Many-Valued Non-Monotonic Modal Logics

Melvin Fitting
mlflc@cunyvm.cuny.edu
Dept. Mathematics and Computer Science
Lehman College (CUNY), Bronx, NY 10468
Depts. Computer Science, Philosophy, Mathematics
Graduate Center (CUNY), 33 West 42nd Street, NYC, NY 10036 *

December 10, 1992

Abstract

Among non-monotonic systems of reasoning, non-monotonic modal logics, and autoepistemic logic in particular, have had considerable success. The presence of explicit modal operators allows flexibility in the embedding of other approaches. Also several theoretical results of interest have been established concerning these logics. In this paper we introduce non-monotonic modal logics based on many-valued logics, rather than on classical logic. This extends earlier work of ours on many-valued modal logics. Intended applications are to situations involving several reasoners, not just one as in the standard development.

1 Introduction

Several kinds of non-monotonic logics have been considered over the past dozen years. Among these, non-monotonic modal logics have been particularly interesting. These, of course, have explicitly occurring modal operators, and this allows for fine control when translating from other formalisms; see [13, 5], for instance. Following the ideas of McDermott and Doyle, [8, 9], every normal modal logic also has a non-monotonic version. In addition Moore, [10], introduced *autoepistemic* logic, which is a kind of modal non-monotonic logic not based directly on a monotonic modal logic. Shvarts showed, [12], that autoepistemic logic was equivalent to one of the non-monotonic modal logics in the McDermott-Doyle family.

Modal logic is not truth-functional, but it is built on a classical logic base, which is the genotypical truth-functional logic. In [3, 4] we introduced a family of modal logics based on many-valued logics besides the classical $\{\mathit{false}, \mathit{true}\}$. In this paper we want to lay the foundations for a non-monotonic version of these logics. First, however, we begin with some background — explaining why one might be interested in many-valued modal logics in the first place.

Suppose we have several situations in which we are interested. Think of these as possible worlds — they can be infinite in number. Suppose also that we have a finite set of experts whose opinions we value. Each expert has his or her opinion on which worlds are accessible from which worlds, and also on which atomic formulas are true in which worlds. In effect, each expert provides us with a Kripke modal model. Further, suppose that some experts may dominate others. If one expert dominates another, anything the first expert says is true will also be judged true by the second

*Research partly supported by NSF Grants CCR-8901489 and CCR-9104015.

expert. This means the various Kripke models are inter-related. It also means that the behavior of connectives at worlds has a kind of Intuitionistic flavor, though this will not be obvious without further explanation which we do not have the space to give here. At any rate, all this yields a structure of some interest: a multiple-expert modal model.

Now suppose that instead of using many Kripke models, we combine them into a single one by broadening the notion of truth value. Specifically, suppose we think of a set of experts as a kind of generalized truth value. The idea is, instead of saying experts a and b call X true at world Γ in their respective Kripke models, we will say X has the truth value $\{a, b\}$ at Γ. Note that the accessibility relation between worlds also takes on these generalized truth values, since the experts may not agree on a common notion of accessibility. Because of the possible domination relations between experts, not all sets of experts are allowed as generalized truth values. In fact, what we get as the space of generalized truth values is a Heyting algebra. Now the question is, can we replace arbitrary multiple-expert modal models by a single many-valued modal model that is, in some sense, equivalent? We showed in [4] that this could be done in a simple, natural way. We do not repeat the proofs here. Further, in [3, 4] we gave a proof procedure for these many-valued modal logics, and proved soundness and completeness.

Now, briefly, the plan of this paper is to extend the earlier work mentioned above to the non-monotonic setting. We do this after first presenting the monotonic version in more detail. We also give some examples, and prove some basic results. It should be emphasized that this work is still in early stages, and is almost certainly susceptible to improvement and extension. If others can be induced to consider such a task, this paper has been a success.

2 Many-valued non-modal logic

Suppose we have a finite set of experts with some dominating others, as sketched in the previous section. Then we can take as generalized truth values those sets of experts that are closed under the domination relation. Such a space of generalized truth values always constitutes a finite Heyting algebra. In fact, a finite lattice is a Heyting algebra if and only if it is distributive. So, in this section we present a formulation of a semantics and a proof procedure for the family of many-valued logics whose truth values constitute a finite, distributive lattice. The formulation is taken from [3].

Let $\mathcal{T}$ be a finite, distributive lattice; we refer to the members of $\mathcal{T}$ as *$\mathcal{T}$-truth values*. Since the lattice is finite, it has a top and a bottom, which we denote by *true* and *false* respectively. We use $\wedge$ and $\vee$ for meet and join in $\mathcal{T}$. And there is a notion of implication that is both standard and useful: set $x \Rightarrow y = \bigvee\{z \mid x \wedge z \leq y\}$. Then $(z \wedge x) \leq y$ if and only if $z \leq (x \Rightarrow y)$. This operation is called *relative pseudo-complement* in [11]. For a finite lattice, the existence of such an operation is equivalent to distributivity. The relative pseudo-complement operation is used semantically for modeling the implication of intuitionistic logic. The structure $\mathcal{T}$ with these operations is a finite Heyting algebra. A negation operation can be introduced via $\neg x = (x \Rightarrow \mathit{false})$, but it will not play a role here.

Once $\mathcal{T}$ has been specified, we set up a formal propositional language, more or less as usual. We have a set of propositional letters, P_1, P_2, ... We also have a set of propositional constants, one for each member of $\mathcal{T}$. Classically, *true* and *false* are definable as $X \supset X$ and $\neg \mathit{true}$, but in a more general lattice not every member may be definable, so we explicitly add constants for them. For simplicity, we just allow members of $\mathcal{T}$ to appear as propositional constants (which we will take as designating themselves). The propositional constants and propositional letters together make up the collection of atomic formulas. Then the set of formulas is built up as usual, allowing $\wedge$, $\vee$ and $\Rightarrow$. The proof procedure below uses the notion of a *sequent*. In the usual way, a sequent is an expression $\Gamma \rightarrow \Delta$ where Γ and Δ are finite sets of formulas.

Now we present a notion of many-valued semantics, taken from [3]. A *valuation* is a mapping from propositional letters to $\mathcal{T}$. Valuations are extended to all formulas in the expected way. If t is a propositional constant (member of $\mathcal{T}$), $v(t) = t$. Further, $v(X \wedge Y) = v(X) \wedge v(Y)$, where the $\wedge$ on the right is the meet operation of $\mathcal{T}$; similarly the connective $\vee$ corresponds to the join of $\mathcal{T}$ and the connective $\supset$ corresponds to relative pseudo-complement. Finally, we say a sequent $\Gamma \rightarrow \Delta$ is *valid* provided, for every valuation v, if every member of Γ maps to *true* under v then some member of Δ also maps to *true* under v. The following proof procedure, from [3], goes with this semantics.

Identity Axiom

$$X \rightarrow X$$

Thinning

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \cup \Gamma' \rightarrow \Delta \cup \Delta'}$$

Cut

$$\frac{\Gamma \rightarrow \Delta, X \quad \Gamma, X \rightarrow \Delta}{\Gamma \rightarrow \Delta}$$

Axiom of Transitivity

$$X \supset Y, Y \supset Z \rightarrow X \supset Z$$

Rule $RI \supset$

$$\frac{\Gamma, t \supset A \rightarrow \Delta, t \supset B \quad (\text{for every } t \in \mathcal{T})}{\Gamma \rightarrow \Delta, A \supset B}$$

Rule $\supset RI$

$$\frac{\Gamma, B \supset t \rightarrow \Delta, A \supset t \quad (\text{for every } t \in \mathcal{T})}{\Gamma \rightarrow \Delta, A \supset B}$$

Conjunction Axioms

$$\rightarrow A \wedge B \supset A$$

$$\rightarrow A \wedge B \supset B$$

$$C \supset A, C \supset B \rightarrow C \supset A \wedge B$$

Disjunction Axioms

$$\rightarrow A \supset A \vee B$$

$$\rightarrow B \supset A \vee B$$

$$A \supset C, B \supset C \rightarrow A \vee B \supset C$$

Implication Axioms

$$(A \wedge B) \supset C \rightarrow A \supset (B \supset C)$$

$$A \supset (B \supset C) \rightarrow (A \wedge B) \supset C$$

Propositional Constant Axioms

$$\rightarrow a \supset b \quad \text{if } a \leq b$$

$$a \supset b \rightarrow \quad \text{if } a \not\leq b$$

In [3] completeness of this system is shown in the form: the sequent $\rightarrow X$ is provable if and only if it is valid. As a special case, if $\mathcal{T}$ is the lattice $\{\mathit{false}, \mathit{true}\}$, with $\mathit{false} \leq \mathit{true}$, the resulting logic is just the usual classical propositional logic.

3 Autoepistemic logic, generalized

In autoepistemic logic, [10], modal operators are added to the language; the operator $\Box$ is intended to be read as 'known' or 'believed.' Then an attempt is made to characterize the beliefs of a perfect reasoner. Such beliefs would contain $\Box X$ whenever they contain X, and would contain $\neg\Box X$ whenever they do not contain X. This leads to the notion of *stable expansion*, which has been much studied in the literature. But now, suppose we have several reasoners, and we want to characterize their collective beliefs. If it is, somehow, the collective wisdom that at least the experts a and b believe X, then it should also be part of the collective wisdom that $\Box X$ is accepted by at least a and b. This can be formalized by using a many-valued logic whose truth values are sets of experts, and saying that if we have $\{a, b\} \supset X$, we should also have $\{a, b\} \supset \Box X$. Considerations like these quickly lead to the following more formal version.

Suppose we extend the language introduced in the previous section by allowing formulas to contain $\Box$ and $\Diamond$ as well as the classical connectives. In general these are not inter-definable using our many-valued machinery, so both are taken as basic. We introduce a generalization of stable expansion. As presented here a stable expansion will be a set of *sequents*, not of formulas. Classically, of course, this difference is unimportant, since a sequent can always be replaced by a formula: conjunction of the formulas on the left of the arrow implies disjunction of the formulas on the right. In our more general setting such a reduction is not always possible. It is convenient to first define some operators, mapping sets of sequents to sets of sequents. The operator $\mathcal{P}$ is the positive introspection operator, and $\mathcal{N}$ is the negative introspection one. The notation $\mathcal{A}$ is intended to suggest 'autoepistemic.' In the following, S and I are sets of sequents, X is an arbitrary formula, and t is a propositional constant. Also **con** is the (non-modal) *consequence* operation: if S is a set of sequents, $\mathbf{con}(S)$ is the set of sequents that are provable in the system that results when the members of S are added as axioms to the sequent calculus of Section 2.

Definition 3.1

$$\begin{array}{rcl}
\mathcal{P}(S) & = & \{\rightarrow (t \supset \Box X) \mid \rightarrow (t \supset X) \in S\} \\
 & & \cup\{\rightarrow (\Diamond X \supset t) \mid \rightarrow (X \supset t) \in S\} \\
\mathcal{N}(S) & = & \{(t \supset \Box X) \rightarrow \mid \rightarrow (t \supset X) \notin S\} \\
 & & \cup\{(\Diamond X \supset t) \rightarrow \mid \rightarrow (X \supset t) \notin S\} \\
\mathcal{A}(I, S) & = & \mathbf{con}(I \cup \mathcal{P}(S) \cup \mathcal{N}(S))
\end{array}$$

Definition 3.2 We say S is a *stable expansion* of I (in the system based on $\mathcal{T}$) if $\mathcal{A}(I, S) = S$.

If $\mathcal{T}$ is classical logic, this definition is equivalent to the usual one.

4 Examples

We consider several examples of stable expansions, but we must do so only briefly because of space constraints. We hope to discuss this more fully elsewhere. In all of the examples, suppose we have two experts, a and b, who are independent of each other. Then we have a lattice $\mathcal{T}$ of truth values whose members are $\emptyset$, $\{a\}$, $\{b\}$ and $\{a, b\}$, ordered by inclusion. Note that in this lattice, *false* $= \emptyset$ and *true* $= \{a, b\}$.

Example We begin with an example taken from [7], set:

$$I_1 = \{\rightarrow (\Box X \supset X)\}.$$

This has four stable expansions. In one we have $\rightarrow (\{a\} \supset X)$ but not $\rightarrow (\{b\} \supset X)$, while in another we have $\rightarrow (\{b\} \supset X)$ but not $\rightarrow (\{a\} \supset X)$. In a third, we have both these sequents, and in the fourth we have neither.

We can think of, say, $\rightarrow (\{a\} \supset X)$ as saying that at least the expert a thinks X is the case. A stable expansion can be thought of as embodying the collective knowledge of a and b, just as in the classical theory a stable expansion embodies the knowledge of a single perfect reasoning agent. Thus, for instance, if $\rightarrow (\{a\} \supset X)$ is in a stable expansion, collective wisdom is that at least a thinks X is the case, and so $\rightarrow (\{a\} \supset \Box X)$ is also part of the collective wisdom, and will also be in the stable expansion.

Note that if we had only one expert instead of two, there would only be two stable expansions of I_1, which can be identified with the two produced by classical autoepistemic theory.

Example The previous example can be further 'refined.' Suppose we set

$$I_2 = I_1 \cup \{(\{a\} \supset X), (\{b\} \supset X) \rightarrow\}.$$

The stable expansions of I_2 are those of I_1 except for the one containing both sequents $\rightarrow (\{a\} \supset X)$ and $\rightarrow (\{b\} \supset X)$.

In a similar way, set

$$I_3 = I_1 \cup \{\rightarrow (\{a\} \supset X), (\{b\} \supset X)\}.$$

The stable expansions of I_3 are those of I_1 except for the one containing neither of the sequents.

Finally, set $I_4 = I_2 \cup I_3$. There are two stable expansions of I_4; each contains exactly one of $\rightarrow (\{a\} \supset X)$ and $\rightarrow (\{b\} \supset X)$.

Example The previous examples did not fully illustrate the expressiveness possible. In each case there was considerable symmetry with respect to a and b. Now, set:

$$I_5 = \{(\{a\} \supset \Box X) \rightarrow (\{a\} \supset X)\}.$$

There are two stable expansions of I_5; one contains $\rightarrow (\{a\} \supset X)$ and the other does not; neither contains $\rightarrow (\{b\} \supset X)$.

5 Monotonic, many-valued modal logic

Autoepistemic logic, as originally presented or in the generalized version above, uses modal formulas but assumes no special facts about the modal operators; the underlying proof procedure is essentially classical. But before autoepistemic logic was introduced in [10] there were non-monotonic modal logics, based on particular well-known modal logics [8, 9]. We want to generalize these to the many-valued setting as well. We begin with a notion of many-valued modal model, taken from [3, 4].

Definition 5.1 A $\mathcal{T}$-*modal model* is a structure $\langle \mathcal{G}, \mathcal{R}, w \rangle$ where $\mathcal{G}$ is a non-empty set of possible worlds, $\mathcal{R}$ is a mapping from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{T}$, and w maps worlds and propositional letters to $\mathcal{T}$.

The mapping $\mathcal{R}$ is a $\mathcal{T}$-valued accessibility relation. Next we extend the valuation w to non-atomic formulas.

Definition 5.2 For any $\Gamma \in \mathcal{G}$:

1. If t is a propositional constant, $w(\Gamma, t) = t$.
2. $w(\Gamma, A \wedge B) = (w(\Gamma, A) \wedge w(\Gamma, B))$.
3. $w(\Gamma, A \vee B) = (w(\Gamma, A) \vee w(\Gamma, B))$.
4. $w(\Gamma, A \supset B) = (w(\Gamma, A) \Rightarrow w(\Gamma, B))$.
5. $w(\Gamma, \Box A) = \bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, A) \mid \Delta \in \mathcal{G}\}$.
6. $w(\Gamma, \Diamond A) = \bigvee\{\mathcal{R}(\Gamma, \Delta) \wedge w(\Delta, A) \mid \Delta \in \mathcal{G}\}$.

The behavior of w is extended to sequents in the usual way. In conventional modal logics $\Box$ and $\Diamond$ are inter-definable. This is not generally the case now, though it will be if $\mathcal{T}$ happens to be not just a Heyting algebra, but a Boolean algebra. As a consequence, we must give separate conditions for each modal operator in what follows, something that is not necessary classically.

In [4] motivation is given for this semantics, and a formal relationship with a multiple-expert semantics is established.

Finally, we extend the sequent calculus of Section 2 to cover the modal operators. This calculus comes from [4]. For convenience, $X \equiv Y$ is taken to abbreviate mutual implication.

Necessitation Rule

$$\frac{\rightarrow A}{\rightarrow \Box A}$$

Modal Axioms

$\rightarrow \Box(A \supset B) \supset (\Box A \supset \Box B)$

For each $t \in \mathcal{T}$, $\rightarrow (t \supset \Box A) \equiv \Box(t \supset A)$

For each $t \in \mathcal{T}$, $\rightarrow (\Diamond A \supset t) \equiv \Box(A \supset t)$

This system is sound and complete in the sense that $\rightarrow X$ is provable if and only if $\rightarrow X$ is valid.

As presented, the many-valued modal logic above is a generalization of the logic K, but other standard modal logics also extend. We introduce a few such extensions, without proof. Suppose we add the following two axioms.

$$\rightarrow (\Box X \supset \Box\Box X)$$

$$\rightarrow (\Diamond\Diamond X \supset \Diamond X)$$

Then we get a proof-theoretic generalization of $K4$, and it corresponds to the imposition of a kind of transitivity condition on models: $\mathcal{R}(\Gamma, \Delta) \wedge \mathcal{R}(\Delta, \Omega) \leq \mathcal{R}(\Gamma, \Omega)$. In a similar way symmetry can be imposed by adding the following.

$$\rightarrow (\Diamond X \supset \Box\Diamond X)$$

$$\rightarrow (\Diamond\Box X \supset \Box X)$$

We will refer to the system that results from the adding of all these sequents as $K45$. It is the natural generalization of the classically-based system of the same name.

6 Non-monotonic modal logics

Autoepistemic logic was introduced by Moore [10] and was intended as an improvement of a family of non-monotonic modal logics due to McDermott and Doyle [8, 9]. As it happened, interest in those non-monotonic modal logics has not disappeared, and Shvarts has shown that autoepistemic logic is, in fact, one of them [12]. In this section we extend the notions of McDermott and Doyle to the many-valued setting.

Suppose L is a many-valued modal logic, arising from adding a set of sequents as axioms to the sequent calculus presented in Section 5. Associated with L is a consequence relation: $\mathbf{con}_L(S)$ is the set of all sequents that can be derived in the sequent calculus of Section 5, with the members of L and the members of S added as axioms. Now we introduce one more operator. It is like the operator $\mathcal{A}$, except that $\mathcal{P}$ no longer plays a role, while the notion of consequence has been strengthened to take modal operators into account directly.

Definition 6.1

$$\mathcal{B}_L(I, S) = \mathbf{con}_L(I \cup \mathcal{N}(S))$$

Definition 6.2 A set S of sequents is an L-*expansion* of I if $\mathcal{B}_L(I, S) = S$.

If $\mathcal{T}$ is classical logic, this notion of L-expansion is equivalent to the usual one.

7 The Shvarts result, extended

In [12] Shvarts showed that autoepistemic logic was essentially the same as non-monotonic $K45$. We have not been able to extend this result to the many-valued setting in full generality, but it does hold with a linearity restriction. If we think of the space $\mathcal{T}$ of truth values as being sets of experts, the linearity assumption says the dominance relation between them is strictly hierarchical. We hope others can find a way to lift, or at least broaden this restriction. We devote this section to a proof of the generalized Shvarts result, which we state first, then prove via a sequence of lemmas.

Definition 7.1 A set S of sequents is *consistent* if it does not contain every sequent.

Theorem 7.2 *Assume $\mathcal{T}$ is linearly ordered. Then a consistent set S of sequents is a stable expansion of I if and only if S is a $K45$ expansion of I.*

The argument is via a sequence of more specialized results, some of which have their proofs presented in full, some just sketched. The first few results do not need the linearity assumption. We begin by considering the $K4$ axioms.

Proposition 7.3 *If $\mathcal{A}(I, S) \subseteq S$ then both $\rightarrow (\Box X \supset \Box\Box X)$ and $\rightarrow (\Diamond\Diamond X \supset \Diamond X)$ are in $\mathcal{A}(I, S)$.*

Proof According to Rule $RI \supset$, it is enough to show that for each $t \in \mathcal{T}$, $(t \supset \Box X) \rightarrow (t \supset \Box\Box X)$ is in $\mathcal{A}(I, S)$. There are two cases.

Case 1. Suppose $\rightarrow (t \supset \Box X) \in S$. Then $\rightarrow (t \supset \Box\Box X) \in \mathcal{P}(S) \subseteq \mathcal{A}(I, S)$.

Case 2. Suppose $\rightarrow (t \supset \Box X) \notin S$. Then $\rightarrow (t \supset \Box X) \notin \mathcal{A}(I, S)$, hence $\rightarrow (t \supset \Box X) \notin \mathcal{P}(S)$, so $\rightarrow (t \supset X) \notin S$. Then $(t \supset \Box X) \rightarrow \in \mathcal{A}(I, S)$.

In either case, the result follows by Thinning. The second half of the Proposition is similar, using Rule $\supset RI$. ■

Proposition 7.4 *Assume S is closed under non-modal consequence, that is, $\mathbf{con}(S) \subseteq S$. Then:*

1. $\rightarrow \Box(X \supset Y) \supset (\Box X \supset \Box Y) \in \mathcal{A}(I, S)$,
2. $\rightarrow \Box(t \supset X) \equiv (t \supset \Box X) \in \mathcal{A}(I, S)$ *for all* $t \in \mathcal{T}$,
3. $\rightarrow \Box(X \supset t) \equiv (\Diamond X \supset t) \in \mathcal{A}(I, S)$ *for all* $t \in \mathcal{T}$.

Proof If $\mathcal{A}(I, S)$ is inconsistent, item 1 holds trivially, so now assume it is consistent. It is enough to show that for each $a \in \mathcal{T}$,

$$(a \supset \Box(X \supset Y)) \rightarrow (a \supset (\Box X \supset \Box Y)) \in \mathcal{A}(I, S)$$

or equivalently,

$$(a \supset \Box(X \supset Y)) \rightarrow ((a \wedge \Box X) \supset \Box Y) \in \mathcal{A}(I, S).$$

To show this, it is enough to show that for any $b \in \mathcal{T}$,

$$(a \supset \Box(X \supset Y)), (b \supset (a \wedge \Box X)) \rightarrow (b \supset \Box Y) \in \mathcal{A}(I, S).$$

It is easy to show that $(b \supset (a \wedge \Box X)) \rightarrow (b \supset a)$ is provable, so if $b \not\leq a$ we are quickly done (we omit details). So assume $b \leq a$ from now on.

Suppose $\rightarrow (a \supset \Box(X \supset Y)) \in \mathcal{A}(I, S)$ and $\rightarrow (b \supset \Box X) \in \mathcal{A}(I, S)$. If we had $\rightarrow (a \supset (X \supset Y)) \notin S$, it would follow that $(a \supset \Box(X \supset Y)) \rightarrow \in \mathcal{A}(I, S)$, from which $\rightarrow \in \mathcal{A}(I, S)$, using Cut, and then the inconsistency of $|CA(I, S)$ follows using Thinning. Hence $\rightarrow (a \supset (X \supset Y)) \in S$. By a similar argument, $\rightarrow (b \supset X) \in S$. Also $\rightarrow (b \supset a) \in S$, since $b \leq a$. Since S is closed under non-modal consequence, it follows that $\rightarrow (b \supset Y) \in S$, and so $\rightarrow (b \supset \Box Y) \in \mathcal{A}(I, S)$.

Now suppose $\rightarrow (a \supset \Box(X \supset Y)) \in \mathcal{A}(I, S)$ but $\rightarrow (b \supset \Box X) \notin \mathcal{A}(I, S)$. Then $\rightarrow (b \supset X) \notin S$, so $(b \supset \Box X) \rightarrow \in \mathcal{A}(I, S)$, from which it follows that $(b \supset (a \wedge \Box X)) \rightarrow \in \mathcal{A}(I, S)$.

Finally suppose $\rightarrow (a \supset \Box(X \supset Y)) \notin \mathcal{A}(I, S)$. Then $\rightarrow (a \supset (X \supset Y)) \notin S$, so $(a \supset \Box(X \supset Y)) \in \mathcal{A}(I, S)$.

This completes the argument for item 1; we omit the arguments for the other two items. ■

Next we have a result that uses linearity in an essential way, and on which the rest of the argument seems to depend.

Lemma 7.5 *Assume $\mathcal{T}$ is linear (and finite, as usual), $a, b \in \mathcal{T}$, a is the predecessor of b in the ordering of $\mathcal{T}$, and S is closed under non-modal deduction (that is, $\mathbf{con}(S) \subseteq S$). Then*

$$(a \supset Z) \rightarrow \in S \qquad \textit{iff} \qquad \rightarrow (Z \supset b) \in S.$$

Proof Suppose first that $(a \supset Z) \rightarrow \in S$. We show that, for every $t \in \mathcal{T}$, $(Z \supset t), (t \supset Z) \rightarrow (Z \supset b) \in S$. The argument for this is as follows.

If $t \leq b$ then $\rightarrow (t \supset b)$ is an axiom. Also $(Z \supset t), (t \supset b) \rightarrow (Z \supset b)$ is an instance of the Axiom of Transitivity. Then $(Z \supset t) \rightarrow (Z \supset b)$ follows by Cut, and the sequent desired by Thinning. On the other hand, if $t \not\leq b$ then $a \leq t$, so by a similar argument we get $(t \supset Z) \rightarrow (a \supset Z)$, from which $(t \supset Z) \rightarrow \in S$, using Cut, and now the desired sequent follows by Thinning again.

Now that we have $(Z \supset t), (t \supset Z) \rightarrow (Z \supset b) \in S$ for each $t \in \mathcal{T}$, $\rightarrow (Z \supset b) \in S$ follows using a Derived Rule from [3].

For the converse direction, suppose $\rightarrow (Z \supset b) \in S$. Using the Transitivity Axiom, $(a \supset Z), (Z \supset b) \rightarrow (a \supset b)$, so by Cut, $(a \supset Z) \rightarrow (a \supset b) \in S$. But since $a \not\leq b$, $(a \supset b) \rightarrow$, and the result follows by Cut. ■

Now we can take up the last of the $K45$ axioms.

Proposition 7.6 *Assume $\mathcal{T}$ is linear, $\mathcal{A}(I, S) \subseteq S$, and S is consistent. Then both $\rightarrow (\Diamond\Box X \supset \Box X) \in \mathcal{A}(I, S)$ and $\rightarrow (\Diamond X \supset \Box\Diamond X) \in \mathcal{A}(I, S)$.*

Proof We only argue for the first of the two sequents. Using Rule $RI \supset$ it is enough to show that for each $a \in \mathcal{T}$, $(a \supset \Diamond\Box X) \rightarrow (a \supset \Box X) \in \mathcal{A}(I, S)$. There are three cases, each of which is concluded by an application of Thinning.

Case 1. Suppose $\rightarrow (a \supset X) \in S$. Then $\rightarrow (a \supset \Box X) \in \mathcal{P}(S) \subseteq \mathcal{A}(I, S)$.

Case 2. Suppose $a = \mathit{false}$. Then $\rightarrow (a \supset \Box X) \in \mathcal{A}(I, S)$, using the fact that $\rightarrow (\mathit{false} \supset Z)$ is provable (shown in [3]).

Case 3. Suppose $a \neq \mathit{false}$ and $\rightarrow (a \supset X) \notin S$. Let b be the predecessor of a. Now, $(a \supset \Box X) \rightarrow \in \mathcal{N}(S) \subseteq \mathcal{A}(I, S)$. By Lemma 7.5, $\rightarrow (\Box X \supset b) \in \mathcal{A}(I, S) \subseteq S$. Then $\rightarrow (\Diamond\Box X \supset b) \in \mathcal{P}(S) \subseteq \mathcal{A}(I, S)$. Finally, by Lemma 7.5 again, $(a \supset \Diamond\Box X) \rightarrow \in \mathcal{A}(I, S)$. ■

Lemma 7.7 *Let L be a many-valued modal logic, not necessarily $K45$. If $S \subseteq \mathcal{B}_L(I, S)$ then $\mathcal{P}(S) \subseteq \mathcal{B}_L(I, S)$ and hence $\mathcal{B}_L(I, S) = \mathcal{B}_L(I \cup \mathcal{P}(S), S)$.*

Proof Assume $S \subseteq \mathcal{B}_L(I, S)$ and $\rightarrow (t \supset \Box X) \in \mathcal{P}(S)$. Then $\rightarrow (t \supset X) \in S \subseteq \mathcal{B}_L(I, S)$. Using the Necessitation Rule, $\rightarrow \Box(t \supset X) \in \mathcal{B}_L(I, S)$, and then using a Modal Axiom, $\rightarrow (t \supset \Box X) \in \mathcal{B}_L(I, S)$. ■

Proposition 7.8 *Again let L be a many-valued modal logic, not necessarily $K45$. If $\mathcal{B}_L(I, S) = S$ then $\mathcal{A}(I, S) \subseteq S$.*

Proof Assume $\mathcal{B}_L(I, S) = S$. Then using Lemma 7.7, together with other straightforward facts, $\mathcal{A}(I, S) \subseteq \mathcal{B}(I \cup \mathcal{P}(S), S) = \mathcal{B}_L(I, S) = S$. ■

Now finally, we are in a position to establish half of Theorem 7.2.

Theorem 7.9 *Assume $\mathcal{T}$ is linearly ordered. If a consistent set S of sequents is a $K45$ expansion of I then S is also a stable expansion of I.*

Proof Assume $\mathcal{T}$ is linear, S is consistent, and $\mathcal{B}_{K45}(I, S) = S$. By Proposition 7.8, $\mathcal{A}(I, S) \subseteq S$. Since S is consistent, so is $\mathcal{A}(I, S)$. It remains for us to show $S \subseteq \mathcal{A}(I, S)$.

If a sequent $\Gamma \rightarrow \Delta$ is in S, it is in $\mathcal{B}_{K45}(I, S)$, so it is enough to show any $K45$ derivation from $I \cup \mathcal{N}(S)$ can be turned into a non-modal derivation from $I \cup \mathcal{P}(S) \cup \mathcal{N}(S)$. Various of the Propositions above show we have the $K45$ axioms available in $\mathcal{A}(I, S)$; so there only remain applications of the Rule of Necessitation to justify.

Assume $\rightarrow X \in \mathcal{A}(I, S)$; we will show $\rightarrow \Box X \in \mathcal{A}(I, S)$. We have $\rightarrow X \in \mathcal{A}(I, S) \subseteq S = \mathcal{B}_{K45}(I, S)$, and since the Rule of Necessitation is available in $K45$, $\rightarrow \Box X \in \mathcal{B}_{K45}(I, S) = S$. Since S is consistent, $\Box X \rightarrow \notin S$, or equivalently, $(\mathit{true} \supset \Box X) \rightarrow \notin S$. Using linearity of $\mathcal{T}$, let b be the predecessor of true. Then by Lemma 7.5, $\rightarrow (\Box X \supset b) \notin S$, and so $(\Diamond\Box X \supset b) \rightarrow \in \mathcal{A}(I, S)$, and so $\rightarrow (\mathit{true} \supset \Diamond\Box X) \in \mathcal{A}(I, S)$, or $\Diamond\Box X \in \mathcal{A}(I, S)$. But we have that $\rightarrow (\Diamond\Box X \supset \Box X) \in \mathcal{A}(I, S)$, and so $\rightarrow \Box X \in \mathcal{A}(I, S)$. ■

The other half of Theorem 7.2 remains to be established. The argument is rather like that above, and is left to the reader.

8 Future work

The material presented here constitutes only the early stages of exploration. There are embeddings of other non-monotonic formalisms into non-monotonic modal logic; how do these extend to the many-valued setting? In particular there has been work on many-valued logic programming, based on bilattices [1, 2]. Does this relate naturally to the generalization of non-monotonic modal logic presented above? There are various results of interest in the literature, concerning non-monotonic modal logics; do these extend to the many-valued setting as well? We addressed only one such result above, and it needed a linearity restriction. Is this restriction essential? Clearly there is much to be done.

References

[1] Melvin C. Fitting. Bilattices and the semantics of logic programming. *Journal of Logic Programming*, 11:91–116, 1991.

[2] Melvin C. Fitting. Well-founded semantics, generalized. In Vijay Saraswat and Kazunori Ueda, editors, *Logic Programming, Proceedings of the 1991 International Symposium*, pages 71–84, Cambridge, MA, 1991. MIT Press.

[3] Melvin C. Fitting. Many-valued modal logics. *Fundamenta Informaticae*, 1992. Forthcoming.

[4] Melvin C. Fitting. Many-valued modal logics, II. *Submitted*, 1992.

[5] Melvin C. Fitting, Wiktor Marek, and Mirosłav Truszczynski. Logic of necessitation. *Journal of Logic and Computation*, Forthcoming.

[6] Matthew L. Ginsberg, editor. *Readings in Nonmonotonic Reasoning.* Morgan Kaufmann, 1987.

[7] Kurt Konolige. On the relation between default and autoepistemic logic. *Artificial Intelligence*, 35:342–382, 1988.

[8] Drew McDermott and Jon Doyle. Nonmonotonic logic I. *Artificial Intelligence*, 13:41–72, 1980. Reprinted in [6].

[9] Drew McDermott and Jon Doyle. Nonmonotonic logic II: Nonmonotonic modal theories. *Journal of the ACM*, 29:33–57, 1982.

[10] Robert C. Moore. Semantical considerations on nonmonotonic logic. *Artificial Intelligence*, 25:75–94, 1985. Reprinted in [6].

[11] Helena Rasiowa and Roman Sikorski. *The Mathematics of Metamathematics.* PWN – Polish Scientific Publishers, Warsaw, third edition, 1970.

[12] Grigori Shvarts. Autoepistemic modal logics. In Rohit Parikh, editor, *Theoretical Aspects of Reasoning About Knowledge (TARK 1990)*, pages 97–109, 1990.

[13] Miroslaw Truszczyński. Embedding default logic into modal nonmonotonic logics. In Anil Nerode, Wiktor Marek, and V. S. Subrahmanian, editors, *Logic Programming and Non-monotonic Reasoning, Proceedings of the First International Workshop*, pages 151–165. MIT Press, 1991.