The Logic of Proofs, Semantically

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Abstract

A new semantics is presented for the logic of proofs (**LP**), [1, 2], based on the intuition that it is a logic of explicit knowledge. This semantics is used to give new proofs of several basic results concerning **LP**. In particular, the realization of **S4** into **LP** is established in a way that carefully examines and explicates the role of the + operator. Finally connections are made with the conventional approach, via soundness and completeness results.

1 Introduction

The *Logic of Proofs* (**LP**) bears the same relationship to explicit proofs in formal arithmetic that *Gödel-Löb* logic (**GL**) bears to arithmetic provability. Instead of the single modal operator, $\Box$, of **GL**, intended to represent arithmetic provability, **LP** has many modal-like operators, with formulas of the form $t{:}X$ informally read "t is a proof of X." The operators, like t, are called *proof polynomials*, and basic machinery is provided for constructing more complex proof polynomials from simpler ones. All this was introduced by Artemov, [1, 2], who also established the chief facts about **LP**. Among his results are: **LP** internalizes its own notion of provability; **S4** embeds in **LP** in a natural way; and there is an arithmetic completeness theorem for **LP** analogous to that of Solovay for **GL**. Most fundamentally, **LP** answers a long standing question about the intended provability semantics for the modal logic **S4** and hence for intuitionistic propositional logic.

Given its intended subject matter, it is not surprising that most of the results concerning **LP** have been established proof theoretically. A semantics for **LP** was introduced in [9]. Among other things, it has been used to provide decidability results, and in [10] to establish cut elimination. In this paper I present a different, but related, semantics for **LP** that, I hope, will have a particular intuitive appeal, and be sufficiently flexible to make it applicable to other **LP**-like logics. One can think of it as appropriate for logics of explicit knowledge, with a proof being one kind of explicit knowledge.

I will give two different versions of a semantics for **LP**. The exact relationship between them is unexpectedly complicated, and this will be discussed. Then, using this semantic machinery, I will re-establish most of the basic results concerning **LP**. As to the embedding of **S4** into **LP**, I will examine this in considerable detail and will prove some new results about the role of the + operation of **LP**. Finally, I will show the equivalence of the semantical characterization with the usual axiomatic and Gentzen calculus formulations. The work presented here had its origins in [5, 6].

2 Syntax

Following [2], the language of **LP** is built from the following basic machinery. (Items 1, 4, and 5 may appear with subscripts.)

1. propositional variables, P, Q, ...
2. propositional constant, $\bot$
3. logical connective, $\supset$
4. proof variables, x, y, ...
5. proof constants, c, d, ...
6. function symbols ! (monadic), $\cdot$, + (binary)
7. operator symbol of the type $\langle$*term*$\rangle$:$\langle$*formula*$\rangle$

Proof polynomials, also called terms, are built up from proof variables and proof constants, using the function symbols. *Ground* proof polynomials are those without variables. *Formulas* are built up from propositional variables and propositional constants using $\supset$ (with other connectives defined in the usual way), and the additional rule of formation: if t is a proof polynomial and X is a formula then $t{:}X$ is a formula. It is appropriate to consider sublanguages with + or ! omitted, though $\cdot$ should always be present.

The formula $t{:}X$ should be read: "t is a proof of X." Proof constants intuitively represent proofs of logical truths—the 'givens.' Proof variables in a formula can be thought of as implicitly universally quantified over proofs. The operation $\cdot$ is most fundamental. If t is a proof of $X \supset Y$ and u is a proof of X, we should think of $t \cdot u$ as a proof of Y. The operation ! is a proof-checker: if t is a proof of X, we should think of $!t$ as a verification that t is such a proof. Finally, the operation + combines proofs in the sense that $t + u$ proves all the things that t proves plus all the things that u proves. Of course, this is all quite informal so far.

3 Semantics

In this section I will present two different versions of models for **LP**. Soundness and completeness will be established for both, but the exact relationship between them is complex, and will be investigated starting in the next section. Interestingly enough, much depends on the role of the proof constants.

While **LP** was motivated by the structure of proofs in arithmetic, it can be placed in a more general context—it is a logic of *explicit* knowledge. We might read $t{:}X$ as "I know X for reason t." If we take this point of view, there are two ways $t{:}X$ might be false: first, X is something I don't know; second, while I know X, the reason is not t.

The logic of knowing or not knowing is something that has been extensively investigated, starting with [7]. In the now-standard approach, it is not actual knowledge that is modeled, but potential knowledge. For instance, if I know X and I know $X \supset Y$, I may not know Y because I have not thought about it, but I potentially know Y—it is something I am entitled to know. This use of potential knowledge simplifies things considerably. One can work with a Kripke structure, thinking of possible worlds as states of knowledge, and the accessibility relation as a kind of indistinguishability notion. If Γ is the way things are, and Δ is accessible from Γ, my knowledge is not sufficient for me to recognize that the world is Γ and not Δ. Then, I (potentially) know X at Γ provided X

is the case at all worlds I can't distinguish from Γ. This makes potential knowledge into a modal operator, usually symbolized by K.

What is now added to this is a notion of *possible justification*, or *possible evidence*. To take an intuitive example: what might serve as possible evidence for the statement, "George Bush is editor of The New York Times?" Clearly the editorial page of any copy of The New York Times would serve, while no page of Mad Magazine would do (although the magazine might very well contain the claim that George Bush does edit the Times). Possible evidence need not be evidence of a fact, nor need it be decisive—it could happen that The New York Times decides to omit its editor's name, or prints the wrong one by mistake. Nonetheless, what the Times prints would count as evidence, and what Mad prints would not.

The two semantics presented in this section are combinations of Kripke-style machinery, and a formalized notion of possible evidence. The Kripke part should be thought of as it was in [7], potential knowledge. The possible evidence notion is essentially from [9].

A *frame* is, as usual, a structure $\langle \mathcal{G}, \mathcal{R} \rangle$, where $\mathcal{G}$ is a non-empty set of *states* or *possible worlds*, and $\mathcal{R}$ is a binary relation on $\mathcal{G}$, called *accessibility*.

Next we need a formal version of possible evidence. If t is a proof polynomial and Γ is a state, t can serve as evidence for certain assertions in state Γ and not for other assertions. Consequently we can identify the 'possible evidence' supplied by t at Γ with a set of formulas. Given a frame $\langle \mathcal{G}, \mathcal{R} \rangle$, a *possible evidence* function $\mathcal{E}$ is a mapping from states and proof polynomials to sets of formulas. We can read $X \in \mathcal{E}(\Gamma, t)$ as "X is one of the formulas that t serves as possible evidence for in state Γ." Of course an evidence function must obey conditions that respect the intended meanings of the operations on proof polynomials. Except for monotonicity, the following have their origins in [9].

Definition 3.1 $\mathcal{E}$ is an evidence function on $\langle \mathcal{G}, \mathcal{R} \rangle$ if, for all proof polynomials s and t, for all formulas X and Y, and for all $\Gamma, \Delta \in \mathcal{G}$:

1. **Application** $(X \supset Y) \in \mathcal{E}(\Gamma, s)$ and $X \in \mathcal{E}(\Gamma, t)$ implies $Y \in \mathcal{E}(\Gamma, s \cdot t)$.
2. **Monotonicity** $\Gamma \mathcal{R} \Delta$ implies $\mathcal{E}(\Gamma, t) \subseteq \mathcal{E}(\Delta, t)$.
3. **Proof Checker** $X \in \mathcal{E}(\Gamma, t)$ implies $t{:}X \in \mathcal{E}(\Gamma, !t)$.
4. **Sum** $\mathcal{E}(\Gamma, s) \cup \mathcal{E}(\Gamma, t) \subseteq \mathcal{E}(\Gamma, s + t)$.

As usual in Kripke semantics, truth of atomic formulas at possible worlds is specified arbitrarily—this is part of the definition of a particular model.

Definition 3.2 A structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ is a *weak* **LP** model provided $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame with $\mathcal{R}$ reflexive and transitive, $\mathcal{E}$ is an evidence function on $\langle \mathcal{G}, \mathcal{R} \rangle$, and $\mathcal{V}$ is a mapping from propositional variables to subsets of $\mathcal{G}$.

Definition 3.3 Given a weak model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$, a forcing relation is defined by the following rules. For each $\Gamma \in \mathcal{G}$:

1. $\mathcal{M}, \Gamma \Vdash P$ for a propositional variable P provided $\Gamma \in \mathcal{V}(P)$.
2. $\mathcal{M}, \Gamma \Vdash \bot$ never holds—written $\Gamma \nVdash \bot$.
3. $\mathcal{M}, \Gamma \Vdash (X \supset Y)$ if and only if $\mathcal{M}, \Gamma \nVdash X$ or $\mathcal{M}, \Gamma \Vdash Y$.
4. $\mathcal{M}, \Gamma \Vdash (t{:}X)$ if and only if $X \in \mathcal{E}(\Gamma, t)$ and, for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$, $\mathcal{M}, \Delta \Vdash X$.

We say X is *true at world* Γ if $\mathcal{M}, \Gamma \Vdash X$, and otherwise X is *false at* Γ.

Item 4 above is the key one. It intuitively says that $t{:}X$ is true at a state provided X is potentially known, and t serves as possible evidence for X at that state.

More restrictions will be placed on models, as the use of the term 'weak' suggests. But first, it is good to stop and see what validities we have at this point, that is, what formulas are true at all worlds of all weak models.

Validity Examples It is easy to see that classical tautologies are true at all worlds. A little more work shows that it is also the case for all formulas of the form $t{:}X \supset X$. This follows from the reflexivity of the accessibility relation. Likewise truth at all worlds holds for formulas of the form $t{:}X \supset\, !t{:}t{:}X$, by the following argument. If $\mathcal{M}, \Gamma \Vdash t{:}X$, it must be that $X \in \mathcal{E}(\Gamma, t)$, and hence $t{:}X \in \mathcal{E}(\Gamma, !t)$, by the Proof Checker condition. Also, let Δ and Ω be arbitrary members of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ and $\Delta\mathcal{R}\Omega$. By the Monotonicity condition, $X \in \mathcal{E}(\Delta, t)$. By transitivity of $\mathcal{R}$, $\Gamma\mathcal{R}\Omega$, and so $\mathcal{M}, \Omega \Vdash X$. Since Ω was arbitrary, it follows that $\mathcal{M}, \Delta \Vdash t{:}X$ and since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash\, !t{:}t{:}X$. I'll leave it to you to verify the similar status of $s{:}X \supset (s+t){:}X$ and $t{:}X \supset (s+t){:}X$, and also of $t{:}(X \supset Y) \supset [s{:}X \supset (t \cdot s){:}Y]$.

Next, there is the matter of proof constants, which turns out to be surprisingly central. In [1, 2] proof constants are intended to represent evidence for elementary truths—those truths we know for reasons we do not further analyze. It is allowed that a proof constant serves as evidence for more than one formula, or for nothing at all. In our context, we simply have a mapping assigning to each proof constant those formulas for which it serves as possible evidence, and those formulas should be universally true.

Definition 3.4 A *constant specification* is a mapping $\mathcal{C}$ from proof constants to sets of formulas (possibly empty). A formula X *has a proof constant* with respect to $\mathcal{C}$ if $X \in \mathcal{C}(c)$ for some proof constant c. A proof constant c is *for* a formula X if $X \in \mathcal{C}(c)$. It is required that any formula having a proof constant with respect to $\mathcal{C}$ must be true at every possible world of every weak **LP** model.

For a trivial example of a constant specification, consider the one that maps every constant to the empty set. For less trivial examples, map constants to sets consisting of any of the formulas shown above to be always true. I note that in [1, 2] constant specifications were treated somewhat differently, essentially being created in the process of providing an axiomatic proof. The differences between the two versions are not fundamental.

Definition 3.5 (Constant Condition) $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ *meets the constant specification* $\mathcal{C}$ provided $\mathcal{C}(c) \subseteq \mathcal{E}(\Gamma, c)$, for each $\Gamma \in \mathcal{G}$.

Weak models provide an adequate semantics, as will be shown in Section 8. But there is one more condition I want to impose, a condition that makes models more useful and, I believe, more appealing intuitively. The additional condition might be summarized as "whatever is known, is known for a reason."

Definition 3.6 A weak **LP** model $\mathcal{M}$ is *Fully Explanatory* provided that, whenever $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, then for some proof polynomial t we have $\mathcal{M}, \Gamma \Vdash (t{:}X)$. If $\mathcal{M}$ is a weak **LP** model, and if the Fully Explanatory condition is also met, then $\mathcal{M}$ is a *strong* **LP** *model*.

Finally, some useful terminology. Let $\mathcal{C}$ be a constant specification. A set S of formulas is weakly $\mathcal{C}$-**LP**-*satisfiable* if there is a weak **LP** model $\mathcal{M}$ that meets $\mathcal{C}$, and a possible world Γ of the model at which all members of S are true, that is, $\mathcal{M}, \Gamma \Vdash X$ for all $X \in S$. A formula X is valid in a weak **LP** model $\mathcal{M}$ if $\mathcal{M}, \Gamma \Vdash X$ for every possible world Γ of $\mathcal{M}$. X is weakly $\mathcal{C}$-**LP** valid if X is valid in every weak **LP** model that meets $\mathcal{C}$. Similar terminology is used for the strong semantics. A set S is strongly $\mathcal{C}$-**LP** satisfiable if there is a strong **LP** model that meets $\mathcal{C}$, and a possible world of the model at which all members of S are true, and so on.

In [9], Mkrtychev gives a semantics for **LP** that essentially coincides with one-world weak models. In future work I plan to consider the consequences of adding a $\Box$ operator to **LP**, based on the semantics above, and when this is done, Mkrtyychev models are no longer applicable. But until then, how might we think about this use of one-world models? Let me speak informally for a moment. Suppose t is not a reason for X. How might this happen. Well, first, X itself might not be something we could know, or second, X might be knowable but t does not account for it. This gives us two possible ways of constructing a countermodel to $t{:}X$. We could create a model in which X fails at some alternate world, and so we cannot know it in this one. For this we need the full Kripke machinery to work with. Or we could create a model in which t is not possible evidence for X, and for this we only need to craft an appropriate $\mathcal{E}$ function. The weak semantics provides considerable flexibility. For all this flexibility, there is one limitation, however. A *strong* completeness theorem will be proved, for the semantics of this section. That does not extend to single world models that are fully explanatory. Consider, for example, the following set of formulas: $S = \{P, \neg(t_1{:}P), \neg(t_2{:}P), \ldots\}$, where P is a propositional variable and t_1, t_2, ... is a list of all proof polynomials. This set is satisfiable in an **LP** model. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ where $\mathcal{G} = \{\Gamma, \Delta\}$, $\mathcal{R}$ is reflexive and $\Gamma\mathcal{R}\Delta$, $\Gamma \Vdash P$, $\Delta \not\Vdash P$, and $\mathcal{E}(\Gamma, t) = \mathcal{E}(\Delta, t)$ is the set of all formulas, for every t. Since $\mathcal{M}, \Delta \not\Vdash P$ then $\mathcal{M}, \Gamma \not\Vdash t{:}P$ for every t. On the other hand, if $\mathcal{N}$ is a single world model, the fully explanatory condition would make it impossible for the set S to be satisfiable at the world of the model $\mathcal{N}$.

4 Weak Semantics Basics

In this section I prove some fundamental results about the weak semantics, concerning substitutions and compactness. In [1, 2] substitutions are allowed for both proof variables and propositional variables. I will only need the notion for proof variables, and so restrict the definition to them.

Definition 4.1 A *substitution* is a mapping from a finite set of proof polynomial variables to proof polynomials. The substitution that maps each x_i to t_i, $i = 1, \ldots, k$, is denoted by $\{x_1/t_1, \ldots, x_k/t_k\}$; it has *domain* $\{x_1, \ldots, x_k\}$. σ, with and without subscripts, will generally be used for substitutions. The result of applying the substitution σ to the **LP** formula Z will be denoted by $Z\sigma$. Similarly $t\sigma$ is the result of applying σ to the proof polynomial t.

I will need to mix the notions of substitution, constant specification, and evidence function. The following covers the machinery.

Definition 4.2 Let $\mathcal{C}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ be constant specifications.

1. $\mathcal{C}_1 \leq \mathcal{C}_2$ means $\mathcal{C}_1(c) \subseteq \mathcal{C}_2(c)$ for every constant c.

2. $\mathcal{C}_1 \cup \mathcal{C}_2$ is the constant specification given by $(\mathcal{C}_1 \cup \mathcal{C}_2)(c) = \mathcal{C}_1(c) \cup \mathcal{C}_2(c)$.

3. For a substitution σ, $\mathcal{C}\sigma$ is the constant specification given by $(\mathcal{C}\sigma)(c) = \{Z\sigma \mid Z \in \mathcal{C}(c)\}$. Also, for an evidence function $\mathcal{E}$, a mapping $(\mathcal{E}\overline{\sigma})$ is given by $(\mathcal{E}\overline{\sigma})(\Gamma, s) = \{Z \mid Z\sigma \in \mathcal{E}(\Gamma, s\sigma)\}$.

Before getting to more fundamental results, here is a simple item that will be of use to us later on.

Proposition 4.3 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be constant specifications, and suppose $\mathcal{C}_1 \leq \mathcal{C}_2$.*

1. *If $\mathcal{M} = \langle, \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$ meets $\mathcal{C}_2$ then it meets $\mathcal{C}_1$.*
2. *Any formula that is weakly $\mathcal{C}_1$-**LP** valid is also weakly $\mathcal{C}_2$-**LP** valid.*

Proof For item 1, we have $\mathcal{C}_1(c) \subseteq \mathcal{C}_2(c) \subseteq \mathcal{E}(\Gamma, c)$. For item 2, if X is not weakly $\mathcal{C}_2$-**LP** valid, it is false at a world of some weak **LP** model that meets $\mathcal{C}_2$. This is also a weak **LP** model that meets $\mathcal{C}_1$, and it serves to show X is not $\mathcal{C}_1$-**LP** valid. ■

Now we move to results that are more basic.

Theorem 4.4 (Substitution) *Let σ be a substitution. If the formula X is weakly $\mathcal{C}$-**LP** valid, then $X\sigma$ is weakly $\mathcal{C}\sigma$-**LP** valid.*

Proof Suppose that $X\sigma$ is not weakly $\mathcal{C}\sigma$-**LP** valid; say $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$ is a weak **LP** model that meets $\mathcal{C}\sigma$ and $\mathcal{M}, \Gamma_0 \not\Vdash X\sigma$, for some $\Gamma_0 \in \mathcal{G}$. Create a new structure $\mathcal{M}' = \langle\mathcal{G}, \mathcal{R}, \mathcal{E}\overline{\sigma}, \mathcal{V}\rangle$; note that the set of possible worlds, the accessibility relation, and the valuation are the same as in $\mathcal{M}$.

Let $\Gamma \in \mathcal{G}$ and suppose $Z \in (\mathcal{E}\overline{\sigma})(\Gamma, s)$. Then $Z\sigma \in \mathcal{E}(\Gamma, s\sigma)$ (by Definition 4.2) so $Z\sigma \in \mathcal{E}(\Gamma, s\sigma + t\sigma)$, since $\mathcal{E}$ is an evidence function. But $s\sigma + t\sigma = (s + t)\sigma$ so $Z\sigma \in \mathcal{E}(\Gamma, (s + t)\sigma)$, and it follows that $Z \in (\mathcal{E}\overline{\sigma})(\Gamma, s + t)$. We have verified that $(\mathcal{E}\overline{\sigma})$ satisfies half of the Sum condition of Definition 3.1. The other parts are similar—$\mathcal{E}\overline{\sigma}$ is an evidence function.

Suppose $Z \in \mathcal{C}(c)$. Then $Z\sigma \in (\mathcal{C}\sigma)(c)$ and, since $\mathcal{M}$ is a weak model with respect to $\mathcal{C}\sigma$, we have $Z\sigma \in \mathcal{E}(\Gamma, c)$ for every $\Gamma \in \mathcal{G}$. Since constants are unchanged under substitution, this is equivalent to $Z\sigma \in \mathcal{E}(\Gamma, c\sigma)$, but then $Z \in (\mathcal{E}\overline{\sigma})(\Gamma, c)$. So $\mathcal{M}'$ meets the constant specification $\mathcal{C}$.

A straightforward induction on degree now shows that, for every **LP** formula Z and every $\Gamma \in \mathcal{G}$,

$$\mathcal{M}, \Gamma \Vdash Z\sigma \Leftrightarrow \mathcal{M}', \Gamma \Vdash Z$$

From this it follows that X is not weakly $\mathcal{C}$-**LP** valid, since $\mathcal{M}', \Gamma_0 \not\Vdash X$. ■

Theorem 4.5 (Compactness) *The weak semantics obeys a compactness condition. More precisely, let $\mathcal{C}$ be any constant specification; then a set of formulas is weakly $\mathcal{C}$-**LP** satisfiable if and only if every finite subset is weakly $\mathcal{C}$-**LP** satisfiable.*

Proof Let $\mathcal{C}$ be a constant specification that is fixed for the argument. I'll say an infinite set of formulas is *$\mathcal{C}$-finitely satisfiable* if every finite subset is weakly $\mathcal{C}$-**LP** satisfiable. What we must show is that every $\mathcal{C}$-finitely satisfiable set is weakly $\mathcal{C}$-**LP** satisfiable.

Being $\mathcal{C}$-finitely satisfiable is a property of finite character—a set has this property if and only if every finite subset has it. Tukey's Lemma then says every $\mathcal{C}$-finitely satisfiable set can be extended to a maximal such set (indeed, a Lindenbaum-like construction can be used to show this). If S is maximally $\mathcal{C}$-finitely satisfiable, then for any formula X, exactly one of X or $\neg X$ must be in S. Certainly we can't have both, because then $\{X, \neg X\}$ would be a finite subset of S and it is not satisfied at any world of any weak model. If neither X nor $\neg X$ were a member of S, then by maximality both $S \cup \{X\}$ and $S \cup \{\neg X\}$ would not be $\mathcal{C}$-finitely satisfiable. Then there would be finite subsets S_1 and S_2 of S such that $S_1 \cup \{X\}$ and $S_2 \cup \{\neg X\}$ are not weakly $\mathcal{C}$ satisfiable. But $S_1 \cup S_2$ is a finite subset of S and so it must be weakly $\mathcal{C}$ satisfiable. Any possible world of any weak model that satisfies $S_1 \cup S_2$ must make either X or $\neg X$ true, and hence must satisfy one of $S_1 \cup \{X\}$ or $S_2 \cup \{\neg X\}$, and this is a contradiction. It then follows that every maximally $\mathcal{C}$-finitely satisfiable set must contain all formulas that are weakly $\mathcal{C}$-**LP** valid, and must be closed under *modus ponens*.

Let $\mathcal{G}$ be the collection of all sets of formulas that are maximally $\mathcal{C}$-finitely satisfiable. For $\Gamma \in \mathcal{G}$, let $\Gamma^{\#} = \{X \mid t{:}X \in \Gamma, \text{ for some } t\}$. And set $\Gamma\mathcal{R}\Delta$ if $\Gamma^{\#} \subseteq \Delta$. The relation $\mathcal{R}$ is reflexive, because $t{:}X \supset X$ is weakly $\mathcal{C}$-**LP** valid, and members of $\mathcal{G}$ are closed under modus ponens. It is also transitive, because $t{:}X \supset !t{:}(t{:}X)$ is weakly $\mathcal{C}$-**LP** valid. Thus $\langle \mathcal{G}, \mathcal{R} \rangle$ is an appropriate frame.

For $\mathcal{E}$, simply set $\mathcal{E}(\Gamma, t) = \{X \mid t{:}X \in \Gamma\}$. The claim is that $\mathcal{E}$ satisfies the conditions for being an evidence function, Definition 3.1.

Consider the Proof Checker condition. We saw in the previous section that $t{:}X \supset !t{:}t{:}X$ is weakly $\mathcal{C}$-**LP** valid. It follows that, for $\Gamma \in \mathcal{G}$, this formula is a member of Γ. Now, if $X \in \mathcal{E}(\Gamma, t)$, then $t{:}X \in \Gamma$. Since Γ is closed under *modus ponens*, $!t{:}t{:}X \in \Gamma$ too, and so $t{:}X \in \mathcal{E}(\Gamma, !t)$.

Next, consider the Monotonicity condition. Suppose $\Gamma\mathcal{R}\Delta$ and $X \in \mathcal{E}(\Gamma, t)$. Then $t{:}X \in \Gamma$. By the argument in the previous paragraph, $!t{:}t{:}X \in \Gamma$ and so by the definition of $\mathcal{R}$, $t{:}X \in \Delta$, and thus $X \in \mathcal{E}(\Delta, t)$.

I'll leave Sum and Application to you. We thus have that $\mathcal{E}$ is an evidence function.

Finally, define a mapping $\mathcal{V}$ by: $\Gamma \in \mathcal{V}(P)$ if $P \in \Gamma$. We now have a weak **LP** model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$.

The model $\mathcal{M}$ meets the constant specification $\mathcal{C}$, by the following argument. Suppose $X \in \mathcal{C}(c)$; we must show $X \in \mathcal{E}(\Gamma, c)$ for every $\Gamma \in \mathcal{G}$. Let $\mathcal{N} = \langle \mathcal{G}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}}, \mathcal{E}_{\mathcal{N}}, \mathcal{V}_{\mathcal{N}} \rangle$ be any weak **LP** model that meets $\mathcal{C}$, and let w be any world of it. Of course we have $X \in \mathcal{E}(w, c)$ since $\mathcal{N}$ meets $\mathcal{C}$. But also X must be true in any world accessible from w, since only universally true formulas can have constants. It follows that $\mathcal{N}, w \Vdash c{:}X$. That is, $c{:}X$ itself is weakly $\mathcal{C}$-**LP** valid. Then $c{:}X$ must be in every set that is maximally $\mathcal{C}$-finitely satisfiable. Thus, for every $\Gamma \in \mathcal{G}$, $X \in \mathcal{E}(\Gamma, c)$.

The main thing remaining is to show a *truth lemma*: for every X and every $\Gamma \in \mathcal{G}$

$$X \in \Gamma \text{ if and only if } \mathcal{M}, \Gamma \Vdash X. \tag{1}$$

Once (1) is shown, if S is any $\mathcal{C}$-finitely satisfiable set, any maximal extension of it will be a member of $\mathcal{G}$ at which all members of S are true, and the proof is finished.

Most of the argument for the truth lemma is straightforward. I'll just check one case: suppose we have the formula $t{:}Z$, and (1) is known to hold for Z. First, if $t{:}Z \in \Gamma$, then $Z \in \mathcal{E}(\Gamma, t)$ by definition of $\mathcal{E}$, and $Z \in \Delta$ for all Δ with $\Gamma\mathcal{R}\Delta$, by definition of $\mathcal{R}$. By the induction hypothesis, $\mathcal{M}, \Delta \Vdash Z$ for all such Δ, and consequently $\mathcal{M}, \Gamma \Vdash t{:}Z$. The converse is rather dull. Suppose $t{:}Z \notin \Gamma$. Then $Z \notin \mathcal{E}(\Gamma, t)$, and so $\mathcal{M}, \Gamma \nVdash t{:}Z$. ■

In the proof above a specific model is constructed, in which the possible worlds are all the sets that are maximal $\mathcal{C}$-finitely satisfiable. One of the consequences of the argument is that these are the same as the maximal weakly $\mathcal{C}$-**LP** satisfiable sets, so I will drop the 'finitely' terminology. Also, from now on, the weak **LP** model $\mathcal{M}$ constructed above will be called canonical. The canonical model is universal in the sense that a formula X is weakly $\mathcal{C}$-**LP** valid if and only if it is valid in the canonical weak $\mathcal{C}$-**LP** model. Usually canonical models are defined using axiomatic provability. Connections with this version will be made once we reach Section 8.

Definition 4.6 The *canonical weak $\mathcal{C}$-**LP** model* is the model constructed in the proof of Theorem 4.5.

5 Conditions and Consequences

We have two versions of an **LP** semantics, and it would be nice to know relationships between them. It is here that the role of the constant specification comes into its own. One of the basic items Artemov showed about **LP** is that it internalizes its own notion of proof—a version of the necessitation rule of modal logic. For the weak semantical version, this becomes: if X is weakly $\mathcal{C}$-**LP** valid, then so is $t{:}X$ for some proof polynomial t (using Theorem 4.4 t can be taken to be ground, provided we are willing to modify $\mathcal{C}$). But internalization is not generally the case. For instance, if $\mathcal{C}$ is the trivial constant specification that maps every formula to the empty set of proof polynomials, then no formula of the form $t{:}X$ is weakly $\mathcal{C}$-**LP** valid. This is so because, with the empty constant specification, models in which the evidence function $\mathcal{E}$ always maps to the empty set are weak **LP** models that meet $\mathcal{C}$, and in such models $t{:}X$ will not be true at any world. Clearly we need stronger conditions on constant specifications.

Definition 5.1 Let $\mathcal{C}$ be a constant specification.

1. $\mathcal{C}$ entails internalization provided, for every formula X, if X is weakly $\mathcal{C}$-**LP** valid then so is $t{:}X$ for some proof polynomial t.

2. $\mathcal{C}$ entails equivalence of semantics provided, for every set S of formulas, if S is weakly $\mathcal{C}$-**LP** satisfiable then S is strongly $\mathcal{C}$-**LP** satisfiable. (The converse is trivial since strong models are also weak models.)

Now, the main fact concerning these notions.

Theorem 5.2 *A constant specification $\mathcal{C}$ entails internalization if and only if $\mathcal{C}$ entails equivalence of semantics if and only if the canonical weak $\mathcal{C}$-**LP** model is a strong $\mathcal{C}$-**LP** model.*

I'll divide the proof up into several separate results, since this will make referring to them easier later on.

Proposition 5.3 *If $\mathcal{C}$ entails equivalence of semantics then $\mathcal{C}$ entails internalization.*

Proof Suppose $\mathcal{C}$ entails equivalence of semantics. Let X be weakly $\mathcal{C}$-**LP** valid, but suppose $t{:}X$ is not weakly $\mathcal{C}$-**LP** valid, for any proof polynomial t. I'll derive a contradiction.

First, I claim the set $\{\neg t_1{:}X, \neg t_2{:}X, \ldots\}$ is weakly $\mathcal{C}$-**LP** satisfiable, where $\{t_1, t_2, \ldots\}$ is the set of all proof polynomials. The argument is as follows. If the set were not weakly $\mathcal{C}$-**LP** satisfiable, using the compactness of weak models (Theorem 4.5) there would be a finite subset $\{\neg t_{i_1}{:}X, \ldots, \neg t_{i_k}{:}X\}$ that is not weakly $\mathcal{C}$-**LP** satisfiable. It follows that $t_{i_1}{:}X \vee \ldots \vee t_{i_k}{:}X$ is weakly $\mathcal{C}$-**LP** valid. It was observed in Section 3 that formulas of the form $s{:}X \supset (s+t){:}X$, and $t{:}X \supset (s+t){:}X$ are weakly $\mathcal{C}$-**LP** valid. It follows, by an easy argument, that $(t_{i_1} + \ldots + t_{i_k}){:}X$ is weakly $\mathcal{C}$-**LP** valid, but this contradicts the assumption that $t{:}X$ is not weakly $\mathcal{C}$-**LP** valid, for any t.

So the set $\{\neg t_1{:}X, \neg t_2{:}X, \ldots\}$ is weakly $\mathcal{C}$-**LP** satisfiable, and since $\mathcal{C}$ entails equivalence of semantics, it is also strongly $\mathcal{C}$-**LP** satisfiable. Say in the strong **LP** model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$, meeting $\mathcal{C}$, for $\Gamma \in \mathcal{G}$, $\mathcal{M}, \Gamma \not\Vdash t_i{:}X$ for each proof polynomial t_i. But our assumption is that X itself is weakly $\mathcal{C}$-**LP** valid, and so for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ we must have $\mathcal{M}, \Delta \Vdash X$. Then the Fully Explanatory condition says we must have $\mathcal{M}, \Gamma \Vdash t{:}X$ for some proof polynomial t, and this is our contradiction. ∎

Next a result that is of independent interest.

Proposition 5.4 *Let $\mathcal{C}$ be a constant specification that entails internalization, and let S be a maximal weakly $\mathcal{C}$-**LP** satisfiable set. If $u{:}X \notin S$ for every proof polynomial u, then $S^{\#} \cup \{\neg X\}$ is weakly $\mathcal{C}$-**LP** satisfiable.*

Proof Suppose $S^{\#} \cup \{\neg X\}$ is not weakly $\mathcal{C}$-**LP** satisfiable. Then, by Theorem 4.5, for some $Y_1, \ldots, Y_k \in S^{\#}$, the finite set $\{Y_1, \ldots, Y_k, \neg X\}$ is not weakly $\mathcal{C}$-**LP** satisfiable. It follows that $Y_1 \supset (Y_2 \supset \cdots (Y_k \supset X) \cdots)$ is weakly $\mathcal{C}$-**LP** valid. Then, since $\mathcal{C}$ entails internalization, there is a proof polynomial t such that $t{:}(Y_1 \supset (Y_2 \supset \cdots (Y_k \supset X) \cdots))$ is weakly $\mathcal{C}$-**LP** valid. For each $i = 1, \ldots, k$, since $Y_i \in S^{\#}$, there is some proof polynomial s_i such that $s_i{:}Y_i \in S$. As noted in Section 3, all formulas of the form $t{:}(X \supset Y) \supset (s{:}X \supset (t{\cdot}s){:}Y)$ are weakly $\mathcal{C}$-**LP** valid. It follows, by the maximality of S, that $(t \cdot s_1){:}(Y_2 \supset \cdots (Y_k \supset X) \cdots) \in S$. Of course this can be repeated, and so in k steps we conclude $(t \cdot s_1 \cdot \ldots \cdot s_k){:}X \in S$ (with the multiplication associated to the right). But this contradicts the fact that $u{:}X \notin S$ for each u. ∎

If weak canonical models are strong models, equivalence of semantics is immediate. So, here is the final piece of the proof.

Proposition 5.5 *Let $\mathcal{C}$ be a constant specification that entails internalization. Then the canonical weak $\mathcal{C}$-**LP** model is a strong $\mathcal{C}$-**LP** model.*

Proof Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ be the canonical weak $\mathcal{C}$-**LP** model. I'll show it is a strong **LP** model, that is, I'll show $\mathcal{M}$ meets the Fully Explanatory condition. I'll show the condition in its contrapositive form. Suppose $\Gamma \in \mathcal{G}$ and, for each proof polynomial u, $\mathcal{M}, \Gamma \not\Vdash u{:}X$. I'll show there is some $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ such that $\mathcal{M}, \Delta \not\Vdash X$. Since $\mathcal{M}, \Gamma \not\Vdash u{:}X$, by (1), $u{:}X \notin \Gamma$, for every u. By Proposition 5.4, $\Gamma^{\#} \cup \{\neg X\}$ is weakly $\mathcal{C}$-**LP** satisfiable. Extend it to a maximal such set Δ. Then $\Delta \in \mathcal{G}$, $\Gamma \mathcal{R} \Delta$, and by (1) again, $\mathcal{M}, \Delta \not\Vdash X$. ∎

It is time for the final definitions setting up our **LP** semantics.

Definition 5.6 A constant specification is *full* if it entails internalization, or equivalently it entails equivalence of semantics. If $\mathcal{C}$ is full, the canonical weak $\mathcal{C}$-**LP** model will also be referred to as the *canonical strong $\mathcal{C}$-**LP** model.*

Theorem 5.7 *Let $\mathcal{C}$ be a full constant specification. Then*

1. *A formula X is strongly $\mathcal{C}$-**LP** valid if and only if it is weakly $\mathcal{C}$-**LP** valid.*
2. *A set S of formulas is strongly $\mathcal{C}$-**LP** satisfiable provided every finite subset is.*

6 Removing Plus

Artemov's axioms for **LP** will be given officially in Section 8. In fact, his axioms were already shown to be weakly $\mathcal{C}$-**LP** valid in Section 3. In his axioms, if one replaces every proof polynomial with $\Box$ one gets an **S4** validity. A typical example is $t{:}P \supset\, !t{:}t{:}P$; under the replacement just mentioned, it turns into $\Box P \supset \Box\Box P$. Each of Artemov's modal axioms converts to a significant **S4** axiom, except the ones for $+$. One of Artemov's axioms for the $+$ operation is $t{:}P \supset (s + t){:}P$, which converts to $\Box P \supset \Box P$, which is simply a *classical tautology.* This should suggest that, somehow,

+ is in a different category from the other operations on proof polynomials. In this section I begin the investigation of the role of +, by introducing a semantics for **LP** without the operation. As to notation, I will systematically use $\mathbf{LP}^-$ for versions of **LP** in which the + operation plays no special role.

Definition 6.1 (Definition 3.2 without plus) A structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ is a *weak* $\mathbf{LP}^-$ *model* provided it meets the conditions of Definition 3.2, except that the evidence function $\mathcal{E}$ is not required to satisfy the Sum condition of Definition 3.1.

The various formulas whose validity was shown in Section 3 are also valid in weak $\mathbf{LP}^-$ models, except for the two involving +. So, the definition of constant specification must be relaxed a bit when working with $\mathbf{LP}^-$ models.

Definition 6.2 (Definition 3.4 without plus) $\mathcal{C}$ is a *constant specification without plus* provided it meets the conditions of Definition 3.4, except that now any formula having a proof constant with respect to $\mathcal{C}$ is required to be true at every possible world of every weak $\mathbf{LP}^-$ model (and not of every weak **LP** model).

Earlier we showed a compactness property for the weak **LP** semantics. A similar result is obtainable for the weak $\mathbf{LP}^-$ semantics. I omit the proof since it is the same as before, except that we no longer show the evidence function obeys the Sum condition.

Theorem 6.3 (Theorem 4.5 without plus) *Let $\mathcal{C}$ be any constant specification without plus. A set of formulas is weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable if and only if every finite subset is weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable.*

The model constructed in the proof of Theorem 4.5 was called canonical. I will use the same terminology here—the model constructed in the (omitted) proof of Theorem 6.3 is the *canonical weak $\mathcal{C}$-$\mathbf{LP}^-$ model.* Its possible worlds are sets of formulas that are maximally $\mathcal{C}$-$\mathbf{LP}^-$ consistent, there is a truth lemma, and so on.

Next we need an appropriate version of the Fully Explanatory condition, without +.

Definition 6.4 (Definition 3.6 without plus) A weak $\mathbf{LP}^-$ model is *Fully Explanatory without plus* provided that, whenever $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, then for some finite set $\{t_1, \ldots, t_n\}$ of proof polynomials $\mathcal{M}, \Gamma \Vdash (t_1{:}X \vee \ldots \vee t_n{:}X)$. (Disjunction is given its usual definition in terms of implication and falsehood.)

If $\mathcal{M}$ is a weak $\mathbf{LP}^-$ model, and if the Fully Explanatory condition without plus is also met, then $\mathcal{M}$ is a *strong* $\mathbf{LP}^-$ *model.*

In Section 5 connections between internalization and equivalence of semantics were shown. Variants of those results continue to hold without +.

Definition 6.5 (Definition 5.1 without plus) Let $\mathcal{C}$ be a minimally acceptable constant specification without plus.

1. $\mathcal{C}$ entails internalization without plus provided, for every formula X, if X is weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid then so is $t_1{:}X \vee \ldots \vee t_n{:}X$ for some proof polynomials $t_1, \ldots, t_n$.

2. $\mathcal{C}$ entails equivalence of semantics without plus provided, for every set S of formulas, if S is weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable then S is strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable.

Theorem 6.6 (Theorem 5.2 without plus) *Let $\mathcal{C}$ be a constant specification without plus. $\mathcal{C}$ entails internalization without plus if and only if $\mathcal{C}$ entails equivalence of semantics without plus if and only if the canonical weak $\mathcal{C}$-$\mathbf{LP}^-$ model is a strong $\mathcal{C}$-$\mathbf{LP}^-$ model.*

The proof of this has non-trivial differences from that of Theorem 5.2, so I give the various pieces of it here.

Proposition 6.7 (Proposition 5.3 without plus) *Let $\mathcal{C}$ be a constant specification without plus that entails equivalence of semantics without plus. Then $\mathcal{C}$ entails internalization without plus.*

Proof Suppose X is weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid, but $t_1{:}X \vee \ldots \vee t_n{:}X$ is not weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid, for any proof polynomials $t_1, \ldots, t_n$. I'll derive a contradiction.

Let $D_1, D_2, \ldots$ be an enumeration of all formulas of the form $u_1{:}X \vee \ldots \vee u_k{:}X$, for arbitrary proof polynomials $u_1, \ldots, u_k$. I claim the set $\{\neg D_1, \neg D_2, \ldots\}$ is weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. If not, using the compactness of weak models without plus (Proposition 6.1), there would be a finite subset $\{\neg D_{i_1}, \ldots, \neg D_{i_k}\}$ that is not weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable, and so $D_{i_1} \vee \ldots \vee D_{i_k}$ would be weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid. But this is actually a disjunction of formulas of the form $t_i{:}X$, and this contradicts our assumption.

So the set $\{\neg D_1, \neg D_2, \ldots\}$ is weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable, and since $\mathcal{C}$ entails equivalence of semantics without plus, it is also strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable; say in the strong $\mathbf{LP}^-$ model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ meeting $\mathcal{C}$, we have $\mathcal{M}, \Gamma \not\Vdash D_i$, for some $\Gamma \in \mathcal{G}$ and all i. But X is weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid, and so for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ we have $\mathcal{M}, \Delta \Vdash X$. But the Fully Explanatory without plus condition says we must have $\mathcal{M}, \Gamma \Vdash D_i$ for some i, and this is a contradiction. ∎

Proposition 6.8 (Proposition 5.4 without plus) *Let $\mathcal{C}$ be a constant specification without plus that entails internalization without plus, and let S be a maximal weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable set. If $(u_1{:}X \vee \ldots \vee u_n{:}X) \notin S$ for all proof polynomials $u_1, \ldots, u_n$, then $S^\# \cup \{\neg X\}$ is weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable.*

Proof Suppose $S^\# \cup \{\neg X\}$ is not weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. Then by compactness, Theorem 6.3, some finite subset is not weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. Then for some $Y_1, \ldots, Y_k \in S^\#$, the formula $Y_1 \supset (Y_2 \supset \cdots (Y_k \supset X) \cdots)$ is weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid. Since $\mathcal{C}$ entails internalization without plus, there are proof polynomials $t_1, \ldots, t_n$ such that the *disjunction* of the formulas $t_i{:}(Y_1 \supset (Y_2 \supset \cdots (Y_k \supset X) \cdots))$, with $i = 1, \ldots, n$, is weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid. For each $i = 1, \ldots, k$, since $Y_i \in S^\#$, there is some proof polynomial s_i such that $s_i{:}Y_i \in S$. It follows by a straightforward argument that $(t_1 \cdot s_1 \cdot \ldots \cdot s_k){:}X \vee \ldots \vee (t_n \cdot s_1 \cdot \ldots \cdot s_k){:}X \in S$ (with association to the right, say). But this contradicts the hypothesis of the Proposition. ∎

Proposition 6.9 (Proposition 5.5 without plus) *Let $\mathcal{C}$ be a constant specification without plus that entails internalization without plus. Then the canonical weak $\mathcal{C}$-$\mathbf{LP}^-$ model is a strong $\mathcal{C}$-$\mathbf{LP}^-$ model.*

Proof Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ be the canonical weak $\mathcal{C}$-$\mathbf{LP}^-$ model. I'll show $\mathcal{M}$ meets the Fully Explanatory condition without plus and, as might be expected, I'll show the condition in its contrapositive form. Suppose $\Gamma \in \mathcal{G}$ and, for all proof polynomials $u_1, \ldots, u_n$, $\mathcal{M}, \Gamma \not\Vdash u_1{:}X \vee \ldots \vee u_n{:}X$. I'll show there is some $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ such that $\mathcal{M}, \Delta \not\Vdash X$. Since $\mathcal{M}, \Gamma \not\Vdash u_1{:}X \vee \ldots \vee u_n{:}X$, by the version of the truth lemma, (1), for $\mathbf{LP}^-$ weak canonical models, $u_1{:}X \vee \ldots \vee u_n{:}X \notin \Gamma$, for every u. By Proposition 6.8, $\Gamma^\# \cup \{\neg X\}$ is weakly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. Extend it to a maximal such set Δ. Then $\Delta \in \mathcal{G}$, $\Gamma \mathcal{R} \Delta$, and by (1) again, $\mathcal{M}, \Delta \not\Vdash X$. ∎

All this leads to the following obvious definition.

Definition 6.10 (Definition 5.6 without plus) A constant specification is *full without plus* if it entails internalization without plus, or equivalently it entails equivalence of semantics without plus. If $\mathcal{C}$ is full without plus, the canonical weak $\mathcal{C}$-$\mathbf{LP}^-$ model will also be called the *strong canonical* $\mathcal{C}$-$\mathbf{LP}^-$ *model*.

7 Relations with S4

In [1, 2] Artemov showed an important embedding theorem connecting **LP** and **S4**, his Realization Theorem. In this section I give a semantic proof of the result, providing a close analysis of the role of constant specifications. As a bonus, this approach provides significant additional information about the role of the + operator.

Let φ be a monomodal formula, by which I mean it is in the standard language of modal logic, with $\Box$ as the only modal operator and no proof polynomials. *Assume φ is fixed for the rest of this section and subsections*—all results and definitions are relative to it. I will make use of φ and its subformulas, but by subformula I really mean subformula *occurrence*. Strictly speaking, I should be working with a parse tree for φ, but I am attempting to keep terminology as simple as possible. So in the following, when you see 'subformula,' understand 'subformula occurrence.'

In what follows, A is any assignment of a proof polynomial *variable* to each subformula of φ of the form $\Box X$ that is in a negative position. It is assumed that A assigns different variables to different subformulas—this plays a role in the proof of Proposition 7.8. Relative to A, two mappings w_A and v_A are defined. The mapping w_A was implicitly used in Artemov's proof of his Realization Theorem. The two mapping definitions differ only in one case—w_A is appropriate for **LP**, while v_A is appropriate for $\mathbf{LP}^-$.

Definition 7.1 w_A and v_A both assign a set of **LP** formulas to each subformula of φ, as follows.

1. If P is an atomic subformula of φ, $w_A(P) = v_A(P) = \{P\}$ (this includes the case that P is $\bot$).
2. If $X \supset Y$ is a subformula of φ,
   $w_A(X \supset Y) = \{X' \supset Y' \mid X' \in w_A(X) \text{ and } Y' \in w_A(Y)\}$
   $v_A(X \supset Y) = \{X' \supset Y' \mid X' \in v_A(X) \text{ and } Y' \in v_A(Y)\}$.
3. If $\Box X$ is a negative subformula of φ,
   $w_A(\Box X) = \{x{:}X' \mid A(\Box X) = x \text{ and } X' \in w_A(X)\}$
   $v_A(\Box X) = \{x{:}X' \mid A(\Box X) = x \text{ and } X' \in v_A(X)\}$.
4. If $\Box X$ is a positive subformula of φ,
   $w_A(\Box X) = \{t{:}X' \mid X' \in w_A(X) \text{ and } t \text{ is any proof polynomial}\}$
   $v_A(\Box X) = \{t{:}(X_1 \vee \ldots \vee X_n) \mid X_1, \ldots, X_n \in v_A(X) \text{ and } t \text{ is any proof polynomial}\}$.

Now, here are the results that will be proved in this section. I assume that the standard notion of validity with respect to Kripke **S4** semantics is understood.

Theorem 7.2 *If there are $\varphi_1, \ldots, \varphi_n \in v_A(\varphi)$ such that $\varphi_1 \vee \ldots \vee \varphi_n$ is weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid, then φ is a validity of* **S4**. *Similarly if there is some $\varphi' \in w_A(\varphi)$ that is weakly $\mathcal{C}$-**LP** valid, then φ is a validity of* **S4**.

Theorem 7.3 *Let $\mathcal{C}$ be a full constant specification without plus. If φ is a valid formula of* **S4** *then there are $\varphi_1, \ldots, \varphi_n \in v_A(\varphi)$ such that $\varphi_1 \vee \ldots \vee \varphi_n$ is strongly $\mathcal{C}$-$\mathbf{LP}^-$ valid.*

Theorem 7.4 *Let $\mathcal{C}$ be a full constant specification. If φ is a valid formula of* **S4** *then there a substitution σ and some $\varphi' \in w_A(\varphi)$ such that φ' is weakly $\mathcal{D}$-**LP** valid, where $\mathcal{D} = \mathcal{C} \cup \mathcal{C}\sigma$. (This is a version of Corollary 9.5 from [2].)*

Before giving the proof, here is an example. The formula $(\Box P \vee \Box Q) \supset \Box(\Box P \vee \Box Q)$ is valid in **S4**. In [2, Example 5.6] it is shown that the following is weakly $\mathcal{C}$-**LP** valid (actually, it is shown to be axiomatically provable, but soundness and completeness results will be shown)

$$(x{:}P \vee y{:}Q) \supset (a{\cdot}!x + b{\cdot}!y){:}(x{:}P \vee y{:}Q) \tag{2}$$

In this, $(x{:}P \supset (x{:}P \vee x{:}Q)) \in \mathcal{C}(a)$ and $(x{:}P \supset (x{:}P \vee x{:}Q)) \in \mathcal{C}(b)$. Formula (2) is a member of $w_A((\Box P \vee \Box Q) \supset \Box(\Box P \vee \Box Q))$, where $A(\Box P) = x$ and $A(\Box Q) = y$. In a similar way, the following is strongly $\mathcal{C}$-$\mathbf{LP}^-$ valid.

$$[(x{:}P \vee y{:}Q) \supset (a{\cdot}!x){:}(x{:}P \vee y{:}Q)] \vee [(x{:}P \vee y{:}Q) \supset (b{\cdot}!y){:}(x{:}P \vee y{:}Q)] \tag{3}$$

Formula (3) is a disjunction of two members of $v_A((\Box P \vee \Box Q) \supset \Box(\Box P \vee \Box Q))$.

7.1 First Part of Proof

Following [2], if Y is a formula of **LP**, then Y° is the monomodal formula that results by replacing each subformula of the form $t{:}Z$ with $\Box Z$.

Lemma 7.5 *Let X be a subformula of φ, and let $Y \in v_A(X)$. Then $Y^\circ \equiv X$ is a validity of* **S4**.

Proof A straightforward induction on the degree of X. The primary tool needed is the validity of the scheme $(A \vee A) \equiv A$.
■

Lemma 7.6 *If X is weakly $\mathcal{C}$-$\mathbf{LP}^-$ valid, for any constant specification $\mathcal{C}$, then X° is* **S4** *valid.*

Proof Suppose X° is not **S4** valid. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ be an **S4** model, in the usual Kripke sense, with $\mathcal{M}, \Gamma \not\Vdash X^\circ$ for some $\Gamma \in \mathcal{G}$. Define an evidence function $\mathcal{E}$ by setting $\mathcal{E}(\Delta, t)$ to be the set of all **LP** formulas, for every $\Delta \in \mathcal{G}$ and every proof polynomial t. It is easy to check that $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ is an $\mathbf{LP}^-$ model that meets every constant specification. And finally, for every formula Z in the language of **LP**, $\mathcal{M}, \Gamma \Vdash Z^\circ$ if and only if $\mathcal{N}, \Gamma \Vdash Z$, by induction on Z. Thus $\mathcal{N}$ provides an $\mathbf{LP}^-$ counter-model to X. ■

Now the following gives us Theorem 7.2. Actually, I'll only prove half; the other part is similar and somewhat simpler.

Proof By Lemma 7.6, weak $\mathcal{C}$-$\mathbf{LP}^-$ validity of $\varphi_1 \vee \ldots \vee \varphi_n$ implies **S4** validity of $(\varphi_1 \vee \ldots \vee \varphi_n)^\circ$, that is, of $\varphi_1^\circ \vee \ldots \vee \varphi_n^\circ$. By Lemma 7.5, this implies **S4** validity of φ. ■

7.2 Second Part of Proof

For this section, let $\mathcal{C}$ be a fixed constant specification that is full without plus. Then the canonical weak $\mathcal{C}$-$\mathbf{LP}^-$ model is also the canonical strong one, weak $\mathcal{C}$-$\mathbf{LP}^-$ satisfiability and strong $\mathcal{C}$-$\mathbf{LP}^-$ satisfiability coincide, and similarly for validity. I will use the 'strong' terminology, though it really does not matter. It will be shown that if φ is valid in **S4**, then $\varphi_1 \vee \ldots \vee \varphi_n$ is strongly $\mathcal{C}$-$\mathbf{LP}^-$ valid for some $\varphi_1, \ldots, \varphi_n \in v_A(\varphi)$. The proof will actually be of the contrapositive.

Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ be the strong canonical $\mathcal{C}$-$\mathbf{LP}^-$ model. Since $\mathcal{R}$ is reflexive and transitive, we can also think of it as an **S4** model ($\mathcal{E}$ plays no role in this). We will need to think of $\mathcal{M}$ both ways in this section. In order to keep it clear how $\mathcal{M}$ is being used, just for this section I will write $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} X$ when X is in the language of $\mathbf{LP}^-$ and the rules for $\Vdash$ are the $\mathbf{LP}^-$ rules, and I will write $\mathcal{M}, \Gamma \Vdash_{\mathbf{S4}} X$ when X is in the conventional mono-modal language, and the rules for $\Vdash$ are those of **S4**.

If X is a subformula of φ I will write $\neg v_A(X)$ for $\{\neg X' \mid X' \in v_A(X)\}$. Note that $\neg v_A(X)$ is very different in meaning from $v_A(\neg X)$. Also, if S is a *set* of $\mathbf{LP}^-$ formulas, I will write $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} S$ if $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} Z$ for every $Z \in S$. Since $\mathcal{M}$ is canonical, $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} S$ is equivalent to $S \subseteq \Gamma$, by the truth lemma.

Proposition 7.7 *For the strong canonical $\mathcal{C}$-$\mathbf{LP}^-$ model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$, for each $\Gamma \in \mathcal{G}$:*

1. *If ψ is a positive subformula of φ and $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(\psi)$ then $\mathcal{M}, \Gamma \not\Vdash_{\mathbf{S4}} \psi$.*

2. *If ψ is a negative subformula of φ and $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} v_A(\psi)$ then $\mathcal{M}, \Gamma \Vdash_{\mathbf{S4}} \psi$.*

Proof The proof, of course, is by induction on the complexity of ψ. The atomic case is trivial. I will cover the remaining cases in detail.

Positive Implication Suppose ψ is $(X \supset Y)$, ψ is a positive subformula of φ, $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(X \supset Y)$, and the result is known for X and Y. Note that X occurs negatively in φ and Y occurs positively.

Let X' be an arbitrary member of $v_A(X)$, and Y' be an arbitrary member of $v_A(Y)$. Then $\neg(X' \supset Y') \in \neg v_A(X \supset Y)$, so $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} \neg(X' \supset Y')$. It follows that $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} X'$ and $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} \neg Y'$. Since X' and Y' were arbitrary, it follows that $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} v_A(X)$ and $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(Y)$. Then by the induction hypothesis, $\mathcal{M}, \Gamma \Vdash_{\mathbf{S4}} X$ and $\mathcal{M}, \Gamma \not\Vdash_{\mathbf{S4}} Y$, and hence $\mathcal{M}, \Gamma \not\Vdash_{\mathbf{S4}} (X \supset Y)$.

Negative Implication Suppose ψ is $(X \supset Y)$, ψ is a negative subformula of φ, $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} v_A(X \supset Y)$, and the result is known for X and Y. In this case, X occurs positively in φ and Y occurs negatively.

If $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(X)$, by the induction hypothesis $\mathcal{M}, \Gamma \not\Vdash_{\mathbf{S4}} X$, and hence $\mathcal{M}, \Gamma \Vdash_{\mathbf{S4}} (X \supset Y)$, and we are done. So now suppose $\mathcal{M}, \Gamma \not\Vdash_{\mathbf{LP}^-} \neg v_A(X)$. Then for some $X' \in v_A(X)$, $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} X'$. Let Y' be an arbitrary member of $v_A(Y)$. Then $(X' \supset Y') \in v_A(X \supset Y)$, hence $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} (X' \supset Y')$. Since $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} X'$, then $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} Y'$. Since Y' was arbitrary, we have that $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} v_A(Y)$, so by the induction hypothesis, $\mathcal{M}, \Gamma \Vdash_{\mathbf{S4}} Y$, and again $\mathcal{M}, \Gamma \Vdash_{\mathbf{S4}} (X \supset Y)$.

Positive Necessity Suppose ψ is $\Box X$, ψ is a positive subformula of φ, $\mathcal{M}, \Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(\Box X)$, and the result is known for X (which also occurs positively in φ).

The key item to show is that $\Gamma^{\#} \cup \neg v_A(X)$ is strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. Then we can extend it to a maximal such set Δ, and since $\mathcal{M}$ is the strong canonical $\mathcal{C}$-$\mathbf{LP}^-$ model we will have

$\Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$ and $\mathcal{M},\Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(X)$. By the induction hypothesis, $\mathcal{M},\Delta \nVdash_{\mathbf{S4}} X$, hence $\mathcal{M},\Gamma \nVdash_{\mathbf{S4}} \Box X$. So I now concentrate on showing this key item.

Suppose $\Gamma^\# \cup \neg v_A(X)$ is not strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. Then by compactness, for some $Y_1,\ldots,Y_k \in \Gamma^\#$ and $X_1,\ldots,X_n \in v_A(X)$, $\{Y_1.\ldots.Y_k, \neg X_1,\ldots,\neg X_n\}$ is not strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. It follows that $\Gamma^\# \cup \{\neg X_1,\ldots,\neg X_n\}$ and so $\Gamma^\# \cup \{\neg(X_1 \vee \ldots \vee X_n)\}$ is not strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. Then by Proposition 6.8, for some $u_1, \ldots, u_m$, the $\mathbf{LP}$ formula

$$u_1{:}(X_1 \vee \ldots \vee X_n) \vee \ldots \vee u_m{:}(X_1 \vee \ldots \vee X_n)$$

is in Γ (recall, strong and weak $\mathcal{C}$-$\mathbf{LP}^-$ satisfiability coincide). By the truth lemma, which holds in canonical models, this formula is true at Γ in $\mathcal{M}$. But $\mathcal{M},\Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(\Box X)$ and so, for each u_i, $\mathcal{M},\Gamma \Vdash_{\mathbf{LP}^-} \neg u_i{:}(X_1 \vee \ldots \vee X_n)$, and we have a contradiction. Thus $\Gamma^\# \cup \neg v_A(X)$ is strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable, and the case is done.

Negative Necessity Suppose ψ is $\Box X$, ψ is a negative subformula of φ, $\mathcal{M},\Gamma \Vdash_{\mathbf{LP}^-} v_A(\Box X)$, and the result is known for X (which also occurs negatively in φ).

Let X' be an arbitrary member of $v_A(X)$. Since $\Box X$ is a negative subformula of φ, $x{:}X' \in v_A(\Box X)$, where $x = A(\Box X)$, and so $\mathcal{M},\Gamma \Vdash_{\mathbf{LP}^-} x{:}X'$. Now if Δ is an arbitrary member of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, we must have $\mathcal{M},\Delta \Vdash_{\mathbf{LP}^-} X'$. Thus $\mathcal{M},\Delta \Vdash_{\mathbf{LP}^-} v_A(X)$, so by the induction hypothesis, $\mathcal{M},\Delta \Vdash_{\mathbf{S4}} X$. Since Δ was arbitrary, $\mathcal{M},\Gamma \Vdash_{\mathbf{S4}} \Box X$.

■

Now Theorem 7.3 can be shown.

Proof Assume $\varphi_1 \vee \ldots \vee \varphi_n$ is not strongly $\mathcal{C}$-$\mathbf{LP}^-$ valid for every $\varphi_1,\ldots,\varphi_n \in v_A(\varphi)$. It follows that $\neg v_A(\varphi)$ is strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable. For otherwise, by compactness, there would be $\varphi_1,\ldots,\varphi_n \in v_A(\varphi)$ such that $\{\neg\varphi_1,\ldots,\neg\varphi_n\}$ was not strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable, and then $\varphi_1 \vee \ldots \vee \varphi_n$ would be strongly $\mathcal{C}$-$\mathbf{LP}^-$ valid, contrary to assumption. Since $\neg v_A(\varphi)$ is strongly $\mathcal{C}$-$\mathbf{LP}^-$ satisfiable, there is some world Γ of the canonical model $\mathcal{M}$ such that $\mathcal{M},\Gamma \Vdash_{\mathbf{LP}^-} \neg v_A(\varphi)$ and so, by the previous Proposition, $\mathcal{M},\Gamma \nVdash_{\mathbf{S4}} \varphi$, hence φ is not $\mathbf{S4}$ valid. ■

7.3 Third Part of Proof

We have Theorem 7.4 to go. For this section $\mathcal{C}$ is a full constant specification (+ included). Note that this implies that strong and weak $\mathcal{C}$-$\mathbf{LP}$ validity coincide, and similarly for satifiability. Recall the notion of substitution from Definition 4.1. If σ_1 and σ_2 are substitutions whose domains do not overlap, then $\sigma_1 \cup \sigma_2$ denotes the obvious combined substitution. Also recall that at the beginning of Section 7 I required the assignment A of variables to negative necessity subformulas of φ to associate distinct variables to different subformulas. Consequently, if X and Y are subformulas of φ with neither a subformula of the other, and $X' \in v_A(X)$ and $Y' \in v_A(Y)$, and $\sigma_{X'}$ and $\sigma_{Y'}$ are substitutions whose domains are the variables of X' and Y' respectively, then $\sigma_{X'}$ and $\sigma_{Y'}$ have non-overlapping domains, and hence $\sigma_{X'} \cup \sigma_{Y'}$ is a well-defined substitution. This plays a role in the proof of the Proposition that follows.

Proposition 7.8 *For every ψ that is a subformula of φ, and for each $\psi_1, \ldots, \psi_n \in v_A(\psi)$, there is a substitution σ and a formula $\psi' \in w_A(\psi)$ such that:*

1. *If ψ is a positive subformula of φ, $(\psi_1 \vee \ldots \vee \psi_n)\sigma \supset \psi'$ is strongly $\mathcal{C}$-$\mathbf{LP}$ valid.*

2. *If ψ is a negative subformula of φ, $\psi' \supset (\psi_1 \wedge \ldots \wedge \psi_n)\sigma$ is strongly $\mathcal{C}$-**LP** valid.*

Proof By induction on the complexity of ψ. If ψ is atomic the result is immediate, since $v_A(\psi)$ and $w_A(\psi)$ are both $\{\psi\}$, so one can take ψ' to be ψ, and use the empty substitution. I'll cover the non-atomic cases in detail.

Positive Implication Suppose ψ is $(X \supset Y)$, ψ is a positive subformula of φ, and the result is known for X and Y. Note that X occurs negatively in φ and Y occurs positively.

For $i = 1, \ldots, n$, say $\psi_i = (X_i \supset Y_i)$, where $X_i \in v_A(X)$ and $Y_i \in v_A(Y)$. By the induction hypothesis there are substitutions σ_X and σ_Y, and there are $X' \in w_A(X)$ and $Y' \in w_A(Y)$ such that $X' \supset (X_1 \wedge \ldots \wedge X_n)\sigma_X$ and $(Y_1 \vee \ldots \vee Y_n)\sigma_Y \supset Y'$ are both strongly $\mathcal{C}$-**LP** valid. Without loss of generality we can assume the domain of σ_X is exactly the set of variables of $X_1 \wedge \ldots \wedge X_n$, and similarly for σ_Y. Then if we let σ be $\sigma_X \cup \sigma_Y$, the following is strongly $\mathcal{C}$-**LP** valid (using only classical logic): $[(X_1 \supset Y_1) \vee \ldots \vee (X_n \supset Y_n)]\sigma \supset (X' \supset Y')$, which establishes this case.

Negative Implication Suppose ψ is $(X \supset Y)$, ψ is a negative subformula of φ, and the result is known for X and Y. Note that X occurs positively in φ and Y occurs negatively.

Again for $i = 1, \ldots, n$ say $\psi_i = (X_i \supset Y_i)$, where $X_i \in v_A(X)$ and $Y_i \in v_A(Y)$. This time, by the induction hypothesis there are substitutions σ_X and σ_Y, and there are $X' \in w_A(X)$ and $Y' \in w_A(Y)$ such that $(X_1 \vee \ldots \vee X_n)\sigma_X \supset X'$ and $Y' \supset (Y_1 \wedge \ldots \wedge Y_n)\sigma_Y$ are strongly $\mathcal{C}$-**LP** valid. As in the previous case, we can assume σ_X and σ_Y have non-overlapping domains. Then again, if we set $\sigma = \sigma_X \cup \sigma_Y$, using classical logic the following is strongly $\mathcal{C}$-**LP** valid: $(X' \supset Y') \supset [(X_1 \supset Y_1) \wedge \ldots \wedge (X_n \supset Y_n)]\sigma$, establishing this case.

Positive Necessity Suppose ψ is $\Box X$, ψ is a positive subformula of φ, and the result is known for X (which also occurs positively in φ).

In this case $\psi_1, \ldots, \psi_n$ are of the form $t_1{:}D_1, \ldots, t_n{:}D_n$, where each t_i is some proof polynomial and D_i is a disjunction of members of $v_A(X)$. Let $D = D_1 \vee \ldots \vee D_n$ be the disjunction of the D_i. D is a disjunction of members of $v_A(X)$, so by the induction hypothesis there is some substitution σ and some member $X' \in w_A(X)$ such that $D\sigma \supset X'$ is strongly $\mathcal{C}$-**LP** valid. Consequently for each i, $D_i\sigma \supset X'$ is strongly $\mathcal{C}$-**LP** valid and so, since $\mathcal{C}$ is full, there is a proof polynomial u_i such that $u_i{:}(D_i\sigma \supset X')$ is strongly $\mathcal{C}$-**LP** valid. But then $(t_i{:}D_i)\sigma \supset (u_i \cdot t_i\sigma){:}X'$ is also strongly $\mathcal{C}$-**LP** valid (this uses the fact that $(t_i{:}D_i)\sigma = (t_i\sigma){:}(D_i\sigma)$. Let s be the proof polynomial $(u_1 \cdot t_1\sigma) + \ldots + (u_n \cdot t_n\sigma)$. For each i we have the strong $\mathcal{C}$-**LP** validity of $(t_i{:}D_i)\sigma \supset s{:}X'$, and hence that of $(t_1{:}D_1 \vee \ldots \vee t_n{:}D_n)\sigma \supset s{:}X'$. Since $s{:}X' \in w_A(\Box X)$, this concludes the positive necessity case.

Negative Necessity Suppose ψ is $\Box X$, ψ is a negative subformula of φ, and the result is known for X (which also occurs negatively in φ).

In this case $\psi_1, \ldots, \psi_n$ are of the form $x{:}X_1, \ldots, x{:}X_n$, where $X_i \in v_A(X)$ and $x = A(\Box X)$. By the induction hypothesis there is some substitution σ and some $X' \in w_A(X)$ such that $X' \supset (X_1 \wedge \ldots \wedge X_n)\sigma$ is strongly $\mathcal{C}$-**LP** valid. Without loss of generality, we can assume x is not in the domain of σ. Now, for each $i = 1, \ldots, n$, the formula $X' \supset X_i\sigma$ is strongly $\mathcal{C}$-**LP** valid, and so there is a proof polynomial t_i such that $t_i{:}(X' \supset X_i\sigma)$ is strongly $\mathcal{C}$-**LP** valid. Let s be the proof polynomial $t_1 + \ldots + t_n$; then $s{:}(X' \supset X_i\sigma)$ is strongly $\mathcal{C}$-**LP** valid, for each i. It follows that $x{:}X' \supset (s \cdot x){:}(X_i\sigma)$ is strongly $\mathcal{C}$-**LP** valid, for each i. If we let σ' be the substitution $\sigma \cup \{x/(s \cdot x)\}$, we have the strong $\mathcal{C}$-**LP** validity of $x{:}X' \supset (x{:}X_i)\sigma'$ for each i, and hence of $x{:}X' \supset (x{:}X_1 \wedge \ldots \wedge x{:}X_n)\sigma'$, which establishes the result in this case.

This concludes the proof. ■

Now Theorem 7.4 follows quickly.

Proof Suppose φ is an **S4** validity. By Theorem 7.3, there are $\varphi_1, \ldots, \varphi_n \in v_A(\varphi)$ such that $\varphi_1 \vee \ldots \vee \varphi_n$ is strongly $\mathcal{C}$-**LP**$^-$ valid, hence also strongly $\mathcal{C}$-**LP** valid, and hence weakly $\mathcal{C}$-**LP** valid, since $\mathcal{C}$ is full. By Proposition 7.8 there is a substitution σ and a formula $\varphi' \in w_A(\varphi)$ such that $(\varphi_1 \vee \ldots \vee \varphi_n)\sigma \supset \varphi'$ is a strong $\mathcal{C}$-**LP** validity, hence also a weak $\mathcal{C}$-**LP** validity. Consequently we have the weak $(\mathcal{C}\sigma)$-**LP** validity of $(\varphi_1 \vee \ldots \vee \varphi_n)\sigma$, by the Substitution Theorem 4.4. But then, for $\mathcal{D} = \mathcal{C} \cup \mathcal{C}\sigma$, we have the weak $\mathcal{D}$-**LP** validity of both $(\varphi_1 \vee \ldots \vee \varphi_n)\sigma$ and $(\varphi_1 \vee \ldots \vee \varphi_n)\sigma \supset \varphi'$, by Proposition 4.3. The $\mathcal{D}$-**LP** validity of φ' follows. ■

8 Axiomatic Soundness and Completeness

The following is a variant of the axiomatic presentation of [1, 2] Axioms are specified by giving axiom schemas, and these are:

A0.	**Classical**	Classical propositional axiom schemes
A1.	**Application**	$t{:}(X \supset Y) \supset (s{:}X \supset (t \cdot s){:}Y)$
A2.	**Reflexivity**	$t{:}X \supset X$
A3.	**Proof Checker**	$t{:}X \supset\, !t{:}(t{:}X)$
A4.	**Sum**	$s{:}X \supset (s+t){:}X$
		$t{:}X \supset (s+t){:}X$

For rules of inference there is, of course, modus ponens. There is also a version of the necessitation rule, for axioms only.

R1.	**Modus Ponens**	$\vdash Y$ provided $\vdash X$ and $\vdash X \supset Y$
R2.	$\mathcal{C}$ **Axiom Necessitation**	$\vdash c{:}X$ where X is an axiom *A0 – A4*
		and $X \in \mathcal{C}(c)$, where $\mathcal{C}$ is a constant specification.

The formula X has an **LP** axiomatic proof using constant specification $\mathcal{C}$ if X is provable using the axioms and rules above, where $\mathcal{C}$ is the constant specification used in *R2*.

Not all constant specifications are reasonable, in this context. The following is the restriction that is appropriate here.

Definition 8.1 A constant specification $\mathcal{C}$ is **LP** *axiomatically appropriate* if it is exactly instances of axiom schemes *A0 – A4* that have proof constants.

There are many axiomatically appropriate constant specifications, since different axiom scheme instances might or might not share the same proof constant, or one instance might or might not have several proof constants. These differences are not major, but are real nonetheless.

The weak validity of a few of the axioms was shown in Section 3, and weak validity of the others is straightforward. It follows that an axiomatically appropriate constant specifications is also a constant specification, according to Definition 3.4. The instances of axiom schemes constitute a family that is closed under substitution, and hence if $\mathcal{C}$ is axiomatically appropriate, $\mathcal{C}\sigma$ only assigns constants to axiom scheme instances, though it might not provide constants for every instance. Of course $\mathcal{C}\sigma$ could be extended to take care of missing schemes—one way of doing this is to combine it with the original $\mathcal{C}$. All this is summarized as follows. (Note the significance of this for Theorem 7.4.)

Significant Fact Let $\mathcal{C}$ be axiomatically appropriate, then

1. $\mathcal{C}$ is a constant specification.
2. For any substitution σ, the constant specification $\mathcal{C} \cup (\mathcal{C}\sigma)$ is axiomaticaly appropriate.

Now the usual methodology for proving axiomatic soundness works fine—show each line of a proof is weakly $\mathcal{C}$-**LP** valid. I omit the details, but state the result.

Theorem 8.2 *If X has an* **LP** *axiomatic proof using constant specification $\mathcal{C}$, where $\mathcal{C}$ is axiomatically appropriate, then X is weakly $\mathcal{C}$-***LP** *valid.*

Completeness proceeds by a canonical model argument, much like that used in Theorem 4.5. Let $\mathcal{C}$ be an axiomatically appropriate constant specification, fixed for the following construction. Axiomatic proofs are assumed to use this constant specification $\mathcal{C}$.

Call a set S of **LP** formulas *inconsistent* if there is some finite subset $\{X_1, \ldots, X_n\} \subseteq S$ such that $(X_1 \wedge \ldots \wedge X_n) \supset \bot$ is a theorem of **LP** (with $\wedge$ defined from $\supset$ and $\bot$ in the usual way). Call S *consistent* if it is not inconsistent. Consistent sets extend to maximal consistent sets, via a standard Lindenbaum Lemma construction. Let $\mathcal{G}$ be the set of all maximally consistent sets of **LP** formulas. If $\Gamma \in \mathcal{G}$, let $\Gamma^{\#} = \{X \mid (t{:}X) \in \Gamma, \text{ for some } t\}$, and set $\Gamma\mathcal{R}\Delta$ if $\Gamma^{\#} \subseteq \Delta$. This gives us a frame, $\langle \mathcal{G}, \mathcal{R} \rangle$. The Reflexivity axiom scheme of **LP** implies the frame is reflexive, and the Proof Checker axiom scheme implies it is transitive. Define a mapping $\mathcal{E}$ by setting $\mathcal{E}(\Gamma, t) = \{X \mid t{:}X \in \Gamma\}$. Finally, define a mapping $\mathcal{V}$ by specifying that for an atomic formula P, $\Gamma \in \mathcal{V}(P)$ if and only if $P \in \Gamma$. This gives us a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$. The claim is that $\mathcal{M}$ is a weak **LP** model that meets $\mathcal{C}$.

First I'll verify two of the conditions of Definition 3.1 for the evidence function $\mathcal{E}$—other conditions are similar. I'll begin with the Application Condition. Suppose we have $X \in \mathcal{E}(\Gamma, t)$ and $(X \supset Y) \in \mathcal{E}(\Gamma, s)$. By the definition of $\mathcal{E}$, we must have $t{:}X \in \Gamma$ and $s{:}(X \supset Y) \in \Gamma$. Since $s{:}(X \supset Y) \supset (t{:}X \supset (s \cdot t){:}Y)$ is an **LP** axiom, and Γ is maximally consistent, it follows that $(s \cdot t){:}Y \in \Gamma$, and hence $Y \in \mathcal{E}(\Gamma, s \cdot t)$.

Next, I'll verify the Monotonicity Condition. Suppose $\Gamma, \Delta \in \mathcal{G}$ and $\Gamma\mathcal{R}\Delta$. Also assume $X \in \mathcal{E}(\Gamma, t)$. By definition of $\mathcal{E}$, we have $t{:}X \in \Gamma$. Since $t{:}X \supset !t{:}t{:}X$ is an **LP** axiom, we have $!t{:}t{:}X \in \Gamma$, and hence $t{:}X \in \Gamma^{\#}$. Since $\Gamma\mathcal{R}\Delta$ we have $\Gamma^{\#} \subseteq \Delta$, so $t{:}X \in \Delta$, and so $X \in \mathcal{E}(\Delta, t)$.

Other conditions are similar. Thus $\mathcal{M}$ is a weak **LP** model that meets $\mathcal{C}$.

Now a Truth Lemma can be shown: for each formula X and each $\Gamma \in \mathcal{G}$

$$X \in \Gamma \Longleftrightarrow \mathcal{M}, \Gamma \Vdash X \tag{4}$$

Most of the cases are familiar. I'll verify only the modal one. Suppose (4) is known for X, and we are considering the formula $t{:}X$.

Suppose first that $t{:}X \in \Gamma$. Then $X \in \Gamma^{\#}$, so if Δ is an arbitrary member of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ we have $\Gamma^{\#} \subseteq \Delta$ and hence $X \in \Delta$. By the induction hypothesis, $\mathcal{M}, \Delta \Vdash X$. Also since $t{:}X \in \Gamma$, we have $X \in \mathcal{E}(\Gamma, t)$. It follows that $\mathcal{M}, \Gamma \Vdash t{:}X$.

Next, suppose $\mathcal{M}, \Gamma \Vdash t{:}X$. This case is trivial. By the general definition of $\Vdash$ we must have $X \in \mathcal{E}(\Gamma, t)$, and by definition of $\mathcal{E}$ for $\mathcal{M}$, we must also have $t{:}X \in \Gamma$.

Thus we have the Truth Lemma. Now, as usual, if X does not have an **LP** axiomatic proof, $\{X \supset \bot\}$ is consistent. Extend it to a maximal consistent set Γ. Then $\Gamma \in \mathcal{G}$ and by (4) $\mathcal{M}, \Gamma \not\Vdash X$. Thus the following has been established.

Theorem 8.3 *Let $\mathcal{C}$ be an axiomatically appropriate proof specification. If X is weakly $\mathcal{C}$-***LP** *valid then X has an* **LP** *axiomatic proof using constant specification $\mathcal{C}$.*

Finally, in [2, Lemma 5.4] it is shown that if X has an **LP** axiomatic proof, using an axiomatically appropriate constant specification $\mathcal{C}$, then so does $t{:}X$ for some proof polynomial t. This means that, via our soundness and completeness results, an axiomatically appropriate proof specification entails internalization. Then Theorem 5.2 immediately gives us the following.

Theorem 8.4 *Let $\mathcal{C}$ be an axiomatically appropriate proof specification. X has an* **LP** *axiomatic proof using $\mathcal{C}$ if and only if X is strongly $\mathcal{C}$-***LP** *valid.*

Similar soundness and completeness results can be established for the Artemov axioms without *A4*, the Sum axioms, relative to the $\mathbf{LP}^-$ semantics. I omit the straightforward details.

9 Tableau and Sequent Systems

In [2] a Gentzen sequent calculus for **LP** is given, and cut elimination is shown, proof theoretically. This Gentzen system plays a role in establishing the connection between **LP** and Peano arithmetic, and cut elimination is needed for this. There is a well-known connection between Gentzen sequent calculi and signed tableau systems—see [11] for details in the classical case. The basic idea is: formulas on the left of a sequent arrow get signed with T, formulas on the right with F, and sequents themselves correspond to stages in the construction of a tableau branch. Thought of this way, tableau proofs and sequent proofs are essentially the same thing, except that one is an upside-down version of the other. In [10] a signed tableau calculus is given for **LP**, corresponding to the sequent calculus of [2], and soundness and completeness is shown relative to the semantics of [9]. As a byproduct, this gives a semantic argument that cut elimination holds for the tableau and the Gentzen system for **LP**. In this section I give a proof, analogous to that of [10], of tableau soundness and completeness relative to the semantics of this paper. Differences between the semantics of [9] and the semantics given here make the argument in this context a bit simpler.

9.1 The Tableau System

A *signed* formula is $T\,X$ or $F\,X$, where X is a formula of **LP**. Intuitively these can be read: X is true, respectively false, in some particular context. A *tableau proof* of X is a closed tableau for $F\,X$—terminology that I will now proceed to explain. A *tableau* for $F\,X$ is a tree, with $F\,X$ at the root, constructed using certain branch extension rules to be given in a moment. Customarily a tableau is written with the root at the top, branching downward. A tableau is *closed* if each branch is closed, and a branch is closed if it contains an explicit contradiction, $T\,Z$ and $F\,Z$ for some formula Z, or $T\,\bot$. What remains is to say what the branch extension rules are. These fall into two categories: those that simply extend branches (often known as α rules) and those that cause branches to split (often known as β rules). The non-branching rules are given in Table 1. If a signed formula of the form shown above the line occurs on a tableau branch, the formula or formulas below the line may be added to the branch end. This is a non-deterministic rule—one may freely choose which signed formula on a branch to apply a rule to.

$$\frac{F\,X \supset Y}{\begin{array}{c}T\,X\\ F\,X\end{array}} \qquad \frac{T\,t{:}X}{T\,X} \qquad \frac{F\,!t{:}(t{:}X)}{F\,t{:}X} \qquad \frac{F\,(s+t){:}X}{F\,s{:}X} \qquad \frac{F\,(s+t){:}X}{F\,t{:}X}$$

Table 1: Non-Branching Tableau Rules

The branching rules are given in Table 2. These are also non-deterministic. If a signed formula of the form shown above the line occurs on a tableau branch, the end of the branch may be split in two, with one child labeled with the first of the signed formulas shown below the line, and the other child with the second. In the rule for times, X can be any formula.

$$\frac{T\,X \supset Y}{F\,X \mid T\,Y} \qquad \frac{F\,(t \cdot s){:}Y}{F\,t{:}X \supset Y \mid F\,s{:}X}$$

Table 2: Branching Tableau Rules

There is one final piece of machinery that is needed. Let $\mathcal{C}$ be a constant specification. I'll say *X has an **LP** tableau proof using $\mathcal{C}$* if it has a proof using the machinery above together with the additional rule: a branch closes if it contains $F\,c{:}Z$ where $Z \in \mathcal{C}(c)$.

9.2 Soundness

Earlier we defined satisfiability for sets of formulas, and now we extend that to sets of signed formulas. Let $\mathcal{C}$ be a constant specification. A set S of signed formulas is weakly $\mathcal{C}$-**LP** satisfiable if there is a weak **LP** model $\mathcal{M}$ that meets $\mathcal{C}$, and a possible world Γ of it at which all T signed members of S are true and all F signed members are false; that is, $\mathcal{M}, \Gamma \Vdash X$ for all $T\,X \in S$ and $\mathcal{M}, \Gamma \not\Vdash X$ for all $F\,X \in S$.

Now the argument for tableau soundness is along the usual lines. A tableau is weakly $\mathcal{C}$ satisfiable if some branch is, and a branch is weakly $\mathcal{C}$ satisfiable if the set of signed formulas on it is. The chief thing is to show the following. The proof is straightforward, and I'll omit it.

Proposition 9.1 *If $\mathcal{T}$ is an **LP** tableau that is weakly $\mathcal{C}$-**LP** satisfiable, and a branch extension rule is applied to $\mathcal{T}$ to produce the tableau $\mathcal{T}'$, then $\mathcal{T}'$ is weakly $\mathcal{C}$-**LP** satisfiable.*

Next we need to take tableau closure into account. Again I'll leave verification to you.

Proposition 9.2 *Suppose $\mathcal{C}$ is a constant specification. If $\mathcal{T}$ is a closed tableau using $\mathcal{C}$, then $\mathcal{T}$ is not weakly $\mathcal{C}$-**LP** satisfiable.*

Combining these, soundness follows: if X has an **LP** tableau proof using $\mathcal{C}$, then X must be weakly $\mathcal{C}$-**LP** valid. For otherwise $\{F\,X\}$ would be weakly $\mathcal{C}$-**LP** satisfiable, and if it were, by Proposition 9.1 every tableau beginning with $F\,X$ would be weakly $\mathcal{C}$-**LP** satisfiable, but such a tableau cannot be closed by Proposition 9.2.

Theorem 9.3 *If X has a tableau proof using the constant specification $\mathcal{C}$, then X is weakly $\mathcal{C}$-**LP** valid.*

9.3 Completeness

Completeness is proved by a variation of the method used in [10]. From now on, we allow a tableau to start with a finite set of signed formulas, instead of with a single one. The order in which these formulas are added to the initial branch does not matter. Let $\mathcal{C}$ be a constant specification, fixed for the rest of the section. Call a set S (not necessarily finite) *tableau consistent* if there is no closed tableau for any finite subset of S, using $\mathcal{C}$. As usual, tableau consistent sets extend to maximal ones.

Suppose S is maximally tableau consistent, and $Z \in S$ for some signed formula Z. If Z is one of the signed formulas above the line in a rule in Table 1, each signed formula below the line in that rule is also in S. As a typical example, if $F\,(s+t){:}X \in S$, then we should also have $F\,s{:}X \in S$. Well if not, since S is maximally tableau consistent, $S \cup \{F\,s{:}X\}$ must not be tableau consistent, and hence there is a closed tableau for a finite subset, say $\{Z_1, \ldots, Z_n, F\,s{:}X\}$. But $\{Z_1, \ldots, Z_n, F\,(s+t){:}X\}$ is a finite subset of S itself, and there is a closed tableau for it, since we can make the first move in the tableau construction the addition of $F\,s{:}X$ using the tableau rule from Table 1, and then proceed as we did in the closed tableau for $\{Z_1, \ldots, Z_n, F\,s{:}X\}$. In a similar way, if Z is one of the signed formulas above the line in a rule in Table 2, and $Z \in S$, then one of the two formulas below the line in that rule is in S.

Construct a weak **LP** model candidate as follows. Let $\mathcal{G}$ be the collection of all maximal tableau consistent sets. For $\Gamma, \Delta \in \mathcal{G}$, let $\Gamma\mathcal{R}\Delta$ provided $\{T\,t{:}X \mid T\,t{:}X \in \Gamma\} \subseteq \Delta$ and $\{F\,t{:}X \mid F\,t{:}X \in \Delta\} \subseteq \Gamma$, for all proof polynomials t and formulas X. Let $X \in \mathcal{E}(\Gamma, t)$ provided $F\,t{:}X \notin \Gamma$. And let $\Gamma \in \mathcal{V}(P)$ provided $T\,P \in \Gamma$, for a propositional letter P. This gives us a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$.

Clearly $\mathcal{R}$ is reflexive and transitive. I'll check two cases of the verification that $\mathcal{E}$ is an evidence function, beginning with monotonicity. Suppose $\Gamma\mathcal{R}\Delta$ and $X \in \mathcal{E}(\Gamma, t)$. By the second supposition, $F\,t{:}X \notin \Gamma$. By the first supposition, $\{F\,t{:}X \mid F\,t{:}X \in \Delta\} \subseteq \Gamma$, and it follows that $F\,t{:}X \notin \Delta$, and so $X \in \mathcal{E}(\Delta, t)$. I'll also check one of the Sum conditions. Suppose $X \notin \mathcal{E}(\Gamma, s+t)$; I'll show $X \notin \mathcal{E}(\Gamma, s)$. But this is easy. It amounts to saying $F\,(s+t){:}X \in \Gamma$ implies $F\,s{:}X \in \Gamma$ and, as we saw above, this is the case since Γ is maximal. Obviously $\mathcal{M}$ meets constant specification $\mathcal{C}$ since no set containing $F\,c{:}X$ is consistent, where $X \in \mathcal{C}(c)$. We thus have that $\mathcal{M}$ is a weak **LP** model that meets $\mathcal{C}$.

The main thing now is a version of the Truth Lemma appropriate for tableaus. Unlike earlier versions, it takes the form of two implications instead of an equivalence. For each $\Gamma \in \mathcal{G}$

$$T\,X \in \Gamma \Longrightarrow \mathcal{M}, \Gamma \Vdash X \tag{5}$$

$$F\,X \in \Gamma \Longrightarrow \mathcal{M}, \Gamma \not\Vdash X \tag{6}$$

The proof is by induction on the degree of X. As usual, I'll just give the modal case. Assume (5) and (6) are known for Z, and now consider $t{:}Z$.

Suppose first that $T\,t{:}Z \in \Gamma$. Since Γ is consistent, $F\,t{:}Z \notin \Gamma$ and so $Z \in \mathcal{E}(\Gamma, t)$. Let Δ be an arbitrary member of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$. By definition of $\mathcal{R}$, $T\,t{:}Z \in \Delta$, and using one of the tableau rules from Table 1, and the maximality of Δ, $T\,Z \in \Delta$. By the induction hypothesis, item (5) gives us $\mathcal{M}, \Delta \Vdash Z$. Since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash t{:}Z$.

Next, suppose that $F\,t{:}Z \in \Gamma$. Then $Z \notin \mathcal{E}(\Gamma, t)$, so trivially $\mathcal{M}, \Gamma \not\Vdash t{:}Z$.

With this established, completeness is as usual. If X does not have a tableau proof, $\{F\,X\}$ is tableau consistent. A maximal extension of this set will be a member of $\mathcal{G}$ at which X is false. We thus have shown the following.

Theorem 9.4 *If $\mathcal{C}$ is a constant specification, any formula that is weakly $\mathcal{C}$-**LP** valid has a tableau proof using $\mathcal{C}$.*

Then of course a full constant specification gives us completeness with respect to **LP** models, and not just weak ones. This is the case for any axiomatically appropriate constant specification, for example.

10 Variations

I have given a semantics for **LP**. This semantics has obvious relationships with **S4**, and in Section 7 the connection was examined at some length. But there are **LP**-style counterparts of various standard sublogics of **S4** as well. These have been considered in [8, 3], and in [4] which investigates a Gentzen-style formulation, and completeness with respect to variations on the semantics of [9] have been established. Here I briefly consider what things are like in the present context. I will use names like **LP**(**T**) and **LP**(**K4**) for these sublogics of **LP**. Using this terminology, **LP** itself is the same as **LP**(**S4**). The basic operation on proof polynomials is $\cdot$ and all logics considered in this section have appropriate semantic conditions for that. In this section a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ is assumed to meet the following conditions: $\mathcal{R}$ is a binary relation on the non-empty set $\mathcal{G}$, $\mathcal{V}$ maps propositional variables to subsets of $\mathcal{G}$, and $\mathcal{E}$ is a function from states and proof polynomials to sets of formulas that meets the Application and Sum conditions from Definition 3.1. Now, various (weak) models are given by the following table.

LP(**K**)	no other conditions
LP(**T**)	$\mathcal{R}$ is reflexive
LP(**K4**)	$\mathcal{R}$ is transitive and $\mathcal{E}$ satisfies Monotonicity and Proof Checker
LP(**S4**)	**LP**(**T**) conditions and **LP**(**K4**) conditions

In Section 7 relationships were shown between **LP** and **S4**. There are similar connections between **LP**(**L**) and **L**, for **L** being any of **K**, **T**, **K4**, or of course **S4**. I omit the details.

The axiom system of Artemov, considered in Section 8, will also be called an **LP**(**S4**) system. There are also systems corresponding to the other semantically defined logics above. For **LP**(**K4**) simply drop axiom *A1*, Reflexivity. **LP**(**K**) and **LP**(**T**) require something a little more complex. Call a constant specification *strongly* **LP** *appropriate* if axioms have proof constants, $c{:}X$ has a proof constant whenever c is a proof constant for X, and apart from these conditions, no other formulas have proof constants. Replace rule *R2*, $\mathcal{C}$ Axiom Necessitation, by the following recursive version, in which it is assumed that $\mathcal{C}$ is a strongly **LP** appropriate proof assignment.

*R2**. $\mathcal{C}$ **Axiom Necessitation** $\vdash c{:}X$ where $X \in \mathcal{C}(c)$
and either X is an axiom *A0* – *A4*
or X is inferable using *R2**

The axiom system for **LP**(**T**) drops axiom *A3*, Proof Checker, and replaces rule *R2* with *R2**. The axiom system for **LP**(**K**) drops *A2* and *A3* and replaces *R2* with *R2**. These axiomatizations appear in [8].

Finally, for **LP**(**L**) being any of the semantics considered above, let $\mathbf{LP}^{-}(\mathbf{L})$ be the similar semantics but in which the evidence function, $\mathcal{E}$, is not assumed to satisfy the Sum condition from Definition 3.1. Dropping axiom *A4*, Sum, from the axiomatization of **LP**(**L**) produces an axiomatization of $\mathbf{LP}^{-}(\mathbf{L})$. Tableau versions are equally straightforward. I'll leave all this to the interested reader.

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