supset (\Box R_2 \supset (\ldots(\Box R_n \supset Y)\ldots)).$$

We may suppose the sequence $R_1, R_2, \ldots, R_n$ is a regular sequence for Y. Now the result follows by lemma 2, axiom λA12 and a result of § 4.

If a constant, f, of IS4 is in C_n but not in any C_k for $k < n$, we say f is of *rank* n.

Theorem. *Let* X *be a formula of* λS4, i.e. $X \in F_0$, *and suppose* X *is a theorem of* IS4. *Then there is a regular set* R *for* X *such that* $\Box R^c \supset X$ *can be proved in the strict propositional part of* λS4.

Proof. X is provable in IS4. Let R be the set consisting of all instances of axioms λA12, IA13 and IA14 used in the proof. Then X is deducible from R in IS4 without any other use of λA12, IA13 or IA14. The deduction theorem for IS4 (and its proof) then gives us: $\Box R^c \supset X$ is provable in IS4 without use of λA12, IA13 or IA14. Let $A = \{A_1, A_2, \ldots, A_n\}$ be the set of IA14 axioms in R and let $B = \{B_1, B_2, \ldots, B_k\}$ be $R - A$. Let ε_1 be the constant chosen by A_1, ε_2 by $A_2, \ldots, \varepsilon_n$ by A_n. Let us suppose the sequence $A_1, A_2, \ldots, A_n$ is arranged so that rank $(\varepsilon_i) \geqq$ rank (ε_{i+1}). Let $\varepsilon_{n+1}, \ldots, \varepsilon_k$ be the other constants of R of rank > 0. Let $f_1, f_2, \ldots, f_n, f_{n+1}, \ldots, f_p$ be constants of rank 0 which do not occur in R or X. For any formula Z, let $Z^* = Z(\varepsilon/f)$. Let $R^* = \{A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_k^*\}$. Then $[\Box R^c \supset X]^* = \Box R^{*c} \supset X^* = \Box R^{*c} \supset X$ since $X \in F_0$. Moreover, clearly $\Box R^{*c} \supset X$ is also provable in IS4 without use of λA12, IA13 or IA14. Then since $\Box R^{*c} \supset X \in F_0$ we have that, $\Box R^{*c} \supset X$ is a theorem of the strict propositional part of λS4. It remains to show that R^* is a regular set for X.

Arrange R^* in the ordering $B_1^*, \ldots, B_k^*, A_1^*, \ldots, A_n^*$. We show this is a regular sequence for X. Certainly each B_i^* is a regular formula. Moreover, A_i^* is of the form $(\exists x)\, W^* \supset (\lambda x W^*)\,(f_i)$ and we claim f_i does not occur in W^*. Suppose it did. A_i is of the form $(\exists x)\, W \supset (\lambda x W)\,(\varepsilon_i)$ so $\varepsilon_i\,[= \varepsilon_W]$ would occur in W. But the rank of ε_W must be greater than the rank of any constant of W, so ε_W is not in W. Thus each A_i^* is regular.

Next, suppose A_j^* is $(\exists x)\,Z^* \supset (\lambda x\, Z^*)\,(f_j)$ where $j > i$. We show f_i does not occur in A_j^*. Otherwise, ε_i would occur in $A_j = (\exists x)\,Z \supset (\lambda x\, Z)\,(\varepsilon_j)$, where $\varepsilon_i = \varepsilon_W$ and $\varepsilon_j = \varepsilon_Z$. But, as above, the rank of ε_Z is greater than the rank of any constant of Z, and by arrangement, $\text{rank}(\varepsilon_i) \geqq \text{rank}(\varepsilon_j)$, so ε_i can not occur in Z. Moreover, if $\varepsilon_i = \varepsilon_j$, $\varepsilon_W = \varepsilon_Z$ and we would have $W = Z$, so $A_i = A_j$ and R would have a redundant formula which we may drop. Thus ε_i is not in A_j, so f_i is not in A_j^*.

Finally, f_i does not occur in X^* since otherwise ε_i would occur in X, but $X \in F_0$.

Thus R^* is regular for X.

Corollary 1. *IS4 is a conservative extension of* λS4.

Corollary 2. *If X is provable in λS4 then there is a regular set R for X such that $\Box R^c \supset X$ can be proved in the propositional (strict propositional) part of λS4.*

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