Many-Valued Modal Logics II

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Abstract

Suppose there are several experts, with some dominating others (expert A dominates expert B if B says something is true whenever A says it is). Suppose, further, that each of the experts has his or her own view of what is possible — in other words each of the experts has their own Kripke model in mind (subject, of course, to the dominance relation that may hold between experts). How will they assign truth values to sentences in a common modal language, and on what sentences will they agree? This problem can be reformulated as one about many-valued Kripke models, allowing many-valued accessibility relations. This is a natural generalization of conventional Kripke models that has only recently been looked at. The equivalence between the many-valued version and the multiple expert one will be formally established. Finally we will axiomatize many-valued modal logics, and sketch a proof of completeness.

1 Motivation

Suppose we have several experts — for present purposes we can take several to be two. And suppose we want the opinion of each, not just on how things are now, but on how they would be under various circumstances, or in various situations. (Think of a situation as a possible world.) Each of the experts will have their own opinion on whether a statement X is true in a situation Γ, but also each expert will have an opinion on whether the situation Γ is worth considering. In other words, each of the experts will have their own truth assignment in possible worlds, and each will have their own accessibility relation as well. The formula $\Box X$ can be read as saying that X is true, 'no matter what,' that is, under all circumstances that are thought worthy of consideration. We are interested in which modal formulas the two experts agree on in 'this' world, even though they may have essentially different modal models in mind.

It is natural to try reformulating the problem as one involving a many-valued logic, taking sets of experts as truth values. Say the experts are simply called 1 and 2. Then there are four truth values: $\emptyset$, $\{1\}$, $\{2\}$ and $\{1,2\}$. To say X has truth value $\{1,2\}$ in some situation corresponds to saying both experts consider X true there, and so on. The goal is to replace the calculation of truth values at possible worlds, in the conventional sense, in the two separate models corresponding to the two experts, by the calculation of truth values at possible worlds of a single, many-valued

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model. How this can be done will be discussed below. The immediate problem is that this example is not sufficiently complicated to bring out essential ideas.

In the two-expert example just presented, the four truth values obtained have a very well-behaved structure: a Boolean algebra, with intersection, union, and complementation available. This is too simple, however, and arises from the fact that the experts are independent. Suppose, instead, that we have two experts, 1 and 2, but 1 is dominant: if expert 1 says something is true, expert 2 will agree. Then the truth value $\{1\}$, indicating that expert 1 alone assigns truth, is no longer possible. We are left with the three values: $\emptyset$, $\{2\}$, and $\{1, 2\}$. We still have closure under intersection and union, but complementation is no longer available to us. In fact, the appropriate structure now is that of a *pseudo-Boolean* or *Heyting* algebra ([11] and Section 2 below). But, in order to see the issue more clearly, it is best to reconsider the experts directly again.

If we assume some experts dominate others, constraints are placed on behavior that gives their logics an intuitionistic, rather than a classical flavor. For instance, consider implication. If there were no relationships between experts, expert 1 would assign $A \supset B$ the value true if A was assigned falsehood by expert 1 or if B was assigned truth. But now suppose expert 1 dominates expert 2, both experts consider B to be false, but expert 1 considers A to be false while expert 2 considers A true. So far, this is compatible with the idea that whatever 1 considers true, 2 will also consider true. However, if we evaluate $A \supset B$ according to the usual rules, expert 1 will take it to be true, while 2 will consider it false, violating the domination of 1 over 2. It is not reasonable to say expert 2 is wrong, since having A without B is clear grounds for rejecting the truth of $A \supset B$. To preserve the constraint that 1 dominates 2 we must say that 1 would not consider $A \supset B$ to be true under these circumstances. That is, an expert should assign truth to an implication $A \supset B$ only if that expert, *and every expert dominated by that one*, either assigns false to A or truth to B. A similar condition applies to negation, while conjunction and disjunction are rather better behaved.

The behavior just worked out for inter-related sets of experts is, in fact, a familiar one. We are describing Kripke intuitionistic logic models, in disguised form, [4]. Then it is no coincidence that the algebraic structure that arose above was a Heyting algebra, since these are also appropriate structures for the semantics of intuitionistic logic.

Now the overall plan of the paper can be loosely described. Suppose we start with a set of experts, with some associated pattern of domination. And suppose each expert has his or her own notion of a modal model, subject to the constraints imposed by the domination relation, of course. Each expert can then assign truth values to sentences in their modal model according to a scheme whereby the connectives $\wedge$, $\vee$, $\supset$ and $\neg$ are interpreted according to the rules of Kripke intuitionistic models, and $\Box$ and $\Diamond$ are interpreted according to the rules of that expert's modal model, modified by the conditions imposed by the dominance relationship between experts. In this way each formula has associated with it, at each possible world of the modal model, the set of experts who assign it truth at that world.

Next, suppose we consider sets of experts to be truth values in a many-valued logic. As observed above, this gives us a Heyting algebra. We introduce a notion of a Heyting-valued modal model, and show that truth value calculations in a single such model can replace the separate calculations of truth values for each of the individual experts.

Finally, we show that for a fixed Heyting algebra $\mathcal{T}$ (equivalently, for a fixed set of experts), the set of modal formulas valid in all $\mathcal{T}$ valued modal models can be given a proof-theoretic formulation, in a sequent calculus, and we prove soundness and completeness of this calculus.

In [10] a closely related system of logic, also based on multiple experts, was investigated. There, the ordering of experts was one of increasing sharpness of perception, which amounts to the converse of the ordering considered here. In the predecessor of the present paper, [3], two families of many-valued modal logics were examined. Semantically speaking, one family allowed formulas to take on values in a many-valued logic at possible worlds, but otherwise the general structure of a Kripke frame was not altered. Such logics, in fact, have a long history, [13, 14, 12, 5, 7, 6, 8]. The other family considered in [3] allowed the accessibility relation itself to be many-valued; something apparently new. In this paper the investigation of the second family of logics is continued. It is my hope that others will take up the study of these logics, for their own sakes and for the sake of possible applications based on the motivation presented here. But in addition, proofs about these logics sometimes generalize the proofs of corresponding results for conventional modal logics, and it is often the case that a better understanding of a subject arises from studying generalizations.

2 Heyting algebras and Kripke models — background

In this section basic ideas of Kripke intuitionistic models and Heyting algebras are sketched. More detailed treatments can be found elsewhere. For this section we assume we have a propositional language with connectives $\wedge$, $\vee$, $\supset$ and $\neg$, but without modal operators.

Definition 2.1 A *Kripke intuitionistic model* is a structure $\langle \mathcal{E}, \mathcal{D}, v\rangle$ where $\mathcal{E}$ is a non-empty set (the set of experts above), $\mathcal{D}$ is a reflexive and transitive relation on $\mathcal{E}$ (the dominance relation above), and v is a mapping from members of $\mathcal{E}$ and atomic formulas to the set $\{\mathit{true}, \mathit{false}\}$ meeting the condition that if $v(e, A) = \mathit{true}$ and $\mathcal{D}(e, f)$ then $v(f, A) = \mathit{true}$.

The mapping v is extended to all formulas as follows: for each $e \in \mathcal{E}$

1. $v(e, A \wedge B) = \mathit{true}$ if and only if $v(e, A) = \mathit{true}$ and $v(e, B) = \mathit{true}$.
2. $v(e, A \vee B) = \mathit{true}$ if and only if $v(e, A) = \mathit{true}$ or $v(e, B) = \mathit{true}$.
3. $v(e, A \supset B) = \mathit{true}$ if and only if, for every $f \in \mathcal{E}$ such that $\mathcal{D}(e, f)$, $v(f, A) = \mathit{false}$ or $v(f, B) = \mathit{true}$.
4. $v(e, \neg A) = \mathit{true}$ if and only if, for every $f \in \mathcal{E}$ such that $\mathcal{D}(e, f)$, $v(f, A) = \mathit{false}$.

Note that if the dominance relation $\mathcal{D}$ of a Kripke intuitionistic model is trivial (that is, it only holds between members of $\mathcal{E}$ and themselves), the truth conditions above reduce to the usual Boolean ones relativized to each member of $\mathcal{E}$. Whether the dominance relation is trivial or not it is now possible to show that, for any formula A, if $v(e, A) = \mathit{true}$ and $\mathcal{D}(e, f)$ then $v(f, A) = \mathit{true}$. Nothing in the definition requires that $\mathcal{E}$ be finite, though of course these are the kinds of examples we have in mind. It can be shown that if A is a theorem of propositional intuitionistic logic, $v(e, A) = \mathit{true}$ for every e of every Kripke intuitionistic model. Further, if A is not a theorem of propositional intuitionistic logic, there is a *finite* Kripke intuitionistic model, $\langle \mathcal{E}, \mathcal{D}, v\rangle$, and an $e \in \mathcal{E}$, with $v(e, A) = \mathit{false}$. This is simply the completeness theorem for intuitionistic logic plus the finite model property. See [4, 2, 1] for proofs. Next we turn to algebraic notions.

Definition 2.2 A *Heyting algebra* is a lattice $\mathcal{T}$ with a bottom element *false*, in which *relative pseudo-complements* exist. The *pseudo-complement* of *a relative to b* exists if there is a greatest member, $c \in \mathcal{T}$, such that $a \wedge c \leq b$. If such an element c exists, it is denoted $a \Rightarrow b$.

A Heyting algebra is a lattice, so meets and joins exist. We use $\wedge$ and $\vee$ to denote them, as we did above. Every Boolean algebra is a Heyting algebra, but not conversely. It can be shown (see [11]) that the existence of relative pseudo-complements implies a lattice is distributive. Further, in a *finite* distributive lattice, relative pseudo-complements and a bottom element must exist. Therefore finite Heyting algebras and finite distributive lattices are the same things. This is not so in the infinite case.

Definition 2.3 A *valuation* in a Heyting algebra $\mathcal{T}$ is a mapping w from the set of formulas to $\mathcal{T}$ such that:

1. $w(A \wedge B) = (w(A) \wedge w(B))$;
2. $w(A \vee B) = (w(A) \vee w(B))$;
3. $w(A \supset B) = (w(A) \Rightarrow w(B))$;
4. $w(\neg A) = (w(A) \Rightarrow \mathit{false})$.

Every Heyting algebra has a top, just set $\mathit{true} = (\mathit{false} \Rightarrow \mathit{false})$. If A is a theorem of propositional intuitionistic logic, $w(A) = \mathit{true}$ for any valuation w in any Heyting algebra. And if A is not a propositional intuitionistic theorem, there is some *finite* Heyting algebra $\mathcal{T}$ and some valuation w in it such that $w(A) \neq \mathit{true}$. Proofs of this can be found in [11, 2, 1].

There are standard connections between these two semantics for propositional intuitionistic logic; [1] can be consulted for details. Below we will establish connections in an extended setting, when modal operators are involved as well. This naturally generalizes the basic relationships, so we say no more about the matter for now.

3 Multiple-expert modal models

In Section 1 a semantics was informally introduced, involving many experts, with some dominating others. Now we return to that notion and present it formally, so that exact relationships with the many-valued semantics of the next section can be established. What the formal version amounts to is the blending of a Kripke intuitionistic model with a Kripke modal model. There is both a set of experts, with a dominance relation among them, and a set of possible worlds or situations. Each expert has his or her own accessibility relation on possible worlds, and the notion of truth at a possible world may be different for each expert, subject, of course, to the dominance relation among them. Incidentally, the ideas presented here are somewhat related to the work of [9], but there the emphasis is on intuitionistic logic, while here we think of a particular set of experts as furnishing us with a many-valued logic, of interest for its own sake.

Definition 3.1 A *multiple-expert modal model* is a structure, $\langle \mathcal{E}, \mathcal{D}, \mathcal{G}, \mathcal{R}, v \rangle$ where:

1. $\mathcal{E}$ is a finite non-empty set. (Think of this as the set of experts.)
2. $\mathcal{D} : \mathcal{E} \times \mathcal{E} \rightarrow \{\mathit{true}, \mathit{false}\}$ is a partial ordering of $\mathcal{E}$. (This is the dominance relation on experts. We will write $\mathcal{D}(e, f)$ as an abbreviation for $\mathcal{D}(e, f) = \mathit{true}$.)
3. $\mathcal{G}$ is a non-empty set. (Think of this as a set of possible worlds or situations.)
4. $\mathcal{R} : \mathcal{E} \times \mathcal{G} \times \mathcal{G} \rightarrow \{\mathit{true}, \mathit{false}\}$. (This is the accessibility relation on possible worlds, which now depends on which expert we are considering. We will write $\mathcal{R}_e(\Gamma, \Delta)$ as an abbreviation for $\mathcal{R}(e, \Gamma, \Delta) = \mathit{true}$.)
5. If $\mathcal{R}_e(\Gamma, \Delta)$ and $\mathcal{D}(e, f)$ then $\mathcal{R}_f(\Gamma, \Delta)$. (If an expert accepts that possible world Δ is accessible from possible world Γ, so must any expert he or she dominates.)

6. $v : \mathcal{E} \times \mathcal{G} \times \textit{set\_of\_propositional\_variables} \rightarrow \{\textit{true}, \textit{false}\}$. (This is the truth assignment, which depends on both which expert we are considering, and which possible world.)

7. If $v(e, \Gamma, P) = \textit{true}$ and $\mathcal{D}(e, f)$ then $v(f, \Gamma, P) = \textit{true}$, for atomic P. (What an expert accepts must also be accepted by any expert he or she dominates.)

Incidentally, the set of experts is required to be finite, though often this plays no particular role. Many of the proofs in this paper hold under more general assumptions. Now, for any multiple-expert modal model, assignment of the classical truth values *true* and *false* can be extended to all formulas of a modal propositional language, relative to experts and to possible worlds, by combining the techniques appropriate to Kripke intuitionistic and Kripke modal models.

Definition 3.2 Let $\langle \mathcal{E}, \mathcal{D}, \mathcal{G}, \mathcal{R}, v \rangle$ be a multiple-expert modal model. The mapping v is extended as follows.

1. $v(e, \Gamma, A \wedge B) = \textit{true}$ if and only if $v(e, \Gamma, A) = \textit{true}$ and $v(e, \Gamma, B) = \textit{true}$.

2. $v(e, \Gamma, A \vee B) = \textit{true}$ if and only if $v(e, \Gamma, A) = \textit{true}$ or $v(e, \Gamma, B) = \textit{true}$.

3. $v(e, \Gamma, A \supset B) = \textit{true}$ if and only if, for every $f \in \mathcal{E}$ such that $\mathcal{D}(e, f)$, $v(f, \Gamma, A) = \textit{false}$ or $v(f, \Gamma, B) = \textit{true}$.

4. $v(e, \Gamma, \neg A) = \textit{true}$ if and only if, for every $f \in \mathcal{E}$ such that $\mathcal{D}(e, f)$, $v(f, \Gamma, A) = \textit{false}$.

5. $v(e, \Gamma, \Box A) = \textit{true}$ if and only if, for every $f \in \mathcal{E}$ with $\mathcal{D}(e, f)$, and for every $\Delta \in \mathcal{G}$ such that $\mathcal{R}_f(\Gamma, \Delta)$, $v(f, \Delta, A) = \textit{true}$.

6. $v(e, \Gamma, \Diamond A) = \textit{true}$ if and only if there is some $\Delta \in \mathcal{G}$ such that $\mathcal{R}_e(\Gamma, \Delta)$ and $v(e, \Delta, A) = \textit{true}$.

Part 4 of the definition above, for instance, says an expert e will consider $\neg A$ to be true in a situation Γ provided no expert that e dominates would consider A to be true in that situation. Similarly part 5 says an expert e will consider $\Box A$ to be true in a situation Γ if every expert f that e dominates takes A to be true in every situation Δ that f thinks is a possible alternative. (We use 'dominates' loosely, and so e dominates e as well.) Incidentally, part 6 can be replaced with: $v(e, \Gamma, \Diamond A) = \textit{true}$ if and only if, for every $f \in \mathcal{E}$ with $\mathcal{D}(e, f)$, there is some $\Delta \in \mathcal{G}$ such that $\mathcal{R}_f(\Gamma, \Delta)$ and $v(f, \Delta, A) = \textit{true}$, which is more symmetric with part 5.

The definition above presupposes that we are using a language in which formulas are built up from propositional variables using the connectives $\wedge$, $\vee$, $\supset$, and $\neg$, and the modal operators $\Box$ and $\Diamond$. In using this language we will generally omit outer parentheses, and we will take $A \equiv B$ as an abbreviation for $(A \supset B) \wedge (B \supset A)$.

It is sometimes convenient to have propositional constants available as well. To this end, we assume there is a propositional constant *for each $\mathcal{D}$-closed subset of $\mathcal{E}$*, where a set S is $\mathcal{D}$-closed provided, if $e \in S$ and $\mathcal{D}(e, f)$, then $f \in S$. If S is a $\mathcal{D}$ closed set, we write $[\![S]\!]$ for the propositional constant corresponding to S. The definition above is extended with the additional condition $v(e, \Gamma, [\![S]\!]) = \textit{true}$ if and only if $e \in S$.

The following Proposition extends one of the conditions of the definition of multiple-expert modal model from the atomic level to all formulas. It essentially says the notion of dominance really has been captured. The straightforward proof is omitted.

Proposition 3.3 *Let $\langle \mathcal{E}, \mathcal{D}, \mathcal{G}, \mathcal{R}, v \rangle$ be a multiple-expert model, and let A be an arbitrary formula. If $v(e, \Gamma, A) = \textit{true}$ and $\mathcal{D}(e, f)$, then $v(f, \Gamma, A) = \textit{true}$.*

4 Many-valued modal models

It was suggested in Section 1 that instead of using two-valued logic, but talking about many experts, we could use a many-valued logic. In this section we present just such a many-valued modal semantics. In subsequent sections we relate it formally to multiple-expert modal models, and to a proof procedure. The semantics presented in this section first appeared in [3], where the models were called *implicational modal models*, to distinguish them from a different many-valued semantics that was also considered in that paper. A treatment of the possibility operator has been added here; it did not appear in [3].

Assume $\mathcal{T}$ is a fixed finite Heyting algebra, whose members will be referred to as *$\mathcal{T}$-truth values*. We continue to use $\wedge$, $\vee$, and $\Rightarrow$ for the meet, join, and relative pseudo-complement of $\mathcal{T}$. The bottom and top members of $\mathcal{T}$ will be denoted *false* and *true* respectively. If these are the only members of $\mathcal{T}$, we have the classical setting, in which case *false* can be defined as $A \wedge \neg A$ and *true* as $A \vee \neg A$. In general, however, not all members of $\mathcal{T}$ are definable using the Heyting algebra machinery, and so we explicitly add propositional constants to our formal language to represent $\mathcal{T}$-truth values. For convenience, we simply take members of $\mathcal{T}$ themselves to be these propositional constants.

We have required that the Heyting algebra $\mathcal{T}$ be finite. This is not necessary at all for many of the results in this paper, while other results can be extended to Heyting algebras that are complete, or meet some other special condition. Rather than try for utmost generality, we have confined our presentation to the finite case where everything works well without qualification, and which is most natural from the point of view of the underlying motivation.

Definition 4.1 A *$\mathcal{T}$-modal model* is a structure $\langle \mathcal{G}, \mathcal{R}, w \rangle$ where $\mathcal{G}$ is a non-empty set of possible worlds, $\mathcal{R}$ is a mapping from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{T}$, and w maps worlds and propositional variables to $\mathcal{T}$.

The mapping $\mathcal{R}$ can be thought of as a $\mathcal{T}$-valued accessibility relation. Now, the key item is how to extend the valuation w to non-atomic formulas. The parts involving $\Box$ and $\Diamond$ need the requirement that $\mathcal{T}$ be finite (or at least, complete).

Definition 4.2 Let $\langle \mathcal{G}, \mathcal{R}, w \rangle$ be a $\mathcal{T}$-modal model. The mapping w is extended as follows. For any $\Gamma \in \mathcal{G}$:

1. If t is a propositional constant (which we identified with a member of $\mathcal{T}$), $w(\Gamma, t) = t$.
2. $w(\Gamma, A \wedge B) = (w(\Gamma, A) \wedge w(\Gamma, B))$.
3. $w(\Gamma, A \vee B) = (w(\Gamma, A) \vee w(\Gamma, B))$.
4. $w(\Gamma, A \supset B) = (w(\Gamma, A) \Rightarrow w(\Gamma, B))$.
5. $w(\Gamma, \neg A) = (w(\Gamma, A) \Rightarrow \mathit{false})$.
6. $w(\Gamma, \Box A) = \bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, A) \mid \Delta \in \mathcal{G}\}$.
7. $w(\Gamma, \Diamond A) = \bigvee\{\mathcal{R}(\Gamma, \Delta) \wedge w(\Delta, A) \mid \Delta \in \mathcal{G}\}$.

Item 6 above says the value of $\Box A$ at Γ is taken to be the meet over every world Δ, of the relative pseudo-complement of the accessibility of Δ with the value of A at Δ. Similarly for $\Diamond A$. It should be noted that if $\mathcal{T}$ is classical two-valued logic, these conditions reduce to the usual Kripke ones.

Not surprisingly, we say a formula A is *valid* in a $\mathcal{T}$-modal model $\langle \mathcal{G}, \mathcal{R}, w \rangle$ if $w(\Gamma, A) = \mathit{true}$ for all $\Gamma \in \mathcal{G}$. Likewise A is $\mathcal{T}$-valid if it is valid in all $\mathcal{T}$-modal models. Many familiar formulas are $\mathcal{T}$-valid in this sense, $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is an example, whose verification is left to the reader. On the other hand, $\Box A \equiv \neg\Diamond\neg A$ is not generally valid, though it will be whenever $\mathcal{T}$ happens to be not just a Heyting algebra, but a Boolean algebra. To illustrate how things work in more detail, we consider an example, and a proposition.

Example Take $\mathcal{T}$ to be the three-element Heyting algebra $\{\mathit{false}, m, \mathit{true}\}$, with $\mathit{false} < m < \mathit{true}$. We show $\neg\Diamond\neg P \supset \Box P$ is not $\mathcal{T}$-valid.

Let $\langle \mathcal{G}, \mathcal{R}, w \rangle$ be the $\mathcal{T}$-modal model such that; $\mathcal{G} = \{\Gamma, \Delta\}$; $\mathcal{R}(\Gamma, \Gamma) = \mathit{false}$; $\mathcal{R}(\Gamma, \Delta) = \mathit{true}$; $\mathcal{R}(\Delta, \Delta) = \mathit{false}$; $w(\Gamma, P) = \mathit{false}$; $w(\Delta, P) = m$. Note that $w(\Delta, \neg P) = (w(\Delta, P) \Rightarrow \mathit{false}) = (m \Rightarrow \mathit{false}) = \mathit{false}$. Now:

$$\begin{aligned} w(\Gamma, \Box P) &= (\mathcal{R}(\Gamma, \Gamma) \Rightarrow w(\Gamma, P)) \wedge (\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, P)) \\ &= (\mathit{false} \Rightarrow \mathit{false}) \wedge (\mathit{true} \Rightarrow m) \\ &= \mathit{true} \wedge m \\ &= m \end{aligned}$$

But also:

$$\begin{aligned} w(\Gamma, \Diamond\neg P) &= (\mathcal{R}(\Gamma, \Gamma) \wedge w(\Gamma, \neg P)) \vee (\mathcal{R}(\Gamma, \Delta) \wedge w(\Delta, \neg P)) \\ &= (\mathit{false} \wedge w(\Gamma, \neg P)) \vee (\mathit{true} \wedge \mathit{false}) \\ &= \mathit{false} \vee \mathit{false} \\ &= \mathit{false} \end{aligned}$$

Then $w(\Gamma, \neg\Diamond\neg P) = \mathit{true}$, so $w(\Gamma, \neg\Diamond\neg P \supset \Box P) = (w(\Gamma, \neg\Diamond\neg P) \Rightarrow w(\Gamma, \Box P)) = (\mathit{true} \Rightarrow m) = m \neq \mathit{true}$.

Next some positive results.

Proposition 4.3 *Let $\mathcal{T}$ be a finite Heyting algebra, and suppose t is a member of $\mathcal{T}$ (that is, t is a propositional constant). The following are valid in every $\mathcal{T}$-modal model.*

1. $(t \supset \Box A) \equiv \Box(t \supset A)$;
2. $(\Diamond A \supset t) \equiv \Box(A \supset t)$.

Proof The verification of item 1 can be found in [3]. Item 2 is established below. In this, and subsequent work, we make free use of standard facts concerning Heyting algebras. For example, in any Heyting algebra, $((a \wedge b) \Rightarrow c) = (a \Rightarrow (b \Rightarrow c))$, and $(\bigvee_i a_i \Rightarrow b) = \bigwedge_i (a_i \Rightarrow b)$, and both facts are used here. Proofs of such items can be found in [11], which is a standard reference.

$$\begin{aligned} w(\Gamma, \Diamond A \supset t) &= w(\Gamma, \Diamond A) \Rightarrow w(\Gamma, t) \\ &= w(\Gamma, \Diamond A) \Rightarrow t \\ &= \bigvee\{\mathcal{R}(\Gamma, \Delta) \wedge w(\Delta, A) \mid \Delta \in \mathcal{G}\} \Rightarrow t \\ &= \bigwedge\{(\mathcal{R}(\Gamma, \Delta) \wedge w(\Delta, A)) \Rightarrow t \mid \Delta \in \mathcal{G}\} \\ &= \bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow (w(\Delta, A) \Rightarrow t) \mid \Delta \in \mathcal{G}\} \\ &= \bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow (w(\Delta, A) \Rightarrow w(\Delta, t)) \mid \Delta \in \mathcal{G}\} \\ &= \bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, A \supset t) \mid \Delta \in \mathcal{G}\} \\ &= w(\Gamma, \Box(A \supset t)) \end{aligned}$$

■

Note that item 2 of Proposition 4.3 has, as a special case, the validity of $(\Diamond A \supset \mathit{false}) \equiv \Box(A \supset \mathit{false})$, which gives the validity of $\neg\Diamond A \equiv \Box\neg A$, and hence also the validity of $\neg\Diamond\neg A \equiv \Box\neg\neg A$.

5 Relationships between semantics

We have now given two different modal semantics, one involving multiple experts, one involving a many-valued logic. In this section we show their essential equivalence by providing embeddings between them. In effect, this extends to the modal setting well-known proofs of the equivalence between the algebraic and the possible world semantics for intuitionistic propositional logic.

Multiple-expert to many-valued. Let $\langle \mathcal{E}, \mathcal{D}, \mathcal{G}, \mathcal{R}, v \rangle$ be a multiple-expert modal model, fixed throughout this argument. We produce a many-valued modal model which is, in a reasonable sense, equivalent to it. We assume the multiple-expert model has a constant symbol $[\![S]\!]$ for each $\mathcal{D}$-closed subset of $\mathcal{E}$.

Take $\mathcal{T}$ to be the collection of $\mathcal{D}$-closed subsets of $\mathcal{E}$, and give it the partial ordering $\subseteq$. This yields a Heyting algebra. Verification of this is omitted — some of the details can be found in [1, Ch 1 §6].

Next, we take over $\mathcal{G}$ unchanged, but define a many-valued relation, $\mathcal{R}'$, on it as follows. $\mathcal{R}'(\Gamma, \Delta) = \{e \in \mathcal{E} \mid \mathcal{R}_e(\Gamma, \Delta)\}$. By Part 5 of Definition 3.1, this is a $\mathcal{D}$-closed set, and so $\mathcal{R}' : \mathcal{G} \times \mathcal{G} \to \mathcal{T}$.

Finally, define a mapping w on worlds and propositional variables by setting $w(\Gamma, P) = \{e \in \mathcal{E} \mid v(e, \Gamma, P) = \mathit{true}\}$. By Part 7 of Definition 3.1, this is a $\mathcal{D}$-closed set, and so the mapping w is to $\mathcal{T}$.

Now a $\mathcal{T}$-modal model, $\langle \mathcal{G}, \mathcal{R}', w \rangle$ has been defined. The essential equivalence of the two models is given in the following.

Proposition 5.1 *For any formula A, $w(\Gamma, A) = \{e \in \mathcal{E} \mid v(e, \Gamma, A) = \mathit{true}\}$.*

Proof The proof is by an induction on degree. The propositional variable case is by definition, and the propositional constant case is immediate. The induction steps concerning the propositional connectives are exactly as in the intuitionistic case and can be found, for instance, in [1, Ch 1 §6] (with somewhat different notation). We show the $\Box$ case, and omit the argument for $\Diamond$. Incidentally, in the Heyting algebra constructed, the meet operation is just intersection, which somewhat simplifies things.

Suppose the result is known for A. Then $w(\Gamma, \Box A) = \bigcap\{\mathcal{R}'(\Gamma, \Delta) \Rightarrow w(\Delta, A) \mid \Delta \in \mathcal{G}\}$, and we must show this is equal to $\{e \mid v(e, \Gamma, \Box A) = \mathit{true}\}$. The argument is in two parts.

1) Choose an arbitrary $\Delta \in \mathcal{G}$. Certainly if $v(e, \Gamma, \Box A) = \mathit{true}$ and $\mathcal{R}_e(\Gamma, \Delta)$ then $v(e, \Delta, A) = \mathit{true}$. Then by the induction hypothesis, if $v(e, \Gamma, \Box A) = \mathit{true}$ and $\mathcal{R}_e(\Gamma, \Delta)$ then $e \in w(\Delta, A)$. That is,

$$\{e \mid v(e, \Gamma, \Box A) = \mathit{true}\} \cap \mathcal{R}'(\Gamma, \Delta) \subseteq w(\Delta, A)$$

from which it follows that

$$\{e \mid v(e, \Gamma, \Box A) = \mathit{true}\} \subseteq \Big(\mathcal{R}'(\Gamma, \Delta) \Rightarrow w(\Delta, A)\Big).$$

Since Δ was arbitrary,

$$\{e \mid v(e, \Gamma, \Box A) = \mathit{true}\} \subseteq \bigcap\{\mathcal{R}'(\Gamma, \Delta) \Rightarrow w(\Delta, A) \mid \Delta \in \mathcal{G}\}.$$

2) Suppose $S \in \mathcal{T}$ and $S \subseteq \mathcal{R}'(\Gamma, \Delta) \Rightarrow w(\Delta, A)$ for all Δ. The second condition is equivalent to assuming $S \cap \mathcal{R}'(\Gamma, \Delta) \subseteq w(\Delta, A)$, for all Δ. We show $S \subseteq \{e \mid v(e, \Gamma, \Box A) = \mathit{true}\}$, which will complete the argument.

Well, suppose not. Say $e \in S$ but $v(e, \Gamma, \Box A) = \mathit{false}$. Then for some $f \in \mathcal{E}$ with $\mathcal{D}(e, f)$, and for some $\Delta_0 \in \mathcal{G}$, $\mathcal{R}_f(\Gamma, \Delta_0)$ and $v(f, \Delta_0, A) = \mathit{false}$. Since S is $\mathcal{D}$-closed, $f \in S$. Also since $\mathcal{R}_f(\Gamma, \Delta_0)$, $f \in \mathcal{R}'(\Gamma, \Delta_0)$. Then $f \in S \cap \mathcal{R}'(\Gamma, \Delta_0) \subseteq w(\Delta_0, A)$, so $f \in w(\Delta_0, A)$. By the induction hypothesis, $v(f, \Delta_0, A) = \mathit{true}$, and this is a contradiction, thus establishing part 2. ■

This concludes the first half of the section.

Many-valued to multiple-expert. For the rest of this section, let $\mathcal{T}$ be a fixed finite Heyting algebra, and let $\langle \mathcal{G}, \mathcal{R}, w \rangle$ be a fixed $\mathcal{T}$-modal model. The conversion of this to a multiple-expert model is somewhat more complicated than the other direction.

A *filter* in a Heyting algebra is a non-empty subset that is closed under $\wedge$ and is upward closed (a filter is upward closed provided it contains T whenever it contains S, and $S \leq T$ in the ordering of the algebra). A filter is *proper* if it is not the entire algebra, or equivalently, if it does not contain the bottom element. Finally, a filter is *prime* provided, whenever it contains $S \vee T$, it also contains one of S or T. (In Boolean algebras, maximal and prime filters are the same thing. In Heyting algebras they need not be.) For information on filters see [11].

Now, take $\mathcal{E}$ to be the set of all proper prime filters in $\mathcal{T}$, with $\mathcal{D}$ being the subset relation. We take over $\mathcal{G}$ as is. For each $e \in \mathcal{E}$ and $\Gamma, \Delta \in \mathcal{G}$, set $\mathcal{R}_e(\Gamma, \Delta) = \mathit{true}$ if and only if $\mathcal{R}(\Gamma, \Delta) \in e$. Finally, for a propositional variable P, set $v(e, \Gamma, P) = \mathit{true}$ if and only if $w(\Gamma, P) \in e$. This gives us a multiple-expert modal model $\langle \mathcal{E}, \mathcal{D}, \mathcal{G}, \mathcal{R}, v \rangle$. The connection between the models is as follows.

Proposition 5.2 *For any formula A, $v(e, \Gamma, A) = \mathit{true}$ if and only if $w(\Gamma, A) \in e$.*

Proof As usual, the proof is by induction on formula degree. And once again, the propositional connective cases can found in [1, Ch 1 §6]. We give the argument for $\Box$, and omit that for $\Diamond$.

Suppose first that $v(e, \Gamma, \Box A) = \mathit{false}$. Then for some $f \in \mathcal{E}$ with $\mathcal{D}(e, f)$, and for some $\Delta_0 \in \mathcal{G}$, $\mathcal{R}_f(\Gamma, \Delta_0)$, but $v(f, \Delta_0, A) = \mathit{false}$. By definition, and the induction hypothesis, $\mathcal{R}(\Gamma, \Delta_0) \in f$ but $w(\Delta_0, A) \notin f$. As a general observation about filters, if $x \in f$ and $x \Rightarrow y \in f$ then $y \in f$ as well, because filters are closed under $\wedge$, in any Heyting algebra $(x \wedge (x \Rightarrow y)) \leq y$, and filters are upward closed. In this case, then, $\mathcal{R}(\Gamma, \Delta_0) \Rightarrow w(\Delta_0, A) \notin f$. Again, filters are upward closed, and so $\bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, A) \mid \Delta \in \mathcal{G}\} \notin f$, that is, $w(\Gamma, \Box A) \notin f$. Finally, since $\mathcal{D}(e, f)$, $e \subseteq f$, so $w(\Gamma, \Box A) \notin e$.

Conversely, assume $w(\Gamma, \Box A) \notin e$. Then $\bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, A) \mid \Delta \in \mathcal{G}\} \notin e$, and so for some Δ_0, $\mathcal{R}(\Gamma, \Delta_0) \Rightarrow w(\Delta_0, A) \notin e$ (the finiteness of the Heyting algebra is used here). It can be shown that if a proper prime filter does not contain $x \Rightarrow y$, there is a proper prime filter that extends it, contains x, but does not contain y. (See [1, Ch 1 §6] for a proof.) Let f be a proper prime filter, extending e, containing $\mathcal{R}(\Gamma, \Delta_0)$ but not $w(\Delta_0, A)$. Then $\mathcal{D}(e, f)$, $\mathcal{R}_f(\Gamma, \Delta_0)$, but $v(f, \Delta_0, A) = \mathit{false}$, and so $v(e, \Gamma, \Box A) = \mathit{false}$. ■

6 A proof procedure

In Section 4 a semantics was given for a family of many-valued modal logics. Now we present a proof procedure to go with that semantics, and sketch a proof of completeness in the following section. In [3], the predecessor of this paper, a proof procedure was also given and it is essentially repeated here, but with certain differences. First, the modal rules of derivation used in the earlier paper are replaced by equivalent ones that should look more familiar. Second, the operator $\Diamond$ is taken into account here, while it was not earlier — an explicit treatment is required since, generally, $\Diamond$ and $\Box$ are not interdefinable in the present setting, unlike in modal logics based on classical logic.

Third, and most important, the completeness proof presented in [3] was incorrect, and a revised (and more complex) one is given here.

For this section, $\mathcal{T}$ is a finite Heyting algebra. Recall, the formal language we are using has the members of $\mathcal{T}$ as propositional constants, and has both $\Box$ and $\Diamond$ primitive. Since we want to concentrate on modal features, we spend little time on the underlying many-valued logic. In [3] a Gentzen sequent calculus was given for this logic, and we simply adopt it now. The basic notion is that of a *sequent*, written

$$A_1, \ldots, A_n \rightarrow B_1, \ldots, B_k$$

and interpreted as asserting that in any many-valued modal model, if all of $A_1, \ldots, A_n$ have the value *true* at some world Γ, then at least one of $B_1, \ldots, B_k$ will also have the value *true* at Γ. For convenience we reproduce the axioms and rules for this (non-modal) sequent calculus, from [3]. The usual structural rules come first.

Identity Axiom

$$X \rightarrow X$$

Thinning

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \cup \Gamma' \rightarrow \Delta \cup \Delta'}$$

Cut

$$\frac{\Gamma \rightarrow \Delta, X \quad \Gamma, X \rightarrow \Delta}{\Gamma \rightarrow \Delta}$$

Next are the rules for implication. The first is a transitivity axiom. The following two make use of the truth values of $\mathcal{T}$.

Axiom of Transitivity

$$X \supset Y, Y \supset Z \rightarrow X \supset Z$$

Rule $RI \supset$

$$\frac{\Gamma, t \supset A \rightarrow \Delta, t \supset B \quad (\text{for every } t \in \mathcal{T})}{\Gamma \rightarrow \Delta, A \supset B}$$

Rule $\supset RI$

$$\frac{\Gamma, B \supset t \rightarrow \Delta, A \supset t \quad (\text{for every } t \in \mathcal{T})}{\Gamma \rightarrow \Delta, A \supset B}$$

Now we have the expected lattice-theoretic rules for the lattice-theoretic connectives, including those peculiar to Heyting algebras.

Conjunction Axioms

$$\rightarrow A \wedge B \supset A$$

$$\rightarrow A \wedge B \supset B$$

$$C \supset A, C \supset B \rightarrow C \supset A \wedge B$$

Disjunction Axioms

$$\rightarrow A \supset A \vee B$$

$$\rightarrow B \supset A \vee B$$

$$A \supset C, B \supset C \rightarrow A \vee B \supset C$$

Implication Axioms

$$(A \wedge B) \supset C \rightarrow A \supset (B \supset C)$$

$$A \supset (B \supset C) \rightarrow (A \wedge B) \supset C$$

Next are the rules that reflect the properties of $\mathcal{T}$ itself.

Propositional Constant Axioms

$$\rightarrow t \supset u \quad \text{for } t, u \in \mathcal{T} \text{ with } t \leq u$$

$$t \supset u \rightarrow \quad \text{for } t \in \mathcal{T} \text{ with } t \not\leq u$$

Finally there are two axioms that were not present in [3]. In that paper, for reasons that are not of concern here, all formulas were implications. The following axioms make the connection with implications available now.

Conversion Axioms

$$X \rightarrow (\mathit{true} \supset X)$$

$$(\mathit{true} \supset X) \rightarrow X$$

This completes the formulation of the underlying system. It is straightforward to verify that, for $a, b, c \in \mathcal{T}$, if $a \wedge b = c$, then both $\rightarrow (a \wedge b) \supset c$ and $\rightarrow c \supset (a \wedge b)$ are provable; and similarly for the other connectives. Likewise Modus Ponens is easily shown to be a derived rule. Finally, it was shown in [3] that the following is a derived rule (where S and T are non-empty sets of formulas):

$$\frac{S, (A \equiv t) \rightarrow T \quad \text{(for all } t \in \mathcal{T})}{S \rightarrow T}$$

Other items will be mentioned, or simply assumed without mention, as needed. Now we give the specifically modal axioms and rules. Several of them will be quite familiar.

Rule of Inference

Necessitation

$$\frac{\rightarrow A}{\rightarrow \Box A}$$

Axioms

$\rightarrow \Box(A \supset B) \supset (\Box A \supset \Box B)$

For each $t \in \mathcal{T}$, $\rightarrow (t \supset \Box A) \equiv \Box(t \supset A)$

For each $t \in \mathcal{T}$, $\rightarrow (\Diamond A \supset t) \equiv \Box(A \supset t)$

We say X is a *theorem* of this proof procedure if $\rightarrow X$ is a provable sequent. Soundness is immediate, using results like Proposition 4.3. Completeness is considerably more technical. We place the argument in a section of its own, so that it can be skimmed, or skipped, if desired.

7 Completeness

In this section completeness of the proof procedure of Section 6 is shown. The proof here is modeled after a similar one in [3]. But it must be pointed out, that earlier proof contained an error which is corrected here — more on this below. As usual with modal completeness proofs, we try to construct a canonical model, whose possible worlds are maximal consistent sets of formulas. It is the definition of consistency that is somewhat unusual. Of course, all this is relative to the choice of an underlying space of truth values. For the rest of this section, $\mathcal{T}$ is a fixed, finite Heyting algebra. Throughout this section we find it convenient to write $\bigwedge S$, where S is a finite set of formulas, to denote the conjunction of the members of the set. We are also using $\bigwedge$ for the meet operation on $\mathcal{T}$, but context will make clear which is meant.

In a classical sequent calculus, $S, A \rightarrow B$ is equivalent to $S \rightarrow (A \supset B)$ (the sequent calculus embodiment of the deduction theorem). It follows that classically a finite set S can be defined to be consistent if either $S \not\rightarrow$ *false*, or if $\not\rightarrow (\bigwedge S \supset$ *false*$)$. The deduction theorem does not hold in general for a many-valued sequent calculus, and so a set on the left of the arrow, and a conjunction in an implication antecedent on the right of the arrow, play quite different roles. Naturally this complicates things. In addition, in classical logic, formulas can be moved across the sequent arrow by adding or dropping a negation symbol. Then, if we want to say $S \not\rightarrow A$, we can say instead that $S, \neg A \not\rightarrow$ *false*. Briefly, questions about the non-derivability of a formula can always be converted into questions about the non-derivability of *false* — into simple consistency issues, in other words. In the many-valued logics being considered here, however, we do not generally have the law of double negation elimination available to us, and so we must talk about the non-derivability of arbitrary formulas, not just of *false*.

Definition 7.1 Let S and T be sets of formulas, and A be a single formula. We call the pair $\langle S, T\rangle$ *A-inconsistent* if there are finite subsets $S_0 \subseteq S$ and $T_0 \subseteq T$ such that

$$S_0 \rightarrow \left(\bigwedge T_0 \supset A\right)$$

is a provable sequent. We say $\langle S, T\rangle$ is *A-consistent* if it is not A-inconsistent.

Then *false*-inconsistency is the direct generalization of inconsistency in the classical sequent calculus. The usual Lindenbaum construction can be applied, to show that if $\langle S, T\rangle$ is A-consistent, it can be extended either to an A-consistent pair $\langle S^*, T\rangle$ with S^* maximal, or to an A-consistent pair $\langle S, T^*\rangle$ with T^* maximal. It is not clear to what extent these two constructions can be combined, but fortunately, we have no need to do so. We only need maximality in the first component, for the completeness proof to go through.

Definition 7.2 We say $\langle S, T\rangle$ is *maximal* A-consistent if it is A-consistent and, if $\langle S', T\rangle$ is also A-consistent, with $S \subseteq S'$, then $S = S'$.

Lemma 7.3 *If $\langle S, T\rangle$ is A-consistent, it can be extended to a maximal A-consistent pair $\langle S^*, T\rangle$. For each formula X there will be exactly one member $t \in \mathcal{T}$ such that $(X \equiv t) \in S^*$.*

Proof As remarked, the usual Lindenbaum construction produces the set S^*. And as usual, such a maximal set will contain every theorem and be closed under modus ponens. We use these facts below, generally without comment. It is the second part of the Lemma that needs work.

If $\langle S^*, T\rangle$ is maximal A-consistent, there can not be two members $t_1, t_2 \in \mathcal{T}$ with both $X \equiv t_1$ and $X \equiv t_2$ in S^*. For otherwise it would follow that $(t_1 \equiv t_2) \in S^*$, but since $t_1 \neq t_2$, $(t_1 \equiv$

$t_2) \neq \mathit{true}$. Then the A-inconsistency of $\langle S^*, T\rangle$ would follow from the sequent calculus rule $t \rightarrow$ for $t \neq \mathit{true}$.

To establish that for at least one $t \in \mathcal{T}$, $(X \equiv t) \in S^*$, it is enough to show that if $\langle S^* \cup \{X \equiv t\}, T\rangle$ is not A-consistent for each $t \in \mathcal{T}$, then $\langle S^*, T\rangle$ is not A-consistent. For notational convenience, say $\mathcal{T} = \{t_1, \ldots, t_n\}$.

Now, suppose $\langle S^* \cup \{X \equiv t\}, T\rangle$ is not A-consistent for each $t \in \mathcal{T}$. Then, for each $1 \leq i \leq n$ there is a finite $S_i \subseteq S^*$, and a finite $T_i \subseteq T$ such that

$$S_i, (X \equiv t_i) \rightarrow (\bigwedge T_i \supset A)$$

is a provable sequent. Let $S_0 = S_1 \cup \ldots \cup S_n$; it follows easily that for each $1 \leq i \leq n$

$$S_0, (X \equiv t_i) \rightarrow (\bigwedge T_i \supset A)$$

is a provable sequent. Finally, let $T_0 = T_1 \cup \ldots \cup T_n$. Then for each $1 \leq i \leq n$

$$S_0, (X \equiv t_i) \rightarrow (\bigwedge T_0 \supset A)$$

is a provable sequent. Now, using a derived rule of the sequent calculus, stated in the previous section,

$$S_0 \rightarrow (\bigwedge T_0 \supset A)$$

is a provable sequent, and this establishes that $\langle S^*, T\rangle$ itself is not A-consistent. ■

Note that in the pair $\langle S, T\rangle$ of sets of formulas, either component can be empty. We identify the formula $\bigwedge \emptyset \supset A$ with A.

Definition 7.4 We call a set S *left consistent* if there is a set T and a formula A such that $\langle S, T\rangle$ is A-consistent. Similarly we say S is *maximal* left consistent if there is a set T and a formula A such that $\langle S, T\rangle$ is maximal A-consistent.

Apology As was mentioned earlier, there is a fundamental error in the completeness proof given in [3] (Theorem 10.4 in that paper). A definition of consistency different from that above is used, essentially keeping only the second component of $\langle S, T\rangle$. It is remarked that a maximal consistent set S will meet the condition that, for each formula X there will be exactly one $t \in \mathcal{T}$ such that $(X \equiv t) \in S$. As it happens, using the notion of consistency from that earlier paper, there must be at least one such t, but it need not be unique. Without this, the entire remainder of the proof fails.

Before getting to the heart of the matter, it is convenient to state and prove some derived rules of the proof procedure. The first of these (without $\Diamond$) was taken as fundamental in [3].

Lemma 7.5 *In the statements below, A_i, B_j, and C are arbitrary formulas, while a_i, b_j and c are propositional constants, that is, members of $\mathcal{T}$. The following are derived rules.*

$$\frac{\rightarrow \left(\bigwedge_{i\in\mathcal{I}}(a_i \supset A_i) \wedge \bigwedge_{j\in\mathcal{J}}(B_j \supset b_j)\right) \supset (c \supset C)}{\rightarrow \left(\bigwedge_{i\in\mathcal{I}}(a_i \supset \Box A_i) \wedge \bigwedge_{j\in\mathcal{J}}(\Diamond B_j \supset b_j)\right) \supset (c \supset \Box C)}$$

$$\frac{\rightarrow \left(\bigwedge_{i\in\mathcal{I}}(a_i \supset A_i) \wedge \bigwedge_{j\in\mathcal{J}}(B_j \supset b_j)\right) \supset (C \supset c)}{\rightarrow \left(\bigwedge_{i\in\mathcal{I}}(a_i \supset \Box A_i) \wedge \bigwedge_{j\in\mathcal{J}}(\Diamond B_j \supset b_j)\right) \supset (\Diamond C \supset c)}$$

Proof Since we have the Rule of Necessitation, and the Axiom allowing distribution of $\Box$ across $\supset$, the usual proofs from conventional modal logics carry over, to show that $\Box(A \wedge B) \equiv (\Box A \wedge \Box B)$ is a theorem, and

$$\frac{\rightarrow A \supset B}{\rightarrow \Box A \supset \Box B}$$

is a derived rule. Using these we can argue as follows. Suppose we have

$$\rightarrow \Big(\bigwedge_{i \in \mathcal{I}} (a_i \supset A_i) \wedge \bigwedge_{j \in \mathcal{J}} (B_j \supset b_j)\Big) \supset (c \supset C).$$

Then

$$\rightarrow \Box\Big(\bigwedge_{i \in \mathcal{I}} (a_i \supset A_i) \wedge \bigwedge_{j \in \mathcal{J}} (B_j \supset b_j)\Big) \supset \Box(c \supset C)$$

$$\rightarrow \Big(\bigwedge_{i \in \mathcal{I}} \Box(a_i \supset A_i) \wedge \bigwedge_{j \in \mathcal{J}} \Box(B_j \supset b_j)\Big) \supset \Box(c \supset C).$$

Finally, using the axioms peculiar to the $\mathcal{T}$-based system,

$$\rightarrow \Big(\bigwedge_{i \in \mathcal{I}} (a_i \supset \Box A_i) \wedge \bigwedge_{j \in \mathcal{J}} (\Diamond B_j \supset b_j)\Big) \supset (c \supset \Box C).$$

The other rule is established in a similar way. ■

Now we are ready for the (hopefully correct) proof of completeness.

Theorem 7.6 (Completeness) *The many-valued modal proof procedure of Section 6 is complete with respect to the semantics of Section 4.*

Proof Take $\mathcal{G}$ to be the collection of all maximal left consistent sets of formulas. For each formula X and each $\Gamma \in \mathcal{G}$, set $w_0(\Gamma, X)$ to be the unique $t \in \mathcal{T}$ such that $(X \equiv t) \in \Gamma$ (Lemma 7.3 is needed here). It can be shown that for $\Gamma \in \mathcal{G}$, $w_0(\Gamma, (X \supset Y)) = (w_0(\Gamma, X) \Rightarrow w_0(\Gamma, Y))$, and similarly for the other connectives. We use this frequently below, without comment. Next, for $\Gamma, \Delta \in \mathcal{G}$, set

$$\begin{array}{rcl} \mathcal{R}(\Gamma, \Delta) & = & \bigwedge\{w_0(\Gamma, \Box Y) \Rightarrow w_0(\Delta, Y) \mid \text{all formulas } Y\} \\ & & \wedge \bigwedge\{w_0(\Delta, Z) \Rightarrow w_0(\Gamma, \Diamond Z) \mid \text{all formulas } Z\}. \end{array}$$

Finally, on atomic formulas P, set $w(\Gamma, P) = w_0(\Gamma, P)$. This determines a many-valued modal model $\langle \mathcal{G}, \mathcal{R}, w \rangle$.

As usual, the mapping w extends to all formulas using Definition 4.2. The chief item to be shown is that w and w_0 agree on all formulas, not just at the atomic level. To keep things relatively easy to follow, we assume this result for the moment, and show how the completeness follows immediately.

Suppose X is not provable, that is, suppose the sequent $\rightarrow X$ does not have a proof. Then $\langle \emptyset, \emptyset \rangle$ is X consistent. Extend $\emptyset$ to a set Γ such that $\langle \Gamma, \emptyset \rangle$ is maximally X-consistent. Then $\Gamma \in \mathcal{G}$. Since $\langle \Gamma, \emptyset \rangle$ is X-consistent, $X \equiv$ *true* can not be in Γ, so $w_0(\Gamma, X) \neq$ *true*. Assuming w and w_0 agree on all formulas, $w(\Gamma, X) \neq$ *true*, and so X is not valid.

Now we turn to the heart of the proof, which we postponed earlier. We will show the following.

Truth Property For every $\Gamma \in \mathcal{G}$ and every formula X, $w(\Gamma, X) = w_0(\Gamma, X)$.

As expected, the Truth Property is established by induction on the degree of X. The ground case is given directly by the definition of w. The propositional connective cases are straightforward, and we omit them. The modal operator cases are essentially new; we present the argument for $\Box$ in some detail, and omit the similar one for $\Diamond$. Several properties of the relative pseudo-complement operator are used frequently; we collect them together here for reference. First, in any Heyting algebra, $x \le y$ iff $true \le (x \Rightarrow y)$. Next, again in any Heyting algebra, $(x \Rightarrow (y \Rightarrow z)) = ((x \wedge y) \Rightarrow z) = (y \Rightarrow (x \Rightarrow z))$. It follows from these that $x \le (y \Rightarrow z)$ if and only if $y \le (x \Rightarrow z)$.

Induction Hypothesis Assume that $w(\Gamma, X) = w_0(\Gamma, X)$, for all worlds Γ.

To Be Shown $w(\Gamma, \Box X) = w_0(\Gamma, \Box X)$, for all worlds Γ.

The argument divides into two parts; we show $w_0(\Gamma, \Box X) \le w(\Gamma, \Box X)$ and $w(\Gamma, \Box X) \le w_0(\Gamma, \Box X)$ separately. We take them in order. Pick an arbitrary world $\Gamma_0 \in \mathcal{G}$, to be held fixed for the remainder of this proof.

Part 1. By definition, for every Δ, $\mathcal{R}(\Gamma_0, \Delta) \le (w_0(\Gamma_0, \Box X) \Rightarrow w_0(\Delta, X))$. It follows that $w_0(\Gamma_0, \Box X) \le (\mathcal{R}(\Gamma_0, \Delta) \Rightarrow w_0(\Delta, X))$. Then, since Δ is arbitrary,

$$\begin{array}{rcll} w_0(\Gamma_0, \Box X) & \le & \bigwedge\{\mathcal{R}(\Gamma_0, \Delta) \Rightarrow w_0(\Delta, X) \mid \Delta \in \mathcal{G}\} & \\ & = & \bigwedge\{\mathcal{R}(\Gamma_0, \Delta) \Rightarrow w(\Delta, X) \mid \Delta \in \mathcal{G}\} & \text{induction hypothesis} \\ & = & w(\Gamma_0, \Box X) & \end{array}$$

Part 2. This is the most complicated part, and will take up the rest of the section. Suppose $w(\Gamma_0, \Box X) \not\le w_0(\Gamma_0, \Box X)$; we derive a contradiction. Let T be the set of all formulas of the forms $w_0(\Gamma_0, \Box Y) \supset Y$ and $Z \supset w_0(\Gamma_0, \Diamond Z)$, and let A be the formula $w(\Gamma_0, \Box X) \supset X$. (Recall that by definition, $w_0(\Gamma_0, \Box Z)$, $w_0(\Gamma_0, \Diamond Z)$, and $w(\Gamma_0, \Box X)$ are members of $\mathcal{T}$, and members of $\mathcal{T}$ are propositional constants of our modal language.)

Claim 1: $\langle \emptyset, T \rangle$ is A-consistent.

The argument for Claim 1 is as follows. Suppose otherwise. Then for some finite sets of formulas $\{Y_1, \ldots, Y_n\}$ and $\{Z_1, \ldots, Z_k\}$ there must be a proof of the sequent:

$$\rightarrow \Big(\bigwedge_{i=1,\ldots,n} (w_0(\Gamma_0, \Box Y_i) \supset Y_i) \wedge \bigwedge_{j=1,\ldots,k} (Z_j \supset w_0(\Gamma_0, \Diamond Z_j)) \Big) \supset \Big(w(\Gamma_0, \Box X) \supset X \Big).$$

Then, using Lemma 7.5, the following sequent would also be provable:

$$\rightarrow \Big(\bigwedge_{i=1,\ldots,n} (w_0(\Gamma_0, \Box Y_i) \supset \Box Y_i) \wedge \bigwedge_{j=1,\ldots,k} (\Diamond Z_j \supset w_0(\Gamma_0, \Diamond Z_j)) \Big) \supset \Big(w(\Gamma_0, \Box X) \supset \Box X \Big)$$

and hence the formula on the right of the arrow, being a theorem, must belong to the maximal left consistent set Γ_0. Now, each of $w_0(\Gamma_0, \Box Y_i) \supset \Box Y_i$ and $\Diamond Z_j \supset w_0(\Gamma_0, \Diamond Z_j)$ must be in Γ_0, by definition of the mapping w_0; hence $w(\Gamma_0, \Box X) \supset \Box X$ is also in Γ_0. But again, by definition of w_0, $\Box X \supset w_0(\Gamma_0, \Box X)$ is in Γ_0, and it follows that $w(\Gamma_0, \Box X) \supset w_0(\Gamma_0, \Box X)$ is in Γ_0. Since we are assuming $w(\Gamma_0, \Box X) \not\le w_0(\Gamma_0, \Box X)$, $(w(\Gamma_0, \Box X) \Rightarrow w_0(\Gamma_0, \Box X)) \ne true$, and it follows that $(w(\Gamma_0, \Box X) \supset w_0(\Gamma_0, \Box X)) \ne true$. Now the left inconsistency of Γ_0 follows immediately, using the rule $t \rightarrow$, for $t \ne true$. This is impossible since $\Gamma_0 \in \mathcal{G}$, and the contradiction establishes Claim 1.

Now extend $\emptyset$ to a set Δ_0 such that $\langle \Delta_0, T \rangle$ is maximally A-consistent. Of course, $\Delta_0 \in \mathcal{G}$.

Claim 2: $\bigwedge w_0(\Delta_0, T) \not\leq w_0(\Delta_0, A)$, where $w_0(\Delta_0, T) = \{w_0(\Delta_0, Z) \mid Z \in T\}$.

Claim 2 is established as follows. Since $\mathcal{T}$ is finite, $w_0(\Delta_0, T) = w_0(\Delta_0, T_0)$ for some finite $T_0 \subseteq T$, and so the set of formulas can be replaced by a single conjunction: $\bigwedge w_0(\Delta_0, T) = w_0(\Delta_0, \bigwedge T_0)$. Now, suppose Claim 2 did not hold.

$$\bigwedge w_0(\Delta_0, T) \leq w_0(\Delta_0, A)$$

$$w_0(\Delta_0, \bigwedge T_0) \leq w_0(\Delta_0, A)$$

$$\Big(w_0(\Delta_0, \bigwedge T_0) \Rightarrow w_0(\Delta_0, A)\Big) = \mathit{true}$$

$$w_0(\Delta_0, \bigwedge T_0 \supset A) = \mathit{true}$$

$$\Big((\bigwedge T_0 \supset A) \equiv \mathit{true}\Big) \in \Delta_0$$

And from this it follows that $\langle \Delta_0, T\rangle$ is A-inconsistent. This contradiction establishes Claim 2.

Since a great deal has intervened, we recall that the basic assumption we have made in Part 2 is: $w(\Gamma_0, \Box X) \not\leq w_0(\Gamma_0, \Box X)$, and we want to derive a contradiction. We are assuming the Truth Property holds for formulas simpler than $\Box X$; in particular, for X itself. Finally we have constructed $\langle \Delta_0, T\rangle$, which is maximal A-consistent. Now, a contradiction is arrived at rather quickly, as follows. First

$$\begin{array}{rcll} w(\Gamma_0, \Box X) & = & \bigwedge\{\mathcal{R}(\Gamma_0, \Delta) \Rightarrow w(\Delta, X) \mid \Delta \in \mathcal{G}\} & \\ & \leq & \mathcal{R}(\Gamma_0, \Delta_0) \Rightarrow w(\Delta_0, X) & \\ & = & \mathcal{R}(\Gamma_0, \Delta_0) \Rightarrow w_0(\Delta_0, X) & \text{induction hypothesis.} \end{array}$$

And so

$$\mathcal{R}(\Gamma_0, \Delta_0) \leq w(\Gamma_0, \Box X) \Rightarrow w_0(\Delta_0, X).$$

Next, $w_0(\Delta_0, A) = w_0(\Delta_0, (w(\Gamma_0, \Box X) \supset X)) = (w(\Gamma_0, \Box X) \Rightarrow w_0(\Delta_0, X))$. Likewise, by definition,

$$\begin{array}{rcl} \mathcal{R}(\Gamma_0, \Delta_0) & = & \bigwedge\{w_0(\Gamma_0, \Box Y) \Rightarrow w_0(\Delta_0, Y) \mid \text{all formulas } Y\} \\ & & \wedge \bigwedge\{w_0(\Delta_0, Z) \Rightarrow w_0(\Gamma_0, \Diamond Z) \mid \text{all formulas } Z\}. \end{array}$$

and so in a similar way, $\mathcal{R}(\Gamma_0, \Delta_0) = \bigwedge w_0(\Delta_0, T)$. Thus we have

$$\bigwedge w_0(\Delta_0, T) \leq w_0(\Delta_0, A)$$

which contradicts Claim 2.

This final contradiction establishes that $w(\Gamma_0, \Box X) \not\leq w_0(\Gamma_0, \Box X)$ is impossible, and concludes the completeness proof. ■

8 Conclusion

The many-valued modal logics considered above are direct analogs of K, the simplest normal modal logic. But analogs of other well-known logics are straightforward to construct as well. For instance, it is easy to verify that the soundness and completeness theorems above specialize to show the axiom schemas $\Box X \supset X$ and $X \supset \Diamond X$ together correspond to the semantic condition $\mathcal{R}(\Gamma, \Gamma) = \mathit{true}$ for all worlds Γ. Thus the classical modal logic T extends to the many-valued case directly. In a similar way the axiom schemas $\Box X \supset \Box\Box X$ and $\Diamond\Diamond X \supset \Diamond X$ correspond to the semantic condition $\mathcal{R}(\Gamma, \Delta) \wedge \mathcal{R}(\Delta, \Omega) \leq \mathcal{R}(\Gamma, \Omega)$, so $K4$ has its analog. Likewise symmetry extends: the semantic condition $\mathcal{R}(\Gamma, \Delta) = \mathcal{R}(\Delta, \Gamma)$ is the counterpart of the axiom schemas $X \supset \Box\Diamond X$ and $\Diamond\Box X \supset X$. Thus we get a many-valued analog of $S5$ by adding all these schemas/conditions.

An examination of the embeddings between many-valued and multiple-expert models, in Section 5, shows that each of the experts will have a transitive Kripke model in mind if and only if the corresponding many-valued model meets the condition $\mathcal{R}(\Gamma, \Delta) \wedge \mathcal{R}(\Delta, \Omega) \leq \mathcal{R}(\Gamma, \Omega)$. Similarly for other conditions. But of course we can consider more general settings — for instance multiple-expert models in which some experts have transitive Kripke models in mind while others have symmetric ones in mind instead. Formulating natural many-valued counterparts of such things seems like an interesting problem, but we will leave it to another time and, perhaps, to another expert. We do not wish to dominate.

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