The Journal of Symbolic Logic
Volume 49, Number 4, Dec. 1984

LINEAR REASONING IN MODAL LOGIC

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§1. Introduction. In [1] Craig introduced a proof procedure for first order classical logic called *linear reasoning*. In it, a proof of $P \supset Q$ consists of a sequence of formulas, each of which implies the next, beginning with P and ending with Q. And one of the formulas in the sequence will be an interpolation formula for $P \supset Q$. Indeed, this was the first proof of the Craig interpolation theorem, some of whose important consequences were demonstrated in a companion paper [2]. In this paper we present systems of linear reasoning for several standard modal logics: K, T, $K4$, $S4$, D, $D4$, and GL. Similar systems can be constructed for several regular, nonnormal modal logics too, though we do not do so here. And just as in the classical case, interpolation theorems are easy consequences. Such theorems are well known for the logics considered here. There is a model theoretic argument in [6], an argument using Gentzen systems in [8], an argument using consistency properties in [4] and [5], and an argument using symmetric Gentzen systems in [5]. This paper presents what seems to be the first modal proof that follows Craig's original methods. We note that if the modal rules given here are dropped, a classical linear reasoning system results, which is related to, but not the same as those in [1] and [10].

Since the basic linear reasoning ideas are fully illustrated by the propositional case, we present that first, to keep the clutter down. Later we show how the techniques can generally be extended to encompass quantifiers. We do not follow [1] in making heavy use of prenex form, since it is not generally available in modal logics. Fortunately, it plays no essential role.

We are not able to deal with all modal logics using linear reasoning techniques. The systems presented below are closely related to semantic tableau systems as presented in [5], so it should not be surprising that we provide linear reasoning systems for the same logics that have cut-free tableau systems. Thus we give no systems for propositional B or $S5$, for instance. Since linear proof procedures supply us with interpolants, we really should not expect things to be nice for those logics not having the interpolation property once quantifiers are added (see [3]).

For a logic L that we consider here, we supply a proof procedure having the following characteristics. A proof of $P \supset Q$ consists of a sequence of formulas: $P_1, \ldots, P_n, R, Q_1, \ldots, Q_k$, where

Received July 26, 1983; revised October 12, 1983.

0022-4812/84/4904-0029/$02.60

1) $P = P_1$ and $Q = Q_k$,
2) each of $P_1, \ldots, P_n$ are *equivalent* in L,
3) each of $Q_1, \ldots, Q_k$ are *equivalent* in L,
4) $P_n \supset R$ is valid in L, and R follows from P_n syntactically by simplification of conjunctions,
5) $R \supset Q_1$ is valid in L, and Q_1 follows from R syntactically by elaboration of disjunctions, and
6) R is an interpolant for $P \supset Q$, that is, in the propositional case the propositional variables of R are common to P and Q, in the first order case the predicate symbols and constants of R are common to P and Q, and $P \supset R$ and $R \supset Q$ are both valid in L.

§2. Propositional background and notation. Although one can make do with a minimum set of propositional connectives, we prefer a rich set. Consequently, we assume the following basic syntax. *Atomic* formulas are the *propositional variables* $A_1, A_2, \ldots$, as well as $\perp$ (falsehood constant) and $\top$ (truth-hood constant). *Formulas* are then built up in the usual way using $\wedge$ (and), $\vee$ (or), $\supset$ (implication), $\sim$ (negation), $\Box$ (necessity) and $\Diamond$ (possibility).

Such a generous syntax requires a large number of rules of derivation in the linear reasoning system. These rules fall into a relatively small number of families, which can be presented succinctly using Smullyan's device of *uniform notation* [9], [10], which we have extended to the modal case [4], [5]. Nonatomic formulas are grouped into four classes: conjunctions (α-formulas), disjunctions (β-formulas), necessitations (ν-formulas) and possibles (π-formulas). For each type, one or two *components* are specified; thus an α-formula has two components, α_1 and α_2, while a ν-formula has one component, ν_0. The classes, and the corresponding components, are specified in the following tables.

α	α_1	α_2
$X \wedge Y$	X	Y
$\sim(X \vee Y)$	$\sim X$	$\sim Y$
$\sim(X \supset Y)$	X	$\sim Y$
$\sim\sim X$	X	X

β	β_1	β_2
$X \vee Y$	X	Y
$\sim(X \wedge Y)$	$\sim X$	$\sim Y$
$X \supset Y$	$\sim X$	Y

ν	ν_0
$\Box X$	X
$\sim\Diamond X$	$\sim X$

π	π_0
$\Diamond X$	X
$\sim\Box X$	$\sim X$

Note that in classical logic, for any α-formula, α is equivalent to $\alpha_1 \wedge \alpha_2$, and for any β-formula, β is equivalent to $\beta_1 \vee \beta_2$. Likewise, in the usual modal logics, for any ν-formula, ν is equivalent to $\Box\nu_0$, and for any π-formula, π is equivalent to $\Diamond\pi_0$.

Although it is not strictly essential, it is very convenient to have generalized conjunctions and disjunctions available. So, we extend the definition of formula by adding the following clause to the usual definition:

If $X_1, X_2, \ldots, X_n$ is a finite sequence of formulas, then $\bigwedge[X_1, X_2, \ldots, X_n]$ and $\bigvee[X_1, X_2, \ldots, X_n]$ are formulas.

We may think of $\bigwedge[X_1, X_2, \ldots, X_n]$ as the conjunction of $X_1, X_2, \ldots, X_n$, parenthesized according to some fixed convention. Similarly for $\bigvee[X_1, X_2, \ldots, X_n]$. The details need not concern us.

In order to minimize elementary syntactic manipulations we will be assuming things like associativity and commutativity of $\bigwedge$ and $\bigvee$ as a matter of course. Thus, if $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_k$ are two sequences of formulas such that $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_k\}$ are equal *sets*, then we take $\bigwedge[X_1, X_2, \ldots, X_n]$ and $\bigwedge[Y_1, Y_2, \ldots, Y_k]$ as interchangable, and similarly for $\bigvee$. Also, to cover the extreme cases, we will take $\bigwedge[X] \equiv \bigvee[X] \equiv X$, and $\bigwedge[\,] \equiv \top$ and $\bigvee[\,] \equiv \bot$.

We will need the notion of a *positive* occurrence of subformula. Informally the idea is this. An occurrence of a subformula Y in X is a positive one if, when X is rewritten in standard ways to translate away instances of $\supset$, then the (rewritten) occurrence of Y will be within the scope of an even number of $\sim$ instances. A more formal characterization follows.

1) The only occurrence of X in the formula X is positive.

2) If X has a positive occurrence in α_1 or α_2 the corresponding occurrence of X in α is positive.

3) If X has a positive occurrence in β_1 or β_2 the corresponding occurrence of X in β is positive.

4) If X has a positive occurrence in ν_0 the corresponding occurrence of X in ν is positive.

5) If X has a positive occurrence in π_0 the corresponding occurrence of X in π is positive.

6) If X has a positive occurrence in Y_i the corresponding occurrence of X in $\bigwedge[Y_1, \ldots, Y_i, \ldots, Y_n]$ is positive; similarly for $\bigvee[Y_1, \ldots, Y_i, \ldots Y_n]$.

Our linear reasoning systems are correct and complete with respect to appropriate Kripke models. We do not introduce a notation for them, since we will not be needing it in this paper. In [5] the reader can find a presentation of Kripke models that makes use of the uniform notational scheme given above.

The logics we will be considering here are quite standard and can be quickly characterized in terms of the accessibility relation of the corresponding Kripke model theory, as follows. (We use *idealization* to mean: for any world, there is some world accessible from it.)

logic	accessibility relation
K	no special conditions
T	reflexivity
$K4$	transitivity
$S4$	reflexivity and transitivity
D	idealization
$D4$	idealization and transitivity
GL	transitivity, irreflexivity and a finite model.

§3. Linear reasoning for propositional K**.** In this section we present the rules of the system for propositional K, the weakest normal modal logic, and we give examples of proofs.

A linear reasoning derivation has four phases:
Phase I: expansion.
Phase II: conjunction simplification.
Phase III: disjunction elaboration.
Phase IV: contraction.
All Phase I rule applications must precede any Phase II rule applications, and so on. In a proof that $P \supset Q$, an interpolant will be produced as the last formula of Phase II.

General Rule. The following rule may be applied in *any* phase of a derivation. We state it first so that the formulation of the other rules can be simplified.

Rule S. If $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_k$ are two sequences of formulas, and $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_k\}$ are the same sets, then any occurrence of $\bigwedge[X_1, X_2, \ldots, X_n]$ can be replaced by an occurrence of $\bigwedge[Y_1, Y_2, \ldots, Y_k]$, and any occurrence of $\bigvee[X_1, X_2, \ldots, X_n]$ can be replaced by an occurrence of $\bigvee[Y_1, Y_2, \ldots, Y_k]$.

Notation and conventions. In the statement of the rules below, X is any single formula, and S, S_1, S_2 are finite sequences of formulas. We write S, X for the sequence consisting of the terms of S, followed by X; and similarly for S_1, S_2, and other combinations. Because of Rule S, $\bigwedge[S_1, X, S_2]$ can be replaced in a derivation by $\bigwedge[S_1, S_2, X]$, so the rules below generally assume the formulas we are concerned with occur at the ends of sequences. In giving examples of proofs we will often tacitly assume Rule S, and will not generally state when it is being used. Finally, we use the following special notation.

$S\#$ is the sequence of ν_0-formulas corresponding to ν-formulas in the sequence S.
$S\flat$ is the sequence of π_0-formulas corresponding to π-formulas in the sequence S.
For example, suppose S is the sequence $A, \Box B, \sim\Box C, \Diamond D, \sim\Diamond E$, where A, B, C, D and E are atomic. Then $S\#$ is the sequence $B, \sim E$ and $S\flat$ is the sequence $\sim C, D$.

Now we give the primary rules of derivation.

Phase I.

I-$\bigvee[\ ]$ $$\frac{X}{\bigvee[X]}$$

I-$\bigwedge[\ ]$ $$\frac{X}{\bigwedge[X]}$$

I-$\bot$ $$\frac{\sim\top}{\bot}$$

I-$\bigwedge\top$ $$\frac{\bigwedge[S]}{\bigwedge[S, \top]}$$

I-C $$\frac{\bigwedge[S, X, \sim X]}{\bot}$$

I-α $$\frac{\bigwedge[S, \alpha]}{\bigwedge[S, \alpha_1, \alpha_2]}$$

I-β $$\frac{\bigwedge[S, \beta]}{\bigvee[\bigwedge[S, \beta_1], \bigwedge[S, \beta_2]]}$$

I–ν $$\frac{\bigwedge[S]}{\bigwedge[S, \Box\bigwedge[S\sharp]]}$$

I–π $$\frac{\bigwedge[S, \pi]}{\bigwedge[S, \Diamond \bigwedge [S\sharp, \pi_0]]}$$

Phase II.

II $$\frac{\bigwedge[S, X]}{\bigwedge[S]}$$

provided $\bigwedge[S, X]$ occurs positively.

Phase III.

III $$\frac{\bigvee[S]}{\bigvee[S, X]}$$

provided $\bigvee[S]$ occurs positively.

Phase IV.

IV–$\bigwedge[\ ]$ $$\frac{\bigwedge[X]}{X}$$

IV–$\bigvee[\ ]$ $$\frac{\bigvee[X]}{X}$$

IV–$\top$ $$\frac{\top}{\sim\bot}$$

IV–$\bigvee\bot$ $$\frac{\bigvee[S, \bot]}{\bigvee[S]}$$

IV–D $$\frac{\top}{\bigvee[S, X, \sim X]}$$

IV–β $$\frac{\bigvee[S, \beta_1, \beta_2]}{\bigvee[S, \beta]}$$

IV–α $$\frac{\bigwedge[\bigvee[S, \alpha_1], \bigvee[S, \alpha_2]]}{\bigvee[S, \alpha]}$$

IV–π $$\frac{\bigvee[S, \Diamond\bigvee[S\flat]]}{\bigvee[S]}$$

IV–ν $$\frac{\bigvee[S, \Box\bigvee[S\flat, \nu_0]]}{\bigvee[S, \nu]}$$

Say Q/R is an instance of one of the rules given above. We say a formula Y *follows from* a formula X by this rule instance if Y is the result of replacing some occurrence of Q in X by an occurrence of R. In the case of Phase II and Phase III rules, the occurrence of Q in X must be a *positive* one.

A *derivation* is a sequence of formulas in which each term (except the first) follows from the preceding term by one of the rules above, and in which all Phase I rule applications precede all Phase II rule applications, which precede all Phase III rule applications, which precede all Phase IV rule applications.

A *proof* of $P \supset Q$ is a derivation of Q from P. Strictly speaking, only implications can be proved in this system, though we can agree to say we have a proof of X if $\top \supset X$ has a proof.

EXAMPLES. $\bigwedge[\;]$ and $\top$ can each be derived from the other very simply, as follows.

$\top$	
$\bigwedge[\top]$	I–$\bigwedge[\;]$
$\bigwedge[\;]$	II

and

$\bigwedge[\;]$	
$\bigwedge[\top]$	I–$\bigwedge\top$
$\top$	IV–$\bigwedge[\;]$

Similarly $\bigvee[\;]$ and $\bot$ are interderivable, as are $\bigwedge[X]$, $\bigvee[X]$ and X.

The following is a proof of $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$; that is, a derivation of $\Box P \supset \Box Q$ from $\Box(P \supset Q)$.

$\Box(P \supset Q)$	
$\bigwedge[\Box(P \supset Q)]$	I–$\bigwedge[\;]$
$\bigvee[\bigwedge[\Box(P \supset Q)]]$	I–$\bigvee[\;]$
$\bigvee[\bigwedge[\Box(P \supset Q), \Box\bigwedge[P \supset Q]]]$	I–ν
$\bigvee[\bigwedge[\Box(P \supset Q), \Box\bigvee[\bigwedge[\sim P], \bigwedge[Q]]]]$	I–β
$\bigvee[\bigwedge[\Box\bigvee[\bigwedge[\sim P], \bigwedge[Q]]]]$	II
$\bigvee[\sim\Box P, \bigwedge[\Box\bigvee[\bigwedge[\sim P], \bigwedge[Q]]]]$	III
$\bigvee[\sim\Box P, \bigwedge[\Box\bigvee[\sim P, \bigwedge[Q]]]]$	IV–$\bigwedge[\;]$
$\bigvee[\sim\Box P, \bigwedge[\Box\bigvee[\sim P, Q]]]$	IV–$\bigwedge[\;]$
$\bigvee[\sim\Box P, \Box\bigvee[\sim P, Q]]$	IV–$\bigwedge[\;]$
$\bigvee[\sim\Box P, \Box Q]$	IV–ν
$\bigvee[\Box P \supset \Box Q]$	IV–β
$\Box P \supset \Box Q$	IV–$\bigvee[\;]$

Finally, here is a proof of $(\Box P \wedge \Diamond Q) \supset \Diamond(Q \vee R)$.

$\Box P \wedge \Diamond Q$	
$\bigwedge[\Box P \wedge \Diamond Q]$	I–$\bigwedge[\;]$
$\bigvee[\bigwedge[\Box P \wedge \Diamond Q]]$	I–$\bigvee[\;]$
$\bigvee[\bigwedge[\Box P, \Diamond Q]]$	I–α
$\bigvee[\bigwedge[\Box P, \Diamond\bigwedge[P, Q]]]$	I–π
$\bigvee[\bigwedge[\Box P, \Diamond\bigvee[\bigwedge[P, Q]]]]$	I–$\bigvee[\;]$

$$\begin{array}{lll}
 & \bigvee[\bigwedge[\diamond\bigvee[\bigwedge[P,Q]]]] & \mathrm{II} \\
(*) & \bigvee[\bigwedge[\diamond\bigvee[\bigwedge[Q]]]] & \mathrm{II} \\
 & \bigvee[\bigwedge[\diamond\bigvee[\bigwedge[Q],R]]] & \mathrm{III} \\
 & \bigvee[\diamond(Q \vee R),\bigwedge[\diamond\bigvee[\bigwedge[Q],R]]] & \mathrm{III} \\
 & \bigvee[\diamond(Q \vee R),\bigwedge[\diamond\bigvee[Q,R]]] & \mathrm{IV}-\bigwedge[\,] \\
 & \bigvee[\diamond(Q \vee R),\diamond\bigvee[Q,R]] & \mathrm{IV}-\bigwedge[\,] \\
 & \bigvee[\diamond(Q \vee R),\diamond\bigvee[Q \vee R]] & \mathrm{IV}-\beta \\
 & \bigvee[\diamond(Q \vee R)] & \mathrm{IV}-\pi \\
 & \diamond(Q \vee R) & \mathrm{IV}-\bigvee[\,]
\end{array}$$

The formula (*) at the end of Phase II is an interpolant. Using the simplifications $\bigwedge[X] \equiv X$ and $\bigvee[X] \equiv X$, the formula in question turns into $\diamond Q$.

REMARKS. Once correctness of this system has been established, it will be immediate that the formula at the end of Phase II of a derivation *must* be an interpolant. For, each line of a derivation will be shown to imply the next; hence any intermediate line is implied by the first, and implies the last. Further, an inspection shows that Phase I rules either do not alter the set of propositional variables present, or else reduce it (in the case of rule I$-C$), while Phase IV rules either leave the set of propositional variables unchanged, or increase it (IV$-D$). And Phase II rules can eliminate some propositional variables but cannot add any, while Phase III rules can introduce some, but cannot remove any. Consequently, any propositional variable present at the end of Phase II of a derivation, before Phase III starts, must be present in all earlier lines, and in all later ones, and hence must be present in the first and last lines of the derivation.

We note that the completeness proof sketched in the next section actually establishes the stronger result that the system remains complete even if *all* rule applications are restricted to the replacement of *positive* subformula occurrences, not just for Phase II and Phase III rules. Also, the formula X in rules I$-C$ and IV$-D$ can be restricted to be *atomic*.

§4. Correctness and completeness. We sketch proofs of the correctness (soundness) and completeness of the system of linear reasoning presented in §3, with respect to Kripke K-models. Correctness depends on the following results.

Semireplacement of implication: Let L be a normal modal logic. Suppose the formula Y results from the replacement of some *positive* occurrences of P in X by occurrences of Q. Then if $P \supset Q$ is L-valid, so is $X \supset Y$.

Replacement of equivalence: Again let L be a normal modal logic. Suppose the formula Y results from the replacement of some (not necessarily positive) occurrences of P in X by Q. Then if $P \equiv Q$ is L-valid, so is $X \equiv Y$. (Here $A \equiv B$ may be thought of as abbreviating $(A \supset B) \wedge (B \supset A)$.)

For a sketch of the proofs in an axiomatic setting see [5, Chapter 4, §4].

It is quite straightforward to check that, for any instance of a Phase I or Phase IV rule, the premise formula (above the line) and the conclusion formula (below the line) are equivalent in K. The same is the case with Rule S. Also trivially, for Phase II and Phase III rule instances, the premise implies the conclusion in K. It then follows from the two results cited above that in any derivation in the system of §3, each line must imply the next in K. Indeed, except where Phase II or Phase III rules are involved, each line will be equivalent to the next.

It follows that if $P \supset Q$ is provable then $P \supset Q$ must be K-valid, so the system is correct.

For showing completeness, rather than constructing a proof from the beginning, we make use of the model existence theorem for K, which we state but do not prove here (see [5, Chapter 2, §5]).

It is most convenient to bring in Smullyan's device of signed formulas [9], [10]. If one does not, the following problem occurs. Suppose, for instance, we have the formula $\sim(X \vee Y)$, and we need to define its "conjugate". Should this be $(X \vee Y)$, or $\sim\sim(X \vee Y)$? Although they are logically equivalent, structurally one is an α-formula while the other is a β-formula. The use of signed formulas avoids this issue, though it does require us to define signed versions of several items already introduced earlier in unsigned form.

Let T and F be two new symbols. By a *signed formula* we mean TX or FX, where X is a formula. Signed nonatomic formulas are grouped into categories, with corresponding notions of components, in the following tables.

α	α_1	α_2
$\mathrm{T}X \wedge Y$	$\mathrm{T}X$	$\mathrm{T}Y$
$\mathrm{F}X \vee Y$	$\mathrm{F}X$	$\mathrm{F}Y$
$\mathrm{F}X \supset Y$	$\mathrm{T}X$	$\mathrm{F}Y$
$\mathrm{F}\sim X$	$\mathrm{T}X$	$\mathrm{T}X$

β	β_1	β_2
$\mathrm{T}X \vee Y$	$\mathrm{T}X$	$\mathrm{T}Y$
$\mathrm{F}X \wedge Y$	$\mathrm{F}X$	$\mathrm{F}Y$
$\mathrm{T}X \supset Y$	$\mathrm{F}X$	$\mathrm{T}Y$
$\mathrm{T}\sim X$	$\mathrm{F}X$	$\mathrm{F}X$

ν	ν_0
$\mathrm{T}\Box X$	$\mathrm{T}X$
$\mathrm{F}\Diamond X$	$\mathrm{F}X$

π	π_0
$\mathrm{T}\Diamond X$	$\mathrm{T}X$
$\mathrm{F}\Diamond X$	$\mathrm{F}X$

If S is a *set* of signed formulas, by $S\#$ we mean $\{\nu_0 \mid \nu \in S\}$. Again, if S is a set of signed formulas, we say S is K-*satisfiable* if there is some Kripke K-model, and some possible world Γ of it such that, at Γ, all the formulas signed with T in S are true, and all the formulas signed with F in S are false.

Let $\mathbf{C}$ be a collection of sets of formulas. $\mathbf{C}$ is a K-*consistency property* if, for each set $S \in \mathbf{C}$:

0) S does not contain both TA and FA for A atomic; S does not contain F $\top$ or T$\bot$;
1) $\alpha \in S \Rightarrow S \cup \{\alpha_1, \alpha_2\} \in \mathbf{C}$;
2) $\beta \in S \Rightarrow S \cup \{\beta_1\} \in \mathbf{C}$ or $S \cup \{\beta_2\} \in \mathbf{C}$; and
3) $\pi \in S \Rightarrow S\# \cup \{\pi_0\} \in \mathbf{C}$.

The *model existence theorem for* K states: if $\mathbf{C}$ is a K-consistency property, then any member of $\mathbf{C}$ is K-satisfiable.

If Z is a *signed* formula, by the *conjugate* of Z, denoted $\bar{Z}$, we mean FX if $Z = \mathrm{T}X$, and TX if $Z = \mathrm{F}X$. For a set S of signed formulas, by $\bar{S}$ we mean $\{\bar{Z} \mid Z \in S\}$. Finally we extend $\bigwedge$ and $\bigvee$ to finite sets of signed formulas in the expected way. Say $S = \{\mathrm{T}X_1, \ldots, \mathrm{T}X_n, \mathrm{F}Y_1, \ldots, \mathrm{F}Y_k\}$. Then $\bigwedge S$ is the formula $\bigwedge[X_1, \ldots, X_n, \sim Y_1, \ldots, \sim Y_k]$, and similarly for $\bigvee S$.

Now we are ready to prove the completeness of the linear reasoning system for K. Let S be a finite set of signed formulas. By a *partition* of S we mean two *disjoint* sets S_1 and S_2 such that $S = S_1 \cup S_2$. Now, let us call a finite set S of signed formulas *consistent* provided, for some partition S_1, S_2 of S, there is no derivation of $\bigvee\overline{S_2}$ from $\bigwedge S_1$. Let $\mathbf{C}$ be the collection of all such consistent sets. We claim $\mathbf{C}$ is a K-consistency property. There are several cases that must be checked to verify this claim. We consider one in detail, involving a β-signed formula.

Suppose neither $S \cup \{\beta_1\}$ nor $S \cup \{\beta_2\}$ is consistent; we show $S \cup \{\beta\}$ is also not consistent. To do this we must consider every partition of $S \cup \{\beta\}$, and this leads to two subcases depending on which part of the partition contains the β-formula.

Subcase 1. $S_1 \cup \{\beta\}, S_2$ is a partition of $S \cup \{\beta\}$. We show there is a derivation of $\bigvee\overline{S_2}$ from $\bigwedge S_1 \cup \{\beta\}$. Note that $S_1 \cup \{\beta_i\}, S_2$ is a partition of $S \cup \{\beta_i\}$, for $i = 1, 2$, so our assumption of inconsistency for $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ says there are derivations of $\bigvee\overline{S_2}$ from each of $\bigwedge S_1 \cup \{\beta_1\}$ and $\bigwedge S_1 \cup \{\beta_2\}$. Then we also have the following derivation.

$\bigwedge S_1 \cup \{\beta\}$	
$\bigvee[\bigwedge S_1 \cup \{\beta_1\}, \bigwedge S_1 \cup \{\beta_2\}]$	I–β
.	using
.	assumed
.	derivations
$\bigvee[\bigvee\overline{S_2}, \bigvee\overline{S_2}]$	
$\bigvee[\bigvee\overline{S_2}]$	Rule S
$\bigvee\overline{S_2}$	IV—$\bigvee[\ ]$

Subcase 2. $S_1, S_2 \cup \{\beta\}$ is a partition. We show there is a derivation of $\bigvee\overline{S_2 \cup \{\beta\}}$ from $\bigwedge S_1$. This is treated similarly to Subcase 1. We note that $\overline{S_2 \cup \{\beta\}} = \overline{S_2} \cup \{\bar{\beta}\}$. Further, the conjugate of a β-formula is always an α-formula, and $\bar{\beta}_i = \bar{\beta}_i (i = 1, 2)$. Then we have the following derivation.

$\bigwedge S_1$	
$\bigwedge[\bigwedge S_1]$	I—$\bigwedge[\ ]$
$\bigwedge[\bigwedge S_1, \bigwedge S_1]$	Rule S
.	using
.	assumed
.	derivations
$\bigwedge[\bigvee S_2 \cup \{\bar{\beta}_1\}, \bigvee S_2 \cup \{\bar{\beta}_2\}]$	
$\bigvee\overline{S_2} \cup \{\bar{\beta}\}$	IV–α

Since the two subcases cover all partitions, $S \cup \{\beta\}$ is not consistent. This concludes the β-case. The remaining cases are similar, and are left to the reader.

Thus $\mathbf{C}$ is a K-consistency property. This gives completeness by the following argument. Suppose $P \supset Q$ is K-valid (true at all worlds of all Kripke K-models). Suppose also that $P \supset Q$ has no proof in our system. Then there is no derivation of Q from P and thus, invoking rules I–$\bigwedge[\ ]$ and IV–$\bigvee[\ ]$, there is no derivation of $\bigvee[Q]$ from $\bigwedge[P]$. Then the set $\{\mathrm{T}P, \mathrm{F}Q\}$ must be consistent because $\{\mathrm{T}P\}, \{\mathrm{F}Q\}$ is a partition of it and there is no derivation of $\bigvee\overline{\{\mathrm{F}Q\}}$ from $\bigwedge\{\mathrm{T}P\}$. By the model existence theorem, $\{\mathrm{T}P, \mathrm{F}Q\}$ must be K-satisfiable, which contradicts the supposed K-validity of $P \supset Q$.

§5. Other modal logics. The linear reasoning system of §3 was for K, the weakest normal modal logic. Simple modifications to it provide us with systems suitable for several other normal propositional modal logics. We describe them briefly.

For $K4$, change the definitions of $\sharp$ and $\flat$ as follows. $S\sharp$ is the sequence consisting of 1) all ν-formulas in the sequence S, together with 2) all ν_0-formulas corresponding to ν-formulas in S. $S\flat$ is the sequence consisting of 1) all π-formulas in S, together with 2) all π_0-formulas corresponding to π-formulas in S. These are the only changes to the K-system that are needed to get $K4$.

For T, add to the K system the following two rules.

I–T
$$\frac{\bigwedge[S, \nu]}{\bigwedge[S, \nu, \nu_0]}$$

IV–T
$$\frac{\bigvee[S, \pi, \pi_0]}{\bigvee[S, \pi]}$$

For $S4$, add the two extra T-rules to the system for $K4$. Alternatively, one can use the T-system, changing the $\sharp$ and $\flat$ definitions as follows. $S\sharp$ is the sequence of ν-formulas in S, and $S\flat$ is the sequence of π-formulas in S.

For D, add to the K-system the following two rules.

I–D
$$\frac{\bigwedge[S]}{\bigwedge[S, \Diamond \bigwedge S\sharp]}$$

IV–D
$$\frac{\bigvee[S, \Box \bigvee S\flat]}{\bigvee[S]}$$

For $D4$, add the two extra D-rules to the system for $K4$.

The correctness of these systems is straightforward, while completeness can be established, as in §4, by using the appropriate model existence theorem [5, Chapter 2, §5].

Finally we mention modifications suitable for GL, the modal logic of provability in Peano arithmetic. For this it is convenient to define an unsigned conjugation, just for ν- and π-formulas, as follows.

ν	$\bar{\nu}$
$\Box X$	$\sim \Box X$
$\sim \Diamond X$	$\Diamond X$

π	$\bar{\pi}$
$\Diamond X$	$\sim \Diamond X$
$\sim \Box X$	$\Box X$

Now, for GL, use the $K4$-system given above, but replace the rules $1-\pi$ and IV$-\nu$ by the following.

I–GL
$$\frac{\bigwedge[S, \pi]}{\bigwedge[S, \Diamond \bigwedge[S\sharp, \bar{\pi}, \pi_0]]}$$

IV–GL
$$\frac{\bigvee[S, \Box \bigvee[S\flat, \bar{\nu}, \nu_0]]}{\bigvee[S, \nu]}$$

Once again, completeness is most easily shown using a model existence theorem [5, Chapter 5, §17].

§6. First order logics. First order version of the logics we have been considering can be specified axiomatically or semantically. Both versions are fairly standard. Axiomatically, one adds the "usual" modal axioms and rules for the particular logic to a "standard" axiomatization of classical first order logic. Semantically one uses Kripke models with a classical first order structure (appropriate for the language) associated with each possible world, subject to the *monotonicity condition*, that any member of the domain of the structure associated with a world is also in the domain of the structure associated with any accessible world. See [5] for proper formulations that make use of the system of uniform notation adopted in the present paper. Correctness and completeness results are well known for K, T, $K4$, $S4$, D and $D4$, and all trace back to [7].

We omitted GL from the list of logics above because we do not know of a good semantic characterization of a first order version of it.

The *Barcan formula* states $(\forall x)\Box A(x) \supset \Box(\forall x)A(x)$. Adding it to an axiomatic formulation of one of the present logics is equivalent to imposing the semantic *constant domain condition*: that the classical structures associated with the various possible worlds all must have the same domains. It is known [3] that such logics do not have the interpolation property, and we do not provide linear reasoning systems for them.

In this section we do provide linear reasoning systems for quantified K, T, $K4$, $S4$, D and $D4$, without the Barcan formula, and we sketch proofs of correctness and completeness. The interpolation theorem for these logics then follows easily.

We note that by adding the quantifier rules of this section to the propositional GL rules given earlier, we do get a first order GL linear reasoning system. We do not know if it is equivalent to an axiomatically formulated first order GL. Without an appropriate model theory available, any equivalence proof would be syntactic in nature. It seems like an interesting problem.

Just as we were generous in our choice of propositional connectives, here we take both $\forall$ (universal) and $\exists$ (existential) quantifiers as basic. We assume formulas are built up in the usual way. We assume an infinite list of constant symbols is available. We refer to constant symbols as *parameters*. We use the term *statement* to mean formula without free variables (though parameters may be present). All lines of proofs will be statements.

Still following [9] and [10] we use uniform notation for quantifiers. The γ-formulas (universals), the δ-formulas (existentials), and their *instances* are specified in the tables below. In stating them we use the following notational convention. $A(x)$ represents a formula with some (possibly none) free occurrences of the variable x, and $A(t)$ represents the result of replacing, in A, all free occurrences of x by occurrences of t (where t is a parameter or a variable).

γ	$\gamma(t)$
$(\forall x)A(x)$	$A(t)$
$\sim(\exists x)A(x)$	$\sim A(t)$

δ	$\delta(t)$
$(\exists x)A(x)$	$A(t)$
$\sim(\forall x)A(x)$	$\sim A(t)$

Generally, not every instance is appropriate. Since t can be a variable here, as well as a parameter as in [10], we have to care about "accidental" quantification.

DEFINITION. Let t be a variable or parameter not occurring in γ. If no occurrence of t in $\gamma(t)$ is within the scope of $(\forall x)$ or $(\exists x)$, we say $\gamma(x)$ is an *allowed instance* of γ. Similarly for allowed instances of δ-formulas.

If $\gamma(x)$ is an allowed instance of a γ-formula, $(\forall x)\gamma(x)$ and γ are equivalent. Similarly, if $\delta(x)$ is an allowed instance of a δ-formula, $(\exists x)\delta(x)$ and δ are equivalent.

Now, the *quantifier rules* are these, where S is any finite sequence of formulas.

Phase I.

I–γ $$\frac{\bigwedge[S, \gamma]}{\bigwedge[S, \gamma, \gamma(t)]}$$

I–δ $$\frac{\bigwedge[S, \delta]}{(\exists x)\bigwedge[S, \delta, \delta(x)]}$$

I–VAC a vacuous quantifier can be added.

Phase IV.

IV–δ $$\frac{\bigvee[S, \delta, \delta(t)]}{\bigvee[S, \delta]}$$

IV–γ $$\frac{(\forall x)\bigvee[S, \gamma, \gamma(t)]}{\bigvee[S, \gamma]}$$

IV–VAC a vacuous quantifier can be dropped.

There are certain restrictions on these rules. First, in I–δ, the $\delta(x)$ must be an allowed instance of δ, and the variable x must not occur free in S or δ. [Free variables are possible because these rules state what replacements are allowed within statements; the "context" in which the replacement takes place supplies the binding quantifiers.] Similarly in IV–γ, $\gamma(x)$ must be an allowed instance of γ, and x must not occur free in S or γ. Second, in I–γ and IV–δ, the rule can only be applied to statements to yield statements. This means the "t" displayed must be either a parameter, or a variable that is within the scope of a quantifier. And further, if t is a variable, the instance must be an allowed one. Finally, in applying the Phase III rule stated in §3, the result must be a statement—that is, any disjuncts added by this rule must be in contexts that quantify their free variables.

We present an example of a derivation using these rules. The particular example chosen does not involve modal features; it was meant to illustrate only the quantifier rules. In it, R is a 2-place relation symbol, and we give a linear proof of $(\exists x)(\forall y)R(x, y) \supset (\forall y)(\exists x)R(x, y)$.

$(\exists x)(\forall y)R(x, y)$	
$\bigwedge[(\exists x)(\forall y)R(x, y)]$	I–$\bigwedge[\ ]$
$(\exists x)\bigwedge[(\exists x)(\forall y)R(x, y), (\forall y)R(x, y)]$	I–δ
$(\forall y)(\exists x)\bigwedge[(\exists x)(\forall y)R(x, y), (\forall y)R(x, y)]$	I–VAC

$(\forall y)(\exists x)\bigwedge[(\exists x)(\forall y)R(x, y), (\forall y)R(x, y), R(x, y)]$	I–γ
$(\forall y)(\exists x)\bigvee[\bigwedge[(\exists x)(\forall y)R(x, y), (\forall y)R(x, y), R(x, y)]]$	I–$\bigvee[\ ]$
$(\forall y)(\exists x)\bigvee[\bigwedge[R(x, y)]]$	II
$(\forall y)(\exists x)\bigvee[(\forall y)(\exists x)R(x, y), (\exists x)R(x, y), \bigwedge[R(x, y)]]$	III
$(\forall y)(\exists x)\bigvee[(\forall y)(\exists x)R(x, y), (\exists x)R(x, y), R(x, y)]$	IV–$\bigwedge[\ ]$
$(\forall y)(\exists x)\bigvee[(\forall y)(\exists x)R(x, y), (\exists x)R(x, y)]$	IV–δ
$(\forall y)\bigvee[(\forall y)(\exists x)R(x, y), (\exists x)R(x, y)]$	IV–VAC
$\bigvee[(\forall y)(\exists x)R(x, y)]$	IV–γ
$(\forall y)(\exists x)R(x, y)$	IV–$\bigvee[\ ]$

The theorem on replacement of equivalences (stated in §4) has a first order analog that says that if the *universal closure* of $P \equiv Q$ is L-valid, so is the *universal closure* of $X \equiv Y$, where Y is the result of replacing some occurrences of P in X by Q (subject to the usual conditions that such replacement turns a statement into a statement, and no free variables get bound that should not be). This result holds for all the logics we are considering (including an axiomatic version of GL). Now, in fact, if P/Q is an instance of any of our six quantifier rules, the universal closure of $P \equiv Q$ is L-valid (indeed, classically valid). Then correctness of the quantified linear reasoning system follows by the same argument as in §4.

To show completeness we can use first order versions of consistency properties, and the model existence theorem. These are available for K, T, $K4$, $S4$, D and $D4$, in their quantified versions. See [5, Chapter 7, §8] for a statement and proof. With this machinery available, completeness of the quantified linear reasoning systems follows by exactly the same argument that worked in §4 in the propositional case.

We note again that we have not established completeness of a quantified GL system—we have no appropriate model existence theorem to use.

Interpolation theorems are again a byproduct, but the argument is a little more complicated than it was in the propositional case. Just as we argued in §3, the statement at the end of Phase II of a derivation will be implied by the first line, and will imply the last. Also any predicate symbols present in such a statement must occur in all earlier and all later lines, and hence in the first and last lines. This is established by the same argument that worked for propositional variables earlier. Thus we immediately have a "weak" interpolation lemma for first order versions of our logics, that takes predicate symbols but not parameters into account. There is a problem with parameters, however. Rule I–γ can introduce a parameter, and rule IV–δ can eliminate one, so it is possible for the statement at the end of Phase II of a derivation to contain parameters not present in the first or last lines. This requires a little extra work.

PROPOSITION. *If $X \supset Y$ has a proof (in one of the first order linear reasoning systems we are considering), then it has a proof in which all parameters involved occur in either X or Y.*

PROOF. Suppose $X \supset Y$ has a proof in which the parameter c appears, while c does not occur in either X or Y. Then c was introduced by either an application of I–γ or in Phase III, and c was eliminated by an application of IV–δ or in Phase II. Say that I–γ and IV–δ are the rules involved; the other two are treated similarly.

Then we have a derivation of Y from X in which in the Phase I part there are two successive lines of the form

$$\cdots \bigwedge[S, \gamma] \cdots$$
$$\cdots \bigwedge[S, \gamma, \gamma(c)] \cdots$$

where c does not occur before these lines, and in the Phase IV part there are two successive lines of the form

$$\cdots \bigvee[S', \delta, \delta(c)] \cdots$$
$$\cdots \bigvee[S', \delta] \cdots$$

where c does not occur after these lines.

Rewrite the derivation as follows. Choose a variable x that was not used in the derivation at all. Replace the first two lines displayed above by the three lines

$$\cdots \bigwedge[S, \gamma] \cdots$$
$$(\forall x) \cdots \bigwedge[S, \gamma] \ldots$$
$$(\forall x) \cdots \bigwedge[S, \gamma, \gamma(x)] \cdots$$

(justified now by I–VAC and I–γ). Replace the last two lines displayed by

$$(\forall x) \cdots \bigvee[S', \delta, \delta(x)] \cdots$$
$$(\forall x) \cdots \bigvee[S', \delta] \cdots$$
$$\cdots \bigvee[S', \delta] \cdots$$

(justified by IV–δ and IV–VAC). And for all intermediate lines, replace every occurrence of c by an occurrence of x, and prefix the quantifier $(\forall x)$.

It is straightforward to check that the resulting sequence of statements is still a derivation, and one in which c does not occur.

INTERPOLATION THEOREM. *Let L be one of the first order logics K, T, K4, S4, D or D4. If $X \supset Y$ is L-valid then there is a statement Z (an interpolant) such that $X \supset Z$ and $Z \supset Y$ are both L-valid, and all predicate symbols and parameters of Z are common to X and Y.*

PROOF. Say X contains the parameters $a_1, \ldots, a_n$ which do not occur in Y, while Y contains the parameters $b_1, \ldots, b_k$ which do not occur in X. We write $X(a_1, \ldots, a_n) \supset Y(b_1, \ldots, b_k)$ to indicate this. Let $x_1, \ldots, x_n, y_1, \ldots, y_k$ be distinct variables that do not occur in X or Y. If $X \supset Y$ is L-valid, so is

$$(\exists x_1) \cdots (\exists x_n) X(x_1, \ldots, x_n) \supset (\forall y_1) \cdots (\forall y_k) Y(y_1, \ldots, y_k).$$

Since this is L-valid, by completeness and the proposition above, it has a linear proof in which all parameters that occur also occur in one of $(\exists x_1) \ldots (\exists x_n) X(x_1, \ldots, x_n)$ or in $(\forall y_1) \ldots (\forall y_k) Y(y_1, \ldots, y_k)$. In this case it means such parameters must occur in both statements. Now it is easy to check that the statement occurring at the end of Phase II in such a proof must be an interpolant for

$$(\exists x_1) \cdots (\exists x_n) X(x_1, \ldots, x_n) \supset (\forall y_1) \cdots (\forall y_k) Y(y_1, \ldots, y_k),$$

and hence for $X(a_1, \ldots, a_n) \supset Y(b_1, \ldots, b_k)$.

§7. Heuristic remarks. We conclude with some remarks and comments. There are two issues we consider: why our linear reasoning rules do not give constant domain models, and what should be the analog of cut elimination in Gentzen systems. There are no proofs in this section, only vague suggestions that may help motivate future work.

The quantifier rules of the previous section are for modal logics whose Kripke models satisfy the monotonicity condition, but need not satisfy the constant domain condition. Where, in the rules themselves, can this distinction be localized? There is an essential difference between rules I$-\gamma$ and I$-\delta$ (and likewise between IV$-\delta$ and IV$-\gamma$, but one pair of rules is enough to talk about). Rule I$-\gamma$ allows replacement of $\bigwedge[S, \gamma]$ by $\bigwedge[S, \gamma, \gamma(x)]$ (where x is a variable that meets specified conditions). In such a replacement, the variable x must be within the scope of a quantifier, but that quantifier could be considerably "further out" in the statement. In particular, $\gamma(x)$ could be within the scope of modal operators that are, themselves, within the scope of the quantifier binding x. On the other hand, rule I$-\delta$ allows the replacement of $\bigwedge[S, \delta]$ by $(\exists x) \bigwedge [S, \delta, \delta(x)]$. Here there can be no modal operators between $\delta(x)$ and its binding quantifier. This difference in rules is the key issue.

Suppose we think of a statement X as a partial description of a Kripke model. Within X a subformula beginning with a modal operator indicates the passage to an alternate world. We can think of quantifiers as ranging over possibly different domains, depending on what modal operators they are within the scope of. Now, the fact that we can replace $\bigwedge[S, \gamma]$ by $\bigwedge[S, \gamma, \gamma(x)]$, where x can be bound by a quantifier "further out" says we can think of this γ as including in its quantifier range things that existed at "earlier" worlds, worlds to which the present one is an alternate. The range of what are essentially universal quantifiers is cumulative: if we go further inside a formula by passing through a modal operator, we still can include in the quantifier range the "things" we had on the outside.

On the other hand, rule I$-\delta$ requires the binding quantifier "instantiating" the δ to be within the same modal operators that δ itself is. That is, if something exists at a certain world, we cannot conclude it existed at any earlier one, something which would correspond to having the binding existential quantifier for $\delta(x)$ outside one or more modal operators covering $\delta(x)$.

Thus the form of the I$-\gamma$ rule suggests that models will obey the monotonicity condition, but the form of the I$-\delta$ rule suggests that they will not be constant domain.

We proved completeness with an appeal to the model existence theorem; the details of a model construction were not seen here. Were one to attempt a direct completeness proof, the ideas suggested above presumably could be used as a guide. The lines of an attempted proof of $P \supset Q$ could be thought of as partial descriptions of a model in the way suggested. If the attempted proof "tries everything", enough material should be generated to construct a counter-model.

The second issue we wish to raise is the possibility of a syntactic completeness argument. That is, it should be possible to give a proof-theoretic argument that establishes the equivalence of one of our linear systems and the corresponding axiomatic version. As usual, the chief difficulty is modus ponens. One needs an analog of cut elimination for linear systems. Specifically, one needs the following.

Suppose we have a linear derivation of Q from P, in the system for logic L, and we also have a linear derivation of R from Q. If we simply write down the steps of the first derivation and follow them by those of the second, we obtain what we might call a *pseudo-derivation* of R from P. Each line still follows from the previous one in a reasonable sense. But it is no longer a proper derivation, because we have lost the feature that all Phase I rule applications must preceed all Phase II rule applications, etc. The completeness proof for L says there must be a proper derivation of R from P. The question is, can a proper derivation of R from P be constructed by purely syntactic means, starting with a pseudo-derivation of R from P? This has obvious similarities with the issue of normalization in natural deduction.

If there is a syntactic means of converting pseudo-derivations into derivations that works for the first order linear GL system, it would amount to a proof that such a system is equivalent to an axiomatically formulated first order GL.

Based on the discussion earlier in this section, a plausible attempt at producing constant domain linear reasoning systems would be to replace the I$-\delta$ and IV$-\gamma$ rules by ones allowing the binding quantifier to be outside the scope of some modal operators. Presumably such a system would not be complete. If a syntactic completeness argument is developed, as suggested in the previous paragraph, it would be interesting to see where it breaks down for the extended versions of I$-\delta$ and IV$-\gamma$.

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