A Second GL Justification Logic

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Abstract

Justification logics are like modal logics, but with each occurrence of the necessity operator replaced by one of an infinite family of terms representing reasons, or justifications, for the necessity of the formula it prefixes. Justification logic has become seen as a rich subject area and two books on it have recently appeared, [5, 10]. It is known that the family of modal logics that have corresponding justification logics is an infinite one. A justification counterpart of a modal logic is one in which, for each theorem of the modal logic there is a related theorem in the justification logic replacing assertions of necessity with justifications of them, where the precise meaning of this will be discussed later in this paper. All known examples of this phenomenon involved canonical modal logics until 2016 when Daniyar Shamkanov showed that Gödel-Löb logic had a justification counterpart, [13]. He did this by using a 'circular' sequent calculus for $\mathsf{S4}$ that he had devised. Not long afterward I also showed the existence of a justification counterpart, but using the more familiar tableau/sequent system found, for instance, in [6]. This system has a rule in which the polarity of a formula is reversed—something that was generally understood to be a block to the creation of a justification counterpart. The way around it was to make use of a more general notion of justification counterpart for a formula than one usually did. The work was extended to Grzegorczyk logic as well. All this was written up at the time, in early 2017, but was never submitted for publication. Since it has features of interest independent from those of [13], including a broader notion of what it means to be a realization, I have brought the references up to date and am making the work public here.

Keywords: Justification Logic, Realization, Gödel-Löb Logic, Grzegorczyk Logic

1 Introduction

Justification logics are *explicit* versions of modal logics. That is, instead of using a necessity operator as in $\Box X$, they use *justification terms* as in $t\colon X$, where t encodes an explanation for why X is necessary (or known, or obligatory, etc.). For example, the fundamental K validity $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$ has as a counterpart the following validity of the justification logic known as J: $t{:}(X \supset Y) \supset (u{:}X \supset [t \cdot u]{:}Y)$ where t and u are justification terms, and the operation symbol in $t \cdot u$ is a symbolic representation of an application of *modus ponens*. The *forgetful functor* is the mapping $X \mapsto X^\circ$, from a justification language to a modal language, that recursively replaces subformulas of the form $t{:}X$ with $\Box X$. A particular modal logic and justification logic pair are *counterparts* if the forgetful functor maps the set of theorems of the justification logic onto the set of theorems of the modal logic. In this sense, modal K and justification J are counterparts. If we have a pair of counterparts, for each modal theorem X there must be a justification theorem Y that 'forgets back' to X, that is, $Y^\circ = X$. In such a case, Y is called a *realization* of X. Which modal logics have justification realizations for all its theorems is a central issue in the study of justification logics.

The first justification logic, LP, was an explicit counterpart of modal S4, and was introduced by Sergei Artemov as part of a project to provide an arithmetic semantics for intuitionistic logic, a project discussed in detail in [1, 2]. For an overall survey of justification logics, see [4]. Since then the family of modal logics having justification logic counterparts has grown steadily and is now known to be infinite, [8]. Two books on justification logic have recently appeared, [5, 10].

Gödel-Löb logic, GL, was at one time thought to be a modal logic that did not have a justification counterpart. This belief was eventually shown to be wrong. In [12] Daniyar Shamkanov devised a variant sequent calculus based on S4 allowing what he called 'circular' proofs, and showed it captured GL. Then in [13] he used this calculus to constructively prove a realization theorem for GL. The following year I also gave a constructive realization proof for GL, but using a more familiar sequent (or rather, tableau) calculus, which can be found in many places including [6]. There are two peculiar features of this that are worth some attention, the first concerning the form our justification logic axiomatization takes, the second concerning the machinery used to prove a realization theorem.

Axioms for justification logics have generally followed those for the corresponding modal logics quite closely. For instance, the modal schema $\Box X \supset \Box\Box X$ is usually made explicit as the justification schema $t{:}X \supset !t{:}t{:}X$, where "!" is an operator mapping justifications to justifications, and is commonly called "justification checker". The subformula X occurs twice in the modal formula, and both occurrences are understood to be identical. An instance of this scheme would have both occurrences replaced by the same formula. These correspond to two occurrences of X in the justification formula, also with both being identical. This pattern has been followed so far for every justification logic/modal logic pair. The characteristic GL axiom schema $\Box(\Box X \supset X) \supset \Box X$ contains three occurrences of subformula X, but in the justification counterpart introduced here they do not simply carry over to three occurrences of the *same* justification formula. Rather they can be three different justification formulas, say A, B, and C, where these formulas must 'forget back' to the same modal formula, that is, $A^\circ = B^\circ = C^\circ$. This seems to be the first time something of the sort has come up. Details appear in Section 2.

Broadly speaking, realization theorems have two kinds of proofs: non-constructive and constructive. The earliest proofs were constructive. All known constructive proofs make use of cut-free proof systems of some kind. Non-constructive proofs have been found to be more general and uniform, as [8] and [5] demonstrates. But they are, well, non-constructive. Up to now, all cases that had constructive realization proofs also had a non-constructive, semantic one. The converse is not true—there are infinitely many modal logics with justification counterparts where the only known proof of realization is a semantic, non-constructive one. As it happens, for GL a non-constructive argument has not been forthcoming. The problem is the lack (so far?) of a Kripke-style semantics for the justification counterpart of GL. But constructive proofs exist, based on a cut-free proof systems, circular in [12] and more conventional here. All this seems to be connected with the fact that GL is not canonical, and a deeper look at some future date is warranted. Also connections between the two versions of realization deserve further study.

This paper is not self-contained, but we have tried to provide background explanations as far as possible. For instance, we make use of *quasi-realizations* as an intermediate step in proving a realization theorem. There is an algorithm converting quasi-realizations to realizations, it is used here, but the details are not presented. Much of the needed background material had been scattered in various papers, but recently has been collected and uniformized in [5]. We refer you to that as providing appropriate details where they have been omitted here.

2 The Logics GL and JGL

A very broad presentation of justification logics can be found in any of [5, 2, 4, 10]. Here we minimize generality and move directly to a formal presentation of the logics of interest.

Gödel-Löb provability logic, GL, is very well-known. Formulas are built up from propositional letters and $\bot$ using boolean connectives and the formation rule: if X is a formula, so is $\Box X$. (Often only $\supset$ is used as primitive, with $\wedge$ and $\vee$ defined. Likewise it is common to take negation as defined using $\supset$ and $\bot$. We are casual about such details here.) Axioms are tautologies and the Löb schema $\Box(\Box X \supset X) \supset \Box X$, and the rules of inference are *Modus Ponens*, $X, X \supset Y \Rightarrow Y$, and *Necessitation*, $X \Rightarrow \Box X$. The modal 4 schema $\Box X \supset \Box\Box X$ is provable, so GL is an extension of modal K4, though not of S4.

The justification analog of GL is called JGL here. We present its syntax and axiomatics. A semantics is not yet known.

As with all justification logics there are at least two basic binary *justification function symbols* $+$ and $\cdot$ in the language. Special to this logic there is also a unary justification function symbol for which we use gl. Justification terms are built up from *justification constants* and *justification variables* using these function symbols. Formulas are built up from propositional letters and $\bot$ using boolean connectives and the following formation rule special to justification logics: if t is a justification term and X is a formula, $t{:}X$ is a formula. It is read: "t justifies X". Note: we sometimes write $[t]{:}X$ instead of $t{:}X$ if t is complicated. This is for reading convenience, and has no formal significance.

As we noted informally earlier, there is a standard mapping from a justification language to modal language, called the *forgetful functor*. Only one propositional connective case is shown, as representative.

$$\begin{aligned} \bot^\circ &= \bot \\ P^\circ &= P \text{ for propositional letters} \\ (X \supset Y)^\circ &= X^\circ \supset Y^\circ \\ (t{:}X)^\circ &= \Box X^\circ \end{aligned}$$

This is a fundamental mapping and plays a basic role in defining the notion of realization. We need it now to formulate our axiom system for JGL, and we will come back to it again later on.

We begin with a restricted version of JGL denoted JGL_0. After its axiomatic presentation we expand the machinery to characterize JGL itself. Axiomatically JGL_0 begins with the following basic axiom schemes.

Classical: All tautologies (or enough of them)

Application: All formulas of the form $s{:}(X \supset Y) \supset (t{:}X \supset [s \cdot t]{:}Y)$

Sum: All formulas of the forms $s{:}X \supset [s + t]{:}X$ and $t{:}X \supset [s + t]{:}X$

Modus Ponens: $X, X \supset Y \Rightarrow Y$

This much defines a logic known in the literature as J_0. We now add one more axiom scheme to the list. (Technically it is an axiom scheme pattern rather than simply an axiom scheme.) In it, gl is a new justification function symbol.

JGL Scheme: All formulas of the form $t{:}(u{:}X_1 \supset X_2) \supset [gl(t)]{:}X_3$ where X_1, X_2, and X_3 are justification formulas such that $X_1^\circ = X_2^\circ = X_3^\circ$.

We write $S \vdash_{\mathsf{JGL}_0} X$ to mean X is derivable from the set S of justification formulas using JGL_0 as defined above. This means, as usual, that X is the last item in a sequence of justification formulas where each member of the sequence is either an axiom of JGL_0, a member of S, or follows from earlier items using *modus ponens*. We write $\vdash_{\mathsf{JGL}_0} X$ for $\emptyset \vdash_{\mathsf{JGL}_0} X$.

Note that while **GL** has a pattern like that of the modal Löb axiom, X_1, X_2, and X_3 may be different justification formulas, making this a very peculiar axiom scheme. I have only some elementary remarks concerning its peculiarity—deeper insight is needed. The modal Löb axiom, $\Box(\Box X \supset X) \supset \Box X$ is commonly understood as a distillation of a feature of formal arithmetic, where $\Box$ represents *provability* expressed using the existential quantifier to assert the existence of a proof. X itself may, and commonly does, contain occurrences of $\Box$, but all occurrences are translated into arithmetic in the same way, using the existential quantifier. But when working with justification logic, each modal $\Box$ occurrence becomes a justification term which we may think of as representing an *explicit proof.* The axiom scheme allows us to bring in more than one justification term for each occurrence of $\Box$ in X itself, X_1, X_2, and X_3 can be different. Loosely the **GL** scheme says there will be a proof of some version of X provided we can show the antecedent using possibly other versions of X in the process. In fact, not only may justification terms in X_3 be different from the corresponding ones in X_1 and X_2, but this is essential for our proof of a realization theorem. It is not essential for Shamkanov's version. A deeper understanding is needed here.

We now bring in the standard and important notion of constant specification. Our axioms are simply assumed and are not analyzed further. The intended role of justification constant symbols is to represent justifications for axioms. If A is an axiom, we can simply announce that constant symbol c plays the role of a justification for it. But note that if A is an axiom, and we assign c as a justification for it, then in effect we are assuming we have $c{:}X$ as an additional axiom, and we may want a justification for this as well. This leads to the following.

A *constant specification* CS for JGL_0 is a set of formulas meeting the following conditions.

1. Members of CS are of the form $c_n{:}c_{n-1}{:}\ldots c_1{:}A$ where $n > 0$, A is an axiom from the set above, and each c_i is a constant symbol.
2. If $c_n{:}c_{n-1}{:}\ldots c_1{:}A$ is in CS where $n > 1$, then $c_{n-1}{:}\ldots c_1{:}A$ is in CS too. Thus CS contains all intermediate specifications for whatever it contains.

We took as axioms of JGL_0 all tautologies, *or enough of them.* This means we have not actually specified an axiom system, but rather a family of them. A constant specification appropriate for one such axiom system may not be appropriate for a different one. So, rather than giving a constant specification once and for all, constant specifications are treated as parameters of the logic formulation. There are a number of different kinds of constant specifications that have been considered, but the only special condition we will need is the following.

Axiomatically Appropriate: For every axiom A of JGL_0 and for every $n > 0$ there are constant symbols c_i so that $c_n{:}c_{n-1}{:}\ldots c_1{:}A \in \mathsf{CS}$.

Now we can formulate the (family of) justification logics that concern us here.

Logic JGL Let CS be a constant specification for JGL_0. We write $S \vdash_{\mathsf{JGL}(\mathsf{CS})} X$ to indicate that $S \cup \mathsf{CS} \vdash_{\mathsf{JGL}_0} X$. We write $S \vdash_{\mathsf{JGL}} X$ to indicate that $S \vdash_{\mathsf{JGL}(\mathsf{CS})} X$ *for some axiomatically appropriate constant specification* CS. As usual, if $S = \emptyset$ we simply leave it out of the notation.

A fundamental feature common to justification logics generally is that they internalize their own proofs. Formally we have the following, stated for JGL. The proof is by a straightforward induction on axiomatic proof length.

Internalization Let CS be an axiomatically appropriate constant specification. If $\vdash_{\mathsf{JGL(CS)}} X$ then for some justification term t, $\vdash_{\mathsf{JGL(CS)}} t{:}X$.

Restated following the convention above for the use of JGL, if $\vdash_{\mathsf{JGL}} X$ then for some justification term t, $\vdash_{\mathsf{JGL}} t{:}X$.

We noted above that $\Box X \supset \Box\Box X$ is provable in GL. There is an analog for JGL as well. In the well-known justification logic LP there is a primitive function symbol ! and $t{:}X \supset !t{:}t{:}X$ is taken as an axiom scheme. Here a version of ! is a definable operation, and we use **!** to represent it.

Proposition 2.1 *Let* CS *be an axiomatically appropriate constant specification. For each justification formula $t{:}A$ there is a justification term **!**t such that* $\vdash_{\mathsf{JGL(CS)}} t{:}X \supset \mathbf{!}t{:}t{:}X$.

Proof To keep the proof relatively uncluttered, we introduce a simple derived rule. Suppose $u{:}(X \supset Y)$ is derivable. Then for any justification term v, $v{:}X \supset [u \cdot v]{:}Y$ is also derivable. This is an easy use of the Application axiom schema. We call this *Distribution*.

Let $t{:}A$ be a justification formula. The following is a proof (derivation from the empty set) in JGL(CS).

1. $(A \wedge t{:}A) \supset A$
2. $c{:}((A \wedge t{:}A) \supset A)$
3. $t{:}(A \wedge t{:}A) \supset [c \cdot t]{:}A$
4. $(A \wedge t{:}(A \wedge t{:}A)) \supset (A \wedge [c \cdot t]{:}A)$
5 $A \supset (t{:}(A \wedge t{:}A) \supset (A \wedge [c \cdot t]{:}A))$
6. $b{:}(A \supset (t{:}(A \wedge t{:}A) \supset (A \wedge [c \cdot t]{:}A)))$
7. $t{:}A \supset [b \cdot t]{:}(t{:}(A \wedge t{:}A) \supset (A \wedge [c \cdot t]{:}A))$
8. $[b \cdot t]{:}(t{:}(A \wedge t{:}A) \supset (A \wedge [c \cdot t]{:}A)) \supset gl(b \cdot t){:}(A \wedge t{:}A))$
9. $t{:}A \supset gl(b \cdot t){:}(A \wedge t{:}A)$
10. $(A \wedge t{:}A) \supset t{:}A$
11. $a{:}((A \wedge t{:}A) \supset t{:}A)$
12. $gl(b \cdot t){:}(A \wedge t{:}A) \supset [a \cdot gl(b \cdot t)]{:}t{:}A$
13. $t{:}A \supset [a \cdot gl(b \cdot t)]{:}t{:}A$

1 is a tautology. 2 is from 1 by Internalization, introducing justification term c. 3 is from 2 by Distribution. 4 is from 3 by classical logic. 5 is from 4 by classical logic. 6 is from 5 by Internalization, introducing b. 7 is from 6 by Distribution. 8 is axiom **GL**; note that $(A \wedge t{:}A)^\circ$ and $(A \wedge [c \cdot t]{:}A)^\circ$ are the same. 9 is from 7 and 8 by classical logic. 10 is a tautology. 11 is from 10 by Internalization, introducing a. 12 is from 11 by Distribution. Finally, 13 is from 9 and 12 by classical logic.

Now take **!**t to be $a \cdot gl(b \cdot t)$. ■

3 What Is Realization

We have a modal logic GL and a justification logic JGL. We will show that these are *counterparts*, a technical term roughly saying the justification logic becomes the modal logic when the details embodied in justification terms are forgotten. Or in the other direction, that the justification logic is the modal logic with reasons for each necessitation operator spelled out. Formally it is characterized using the forgetful functor from Section 2.

1. If X is provable in JGL using any constant specification then X° is a theorem of GL.

2. If Y is a theorem of GL then there is some justification formula X so that $X^\circ = Y$, where X is provable in JGL using some axiomatically appropriate constant specification.

As we phrased it earlier, this says the forgetful functor is a mapping from the set of theorems of JGL *onto* the set of theorems of GL.

The first example of a counterpart pair matched modal $\mathsf{S4}$ with justification logic LP, the logic of proofs, [1], and constituted an essential part of Sergei Artemov's arithmetic semantics for intuitionistic logic. Since then the family of modal logics having justification counterparts has been shown to be infinite, [8], though all examples considered there involved canonical modal logics.

The proof of item 1 above is simple, as it has been for all justification logics known so far. It is easy to check that the forgetful functor maps each axiom of JGL to a theorem of GL (indeed to an axiom). The same is true for members of any constant specification. And modus ponens preserves this feature. So every line of an axiomatic JGL proof maps to a theorem of GL, in particular every theorem does.

Item 2 is not simple. Given a formula Y of GL, a justification formula X such that $X^\circ = Y$ is called a *potential realization* of Y. In effect, X is like Y but with every necessitation operator replaced by some justification term. A potential realization is called *normal* if it meets the condition that every *negative* occurrence of necessity in the formula being realized is replaced by a distinct justification variable. What is desired is to show that every theorem of GL has a *provable* potential realization in JGL—a Realization Theorem. In fact essentially all known proofs of realization theorems produce normal realizations. Normality is an important feature. We can think of justification variables as representing justifications supplied 'from the outside' and more complex justification terms as being computed from these. Normality says theorems of GL have a certain kind of input/output structure.

4 Tableaus for GL

Realization theorems have been proved both constructively and non-constructively. Almost universally, non-constructive proofs have been found for a strictly larger family of modal logics than have constructive ones, but GL is a counter-example, at least so far. The proof we give and the proof in [13] are constructive, and a non-constructive, semantic based, proof has not been found. All constructive realization theorem proofs so far have made use of some cut-free proof system. Here we use a tableau system for GL, though a sequent formulation would work in the same way. Tableaus and sequents are, essentially, notational variants of each other, but tableaus offer us some conveniences.

Commonly tableau proofs take the form of trees with formulas labeling nodes. Here it is convenient to use *block* tableaus, with *a set* of formulas as a label. Roughly speaking, such a set represents the contents of an entire branch of a conventional tableau, at a particular stage of its construction. To keep notational clutter down, we will write such sets without enclosing curly brackets. Also it is convenient for us to use *signed* formulas. Two special symbols, T and F, are introduced, and a signed formula is one of $T\,X$ or $F\,X$, informally asserting that X is true or false respectively. The proof system is refutation based. A proof of X begins with a tree having only a root node, with the block consisting of just $F\,X$ labeling the root. The tree is grown using *branch extension rules.* The rules are formulated so that a block is like a conjunction, and the tree itself is like the disjunction of its branches. Some of the rules are *destructive*—they eliminate information from a tree node. A tableau branch is closed if it has a block containing both $T\,A$ and $F\,A$, where A is atomic, or if it has a block containing $T\,\bot$. (Please note that we require *atomic* closure.) A tableau with every branch closed is a closed tableau. A proof of X is a closed tableau for $F\,X$.

Informally, the assumption that X is false under some circumstances leads to a contradiction, hence X must be universally true.

We typically use $\mathcal{B}$ for a block, and write $\mathcal{B}, T\,X$ as short for $\mathcal{B} \cup \{T\,X\}$, and similarly for F-signed formulas. Propositional tableau rules are quite standard; and are given in Figure 1. A signed tableau system for GL consists of the standard propositional rules with one additional modal rule, given in Figure 2.

$$\frac{\mathcal{B}, T\,\neg X}{\mathcal{B}, F\,X} \qquad \frac{\mathcal{B}, F\,\neg X}{\mathcal{B}, T\,X}$$

$$\frac{\mathcal{B}, T\,X \supset Y}{\mathcal{B}, F\,X \mid \mathcal{B}, T\,Y} \qquad \frac{\mathcal{B}, F\,X \supset Y}{\begin{array}{c}\mathcal{B}, T\,X \\ \mathcal{B}, F\,Y\end{array}}$$

$$\frac{\mathcal{B}, T\,X \wedge Y}{\begin{array}{c}\mathcal{B}, T\,X \\ \mathcal{B}, T\,Y\end{array}} \qquad \frac{\mathcal{B}, F\,X \wedge Y}{\mathcal{B}, F\,X \mid \mathcal{B}, F\,Y}$$

$$\frac{\mathcal{B}, T\,X \vee Y}{\mathcal{B}, T\,X \mid \mathcal{B}, T\,Y} \qquad \frac{\mathcal{B}, F\,X \vee Y}{\begin{array}{c}\mathcal{B}, F\,X \\ \mathcal{B}, F\,Y\end{array}}$$

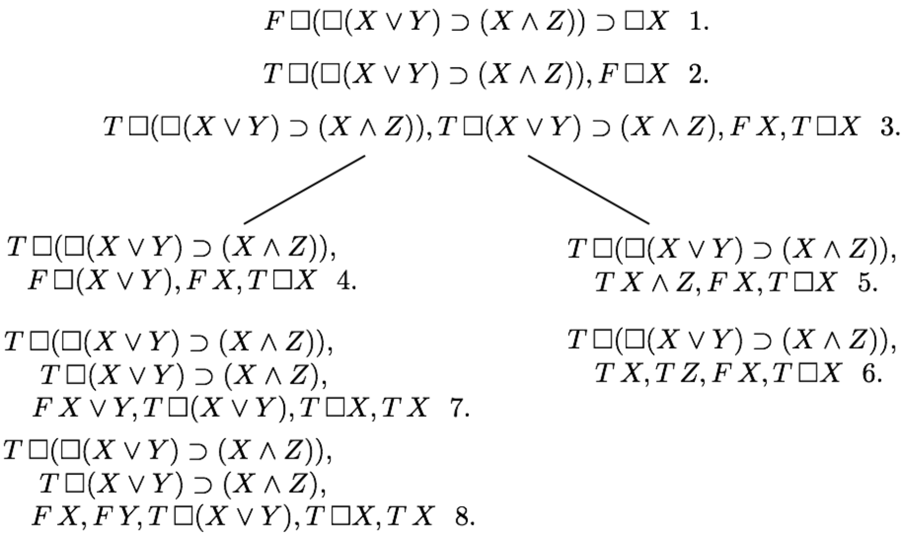
Figure 1: Propositional Block Tableau Rules

$$\frac{\mathcal{B}, F\,\Box X}{\mathcal{B}^{\sharp}, F\,X, T\,\Box X}$$

where $\mathcal{B}^{\sharp} = \{T\,Y, T\,\Box Y \mid T\,\Box Y \in \mathcal{B}\}$.

Figure 2: GL Block Tableau Rule

Example 4.1 Here is a GL block tableau proof of $\Box(\Box(X \vee Y) \supset (X \wedge Z)) \supset \Box X$. Assume X, Y, and Z are atomic, to keep it simple.



2 is 1 by $F \supset$. 3 is from 2 by $F\,\Box$. 4 and 5 are from 3 by $T \supset$. 6 is from 5 by $T\,\wedge$. 7 is from 4 by $F\,\Box$. 8 is from 7 by $F\,\vee$. Blocks 6 and 8 contain atomic contradictions so both branches, and hence the tableau, closes.

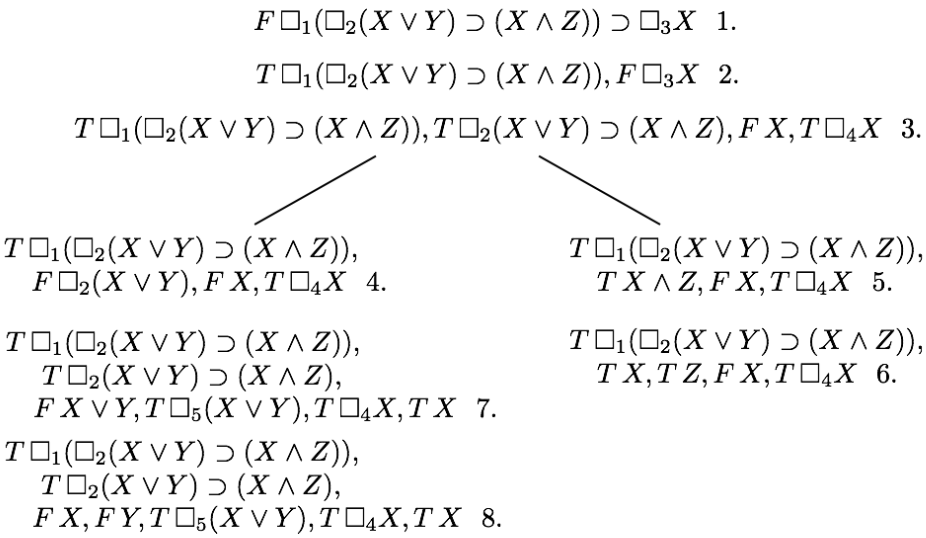
As presented in [8, 9, 5], *annotated* tableau proofs are needed for realization constructions. In annotated tableaus necessitation occurrences are assigned indexes, positive integers, to help track occurrences. To annotate a GL tableau proof, begin by assigning distinct positive integers to necessitation operators at the root. Propagate these annotations downward through propositional rules in the obvious way.—essentially just read the propositional rules in Figure 1 as applying to annotated formulas. For the GL tableau rule for $F\,\Box$, Figure 2, propagation is according to the scheme in Figure 3.

$$\frac{\mathcal{B}, F\,\Box_n X}{\mathcal{B}^\sharp, F\,X^*, T\,\Box_m X^{**}}$$

where X^* and X^{**} are like X
but with all annotations replaced by new distinct ones
that have not previously occurred in the tree
and also m is new and distinct.

Figure 3: Annotated GL Block Tableau Rule

Example 4.2 This is Example 4.1, but with the block tableau annotated.



Since X is atomic, $X^* = X^{**} = X$, so the step from 2 to 3, involving the $F\,\Box$ rule, is simpler than it would be in a bigger example.

5 Some Technical Items

We will need quite a bit of formal machinery but most is already in the literature and would be very lengthy to repeat in detail here. We just summarize what we need; fuller information can be found in [1, 8, 9, 5]. We do not state the results in full generality, but only as they apply to our current interest, JGL and GL.

We have already noted Internalization is fundamental. There is a direct corollary that is also very helpful. It is often called the *Lifting Lemma.*

Proposition 5.1 (Lifting Lemma) *Assume* CS *is an axiomatically appropriate constant function for* JGL *and* $t_1, \ldots, t_n, t_{n+1} \ldots t_{n+k}$ *are justification terms. If*

$$X_1, \ldots, X_n, t_{n+1}{:}X_{n+1}, \ldots, t_{n+k}{:}X_{n+k} \vdash_{\mathsf{JGL(CS)}} Y$$

then for any justification terms $t_1, \ldots, t_n$ *there is a justification term* u *so that*

$$t_1{:}X_1, \ldots, t_n{:}X_n, t_{n+1}{:}X_{n+1}, \ldots, t_{n+k}{:}X_{n+k} \vdash_{\mathsf{JGL(CS)}} u{:}Y.$$

This result originated in [1], where it was proved for LP. The logic LP has a proof-checker operation, !, with the characteristic axiom schema $t{:}X \supset !t{:}t{:}X$, and this plays an important role in the Lifting Lemma proof for LP. The same proof applies to JGL, but using the defined operation $\mathbf{!}t$ from Proposition 2.1. Details can also be found in [5], and are omitted here.

Constructive proofs of realization make use of cut-free proof procedures—in our case block tableaus for GL, annotated to track necessitation occurrences. There are several different such proofs. We follow the argument in [9] and in [5, Chapter 7], which was originally given for S4 in LP. The general structure of the proof makes use of *Quasi-Realizations.* Roughly, a quasi-realization is like a realization except that it admits more complicated formulas, involving disjunctions. We sketch the basic notation, and refer to the literature for full definitions and proofs of results.

Justification variables play a role. We fix an ordering of them, once and for all: v_1, v_2, ... When an annotated necessitation operator, say $\Box_n$, is replaced with a justification variable we will always use v_n, so the variable subscripts match the necessitation indexes. This convention simplifies things quite a bit.

We begin with quasi-realizers, then modify things to realizers. Propositionally we just give the implication case, as sufficiently representative.

Definition 5.2 (Potential Quasi-Realizers) The mapping $\langle\!\langle\cdot\rangle\!\rangle$ maps annotated, signed modal formulas to sets of signed justification formulas called potential quasi-realizers. It is defined recursively, as follows.

1. If A is atomic, a propositional letter or $\bot$, $\langle\!\langle T\,A\rangle\!\rangle = \{T\,A\}$ and $\langle\!\langle F\,A\rangle\!\rangle = \{F\,A\}$.
2. $\langle\!\langle T\,X \supset Y\rangle\!\rangle = \{T\,U \supset V \mid F\,U \in \langle\!\langle F\,X\rangle\!\rangle \text{ and } T\,V \in \langle\!\langle T\,Y\rangle\!\rangle\}$
   $\langle\!\langle F\,X \supset Y\rangle\!\rangle = \{F\,U \supset V \mid T\,U \in \langle\!\langle T\,X\rangle\!\rangle \text{ and } F\,V \in \langle\!\langle F\,Y\rangle\!\rangle\}$
3. $\langle\!\langle T\,\Box_n X\rangle\!\rangle = \{T\,v_n{:}U \mid T\,U \in \langle\!\langle T\,X\rangle\!\rangle\}$.
   $\langle\!\langle F\,\Box_n X\rangle\!\rangle = \{F\,t{:}(U_1 \vee \ldots \vee U_k) \mid F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,X\rangle\!\rangle \text{ and } t \text{ is any justification term}\}$.
4. The mapping is extended to *sets* of signed annotated formulas by letting $\langle\!\langle S\rangle\!\rangle = \cup\{\langle\!\langle Z\rangle\!\rangle \mid Z \in S\}$.

Potential realizers have a similar definition with one case simplified—the second part of item 3.

Definition 5.3 (Potential Realizers) The mapping $[\![\cdot]\!]$ also maps annotated, signed formula to sets of signed justification formulas, now called potential *realizers*.

1. If A is atomic, $[\![T\,A]\!] = \{T\,A\}$ and $[\![F\,A]\!] = \{F\,A\}$
2. $[\![T\,X \supset Y]\!] = \{T\,U \supset V \mid F\,U \in [\![F\,X]\!] \text{ and } T\,V \in [\![T\,Y]\!]\}$
   $[\![F\,X \supset Y]\!] = \{F\,U \supset V \mid T\,U \in [\![T\,X]\!] \text{ and } F\,V \in [\![F\,Y]\!]\}$
3. $[\![T\,\Box_n X]\!] = \{T\,v_n{:}U \mid T\,U \in [\![T\,X]\!]\}$.
   $[\![F\,\Box_n X]\!] = \{F\,t{:}U \mid F\,U \in [\![F\,X]\!] \text{ and } t \text{ is any justification term}\}$.
4. The mapping is extended to *sets* of signed annotated formulas by letting $[\![S]\!] = \cup\{[\![Z]\!] \mid Z \in S\}$.

It is easy to show, by induction on formula complexity, that every potential realization is also a potential quasi-realization, that is, $[\![Z]\!] \subseteq \langle\!\langle Z\rangle\!\rangle$. There is an algorithm, not reproduced here, converting a provable potential quasi-realization into a provable potential realization. It works for all justification logics meeting certain elementary conditions, in particular it works for JGL. Originally this appeared in [9], and in a larger context, in [5, Section 7.6]. The rest of this section is a summary of the consequences of the conversion algorithm, without proof. Assuming this, what is left here is to show there is a provable potential quasi-realization for each theorem of GL, which we do in Section 6.

Substitutions of justification terms for justification variables play a central role. We call the set of variables changed by a substitution its *domain*, and we assume domains are finite. If σ is a substitution we write $X\sigma$ for the result of applying σ to the formula X, and similarly $t\sigma$ for the result of applying σ to justification term t.

Definition 5.4 Let σ be a substitution and let A be an annotated modal formula.

1. *σ lives on A* if, for every justification variable v_k in the domain of σ, $\Box_k$ occurs in A;

2. σ *lives away from* A if, for every justification variable v_k in the domain of σ, $\Box_k$ does not occur in A;

3. σ meets the *no new variable* condition if, for every v_k in the domain of σ, the justification term $v_k\sigma$ contains no variables other than v_k.

Theorem 5.5 *Assume σ_A is a substitution that lives on annotated modal formula A, and σ_Z is a substitution that lives away from A.*

1. *If $T\,U \in \llbracket T\,A \rrbracket$ then $T\,U\sigma_Z \in \llbracket T\,A \rrbracket$.*
   If $F\,U \in \llbracket F\,A \rrbracket$ then $F\,U\sigma_Z \in \llbracket F\,A \rrbracket$.

2. *If $T\,U \in \langle\!\langle T\,A \rangle\!\rangle$ then $T\,U\sigma_Z \in \langle\!\langle T\,A \rangle\!\rangle$.*
   If $F\,U \in \langle\!\langle F\,A \rangle\!\rangle$ then $F\,U\sigma_Z \in \langle\!\langle F\,A \rangle\!\rangle$.

3. *If both σ_A and σ_Z meet the no new variable condition, then $\sigma_A\sigma_Z = \sigma_Z\sigma_A$.*

Remark: Items 1 and 3 above appear in [8, 5], where they are shown for arbitrary justification logics. Item 2 is new, but has a similar proof to that of item 1. Here is the thing of main interest, called Condensing, because it replaces a finite set of justification formulas with a single formula (with a substitution also involved). First we introduce notation, then the Condensing Theorem itself.

Definition 5.6 (Condensing for JGL) Let A be an annotated modal formula, $\mathcal{A}$ be a finite set of justification formulas, A' be a single justification formula, and σ be a substitution that lives on A and meets the no new variable condition. We write $T\,\mathcal{A}$ for $\{T\,X \mid X \in \mathcal{A}\}$, and similarly for $F\,\mathcal{A}$.

Notation	Meaning
$\mathcal{A} \xrightarrow{T\,A} (A', \sigma)$	$T\,\mathcal{A} \subseteq \langle\!\langle T\,A \rangle\!\rangle$
	$T\,A' \in \llbracket T\,A \rrbracket$
	$\vdash_{\mathsf{JGL}} A' \supset (\bigwedge \mathcal{A})\sigma$
$\mathcal{A} \xrightarrow{F\,A} (A', \sigma)$	$F\,\mathcal{A} \subseteq \langle\!\langle F\,A \rangle\!\rangle$
	$F\,A' \in \llbracket F\,A \rrbracket$
	$\vdash_{\mathsf{JGL}} (\bigvee \mathcal{A})\sigma \supset A'$

And finally the basic Condensing result, which informally says that any finite set of potential quasi-realizers condenses to a potential realizer. As we said, the result has an algorithmic proof that is omitted here.

Theorem 5.7 (Condensing Theorem) *Let A be an annotated modal formula. For each non-empty finite set $\mathcal{A}$ of justification formulas:*

1. *If $T\,\mathcal{A} \subseteq \langle\!\langle T\,A \rangle\!\rangle$ then there are A' and σ so that $\mathcal{A} \xrightarrow{T\,A} (A', \sigma)$.*

2. *If $F\,\mathcal{A} \subseteq \langle\!\langle F\,A \rangle\!\rangle$ then there are A' and σ so that $\mathcal{A} \xrightarrow{F\,A} (A', \sigma)$.*

This completes our summary of the background machinery we will need for what follows.

6 Realization

We sketched a proof in Section 3 that for any theorem X of JGL, X° is a theorem of GL. Realization goes the other way. An algorithm is given in both [9] and in [5, Section 7.8] for converting a closed annotated S4 block tableau into a tree whose labels are sets of potential quasi-realizers. Specifically, each signed modal formula Z is replaced with a finite, non-empty subset of $\langle\!\langle Z \rangle\!\rangle$ of potential quasi-realizers. This is done in such a way that the conjunction of the T-signed formulas at a node implies the disjunction of the F-signed formulas there; that is, the implication is provable in the justification logic LP. This is called being *justifiction sound*. The tableau root node has no T-signed formulas, so the disjunction of the F-signed formulas at the root is an LP *provable* potential quasi-realizer for the modal formula of the original tableau.

The construction for JGL and GL follows the same pattern. We begin with a closed annotated GL block tableau proof. Then a justification sound tree of finite sets of potential quasi-realizers is constructed from it. The conversion from tableau proof to quasi-realizer tree works from leaves upward, and justification soundness is established simultaneously with the construction. The propositional connective and closure steps are exactly the same as for S4, and are not repeated here. Unlike with S4 tableaus, there is no $T\,\Box$ case now. We describe what to do with the $F\,\Box$ case, using the GL tableau rule given in Section 4. This is the most complicated case.

Suppose we have applied the following annotated block modal rule, where the annotations in X^* and X^{**} are new and distinct and m is new.

$$\frac{\mathcal{B}, F\,\Box_n X}{\mathcal{B}^\sharp, F\,X^*, T\,\Box_m X^{**}}$$

And suppose a counterpart for the consequent has been constructed, in the tree of quasi-realizers, and that counterpart is justification sound. We show that it can be converted into a justification sound counterpart of the antecedent. This will occupy the rest of the section.

It helps to be more specific about the formulas involved in the modal rule application. Let us say that

$$\mathcal{B} = \{T\,\Box_{i_1} U_1, T\,\Box_{i_2} U_2, \ldots, T\,V_1, T\,V_2, \ldots, F\,W_1, F\,W_2, \ldots\}$$

where $V_1, V_2, \ldots$ are not necessitated formulas. Then the rule for this case looks like the following, where X^* and X^{**} are like X but with all annotations replaced by new distinct ones that do not occur in the antecedent and m is new and distinct..

$$\frac{T\,\Box_{i_1} U_1, T\,\Box_{i_2} U_2, \ldots, T\,V_1, T\,V_2, \ldots, F\,W_1, F\,W_2, \ldots, F\,\Box_n X}{T\,\Box_{i_1} U_1, T\,U_1, T\,\Box_{i_2} U_2, T\,U_2, \ldots, F\,X^*, T\,\Box_m X^{**}} \tag{1}$$

As noted, we want to convert a justification sound potential quasi-realizer for the block below the line in (1) into a justification sound potential quasi-realizer for the block above. Again, a justification sound potential quasi-realizer assigns a set of signed justification formulas to each signed modal formula so that the conjunction of the T-signed formulas implies the disjunction of the F-signed formulas. Now, $T\,V_1, T\,V_2, \ldots, F\,W_1, F\,W_2, \ldots$ appear above the line, but have vanished below. Sets of justification formulas must be assigned to these as potential quasi-realizers, but we cannot expect them to play a role in provability of the T's implying the F's, since they are irrelevant to the behavior of what is below the line. Something must be done with them, but the details can't be expected to matter. In [9, 5], *trivial* quasi-realizers were introduced for this purpose, and we carry the idea over to here. A trivial quasi-realizer for, say, $T\,V_1$ simply assigns a singleton set in which each necessitated subformula of V_1, $\Box_k Z$ is replaced with $v_k{:}Z$. This is purely a formal device, but it does the job.

More importantly, similar considerations apply to $F\,\Box_n X$, which appears above the line. It has been replaced below the line with $F\,X^*$ and $T\,\Box_m X^{**}$, but by design these share no indexes with $F\,\Box_n X$. Then what potential quasi-realizers we assign to $F\,X^*$ and $T\,\Box_m X^{**}$ give us no information about what to do with $F\,\Box_n X$. In fact, any potential quasi-realizer will do for $F\,\Box_n X$, and we may as well use a trivial quasi-realizer here too. Then what is the role of the formulas $F\,X^*$ and $T\,\Box_m X^{**}$ below the line? In fact, they play an essential role in showing that the potential quasi-realizers we use for the signed formulas above the line constitute a justification sound set—conjunction of the T's implies disjunction of the F's. The result is actually a very strong one: whatever potential quasi-realizers we use for $F\,\Box_n X$ will have the appropriate provability property! Now the details.

Since the consequent is assumed to be justification sound, for each item Z below the line we have a finite subset of $\langle\!\langle Z \rangle\!\rangle$ consisting of potential quasi-realizers for Z, let us call the finite subset Z^q. Let us say we have the following, making use of the fixed enumeration of justification variables mentioned in Section 5.

$$\begin{aligned}
(T\,\Box_{i_1} U_1)^q &= \{T\,v_{i_1}{:}U_1^1, T\,v_{i_1}{:}U_1^2, \ldots\} \subseteq \langle\!\langle T\,\Box_{i_1} U_1 \rangle\!\rangle \\
(T\,U_1)^q &= \{T\,\hat{U}_1^1, T\,\hat{U}_1^2, \ldots\} \subseteq \langle\!\langle T\,U_1 \rangle\!\rangle \\
(T\,\Box_{i_2} U_2)^q &= \{T\,v_{i_2}{:}U_2^1, T\,v_{i_2}{:}U_2^2, \ldots\} \subseteq \langle\!\langle T\,\Box_{i_2} U_2 \rangle\!\rangle \\
(T\,U_2)^q &= \{T\,\hat{U}_2^1, T\,\hat{U}_2^2, \ldots\} \subseteq \langle\!\langle T\,U_2 \rangle\!\rangle \\
&\vdots \\
(F\,X^*)^q &= \{F\,X^{*1}, F\,X^{*2}, \ldots\} \subseteq \langle\!\langle F\,X^* \rangle\!\rangle \\
(T\,\Box_m X^{**})^q &= \{T\,v_m{:}X^{**1}, T\,v_m{:}X^{**2}, \ldots\} \subseteq \langle\!\langle T\,\Box_m X^{**} \rangle\!\rangle
\end{aligned} \tag{2}$$

Since each $T\,v_{i_a}{:}U_a^b \in \langle\!\langle T\,\Box_{i_a} U_a \rangle\!\rangle$ then using the definition of $\langle\!\langle \cdot \rangle\!\rangle$, each $T\,U_a^b \in \langle\!\langle T\,U_a \rangle\!\rangle$. Similarly each $T\,X^{**b} \in \langle\!\langle T\,X^{**} \rangle\!\rangle$.

Justification soundness for the consequent tells us the conjunction of the T-signed quasi-realizers in the consequent implies the disjunction of the F-signed ones, so we have the following.

$$\begin{aligned}
&v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots \\
&\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad \wedge \ldots \\
&\qquad\qquad \wedge v_m{:}X^{**1} \wedge v_m{:}X^{**2} \wedge \ldots \vdash_{\mathsf{JGL}} X^{*1} \vee X^{*2} \vee \ldots
\end{aligned}$$

The Deduction Theorem holds for JGL and gives us the following.

$$\begin{aligned}
&v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots \\
&\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots \\
&\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGL}} [v_m{:}X^{**1} \wedge v_m{:}X^{**2} \wedge \ldots] \supset [X^{*1} \vee X^{*2} \vee \ldots]
\end{aligned} \tag{3}$$

We have Condensing available, as described in Section 5. By (2) we have $\{F\,X^{*1}, F\,X^{*2}, \ldots\} \subseteq \langle\!\langle F\,X^* \rangle\!\rangle$ so there is a substitution σ^* that lives on X^*, and a formula $X^{*\prime}$ so that

$$F\,X^{*\prime} \in [\![F\,X^*]\!] \text{ and } \vdash_{\mathsf{JGL}} (X^{*1} \vee X^{*2} \vee \ldots)\sigma^* \supset X^{*\prime}. \tag{4}$$

Likewise, because $\{T\,v_m{:}X^{**1}, T\,v_m{:}X^{**2}, \ldots\} \subseteq \langle\!\langle T\,\Box_m X^{**} \rangle\!\rangle$, there is a substitution σ^{**} that lives on $\Box_m X^{**}$, and a formula $v_m{:}X^{**\prime}$ so that

$$T\,v_m{:}X^{**\prime} \in [\![T\,\Box_m X^{**}]\!] \text{ and } \vdash_{\mathsf{JGL}} v_m{:}X^{**\prime} \supset (v_m{:}X^{**1} \wedge v_m{:}X^{**2} \wedge \ldots)\sigma^{**}. \tag{5}$$

Moreover both σ^* and σ^{**} meet the no new variable condition.

We have (3) and since theorems of justification logics are closed under substitution, though with a change of constant specification, we also have the following.

$$\begin{aligned}
&[v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots \\
&\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots \\
&\qquad\qquad \wedge \ldots]\sigma^*\sigma^{**} \\
&\qquad\qquad\qquad \vdash_{\mathsf{JGL}} \big[[v_m{:}X^{**1} \wedge\, v_m{:}X^{**2} \wedge \ldots] \supset [X^{*1} \vee X^{*2} \vee \ldots]\big]\, \sigma^*\sigma^{**}
\end{aligned}$$

By the newness conditions on annotations, since $\sigma**$ lives on $\Box_m X^{**}$ it must live away from X^*, and since σ^* lives on X^* it must live away from $\Box_m X^{**}$. It follows from Theorem 5.5 that σ^* and σ^{**} commute. Thus we have the following.

$$\begin{aligned}
&\Big[v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots \\
&\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots \\
&\qquad\qquad \wedge \ldots \Big]\, \sigma^{**}\sigma^* \\
&\qquad\qquad\qquad \vdash_{\mathsf{JGL}} [v_m{:}X^{**1} \wedge v_m{:}X^{**2} \wedge \ldots]\sigma^{**}\sigma^* \supset [X^{*1} \vee X^{*2} \vee \ldots]\sigma^*\sigma^{**}
\end{aligned} \tag{6}$$

We also have (5) hence

$$\vdash_{\mathsf{JGL}} (v_m{:}X^{**\prime})\sigma^* \supset (v_m{:}X^{**1} \wedge v_m{:}X^{**2} \wedge \ldots)\sigma^{**}\sigma^* \tag{7}$$

and we have (4) hence

$$\vdash_{\mathsf{JGL}} (X^{*1} \vee X^{*2} \vee \ldots)\sigma^*\sigma^{**} \supset (X^{*\prime})\sigma^{**} \tag{8}$$

Combining (6), (7), and (8), we have

$$\begin{aligned}
&[v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots \\
&\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots \\
&\qquad\qquad \wedge \ldots]\sigma^{**}\sigma^* \vdash_{\mathsf{JGL}} (v_m{:}X^{**\prime})\sigma^* \supset (X^{*\prime})\sigma^{**}
\end{aligned}$$

or equivalently

$$\begin{aligned}
&(v_{i_1}{:}U_1^1)\sigma^{**}\sigma^* \wedge (v_{i_1}{:}U_1^2)\sigma^{**}\sigma^* \wedge \ldots \wedge (\hat{U}_1^1)\sigma^{**}\sigma^* \wedge (\hat{U}_1^2)\sigma^{**}\sigma^* \wedge \ldots \\
&\wedge (v_{i_2}{:}U_2^1)\sigma^{**}\sigma^* \wedge (v_{i_2}{:}U_2^2)\sigma^{**}\sigma^* \wedge \ldots \wedge (\hat{U}_2^1)\sigma^{**}\sigma^* \wedge (\hat{U}_2^2)\sigma^{**}\sigma^* \wedge \ldots \\
&\qquad\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGL}} (v_m{:}X^{**\prime})\sigma^* \supset (X^{*\prime})\sigma^{**}
\end{aligned} \tag{9}$$

This can be cleaned up in an important way. $\Box_m X^{**}$, X^*, and $\Box_{i_1} U_1$ share no annotations, since $\Box_m X^{**}$ and X^* were introduced with new and distinct annotations. Also σ^{**} lives on $\Box_m X^{**}$ and σ^* lives on X^*. Since $T\, v_{i_1}{:}U_1^1 \in \langle\!\langle T\, \Box_{i_1} U_1 \rangle\!\rangle$, it follows that v_{i_1} is not in the domain of σ^* or of σ^{**} so $v_{i_1}\sigma^* = v_{i_1}\sigma^{**} = v_{i_1}$. Thus $v_{i_1}\sigma^{**}\sigma^* = v_{i_1}$. The situation with v_{i_2} is similar, and so on, so (9) is equivalent to the following.

$$\begin{aligned}
&v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(U_1^2\sigma^{**}\sigma^*) \wedge \ldots \wedge (\hat{U}_1^1\sigma^{**}\sigma^*) \wedge (\hat{U}_1^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\wedge v_{i_2}{:}(U_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(U_2^2\sigma^{**}\sigma^*) \wedge \ldots \wedge (\hat{U}_2^1\sigma^{**}\sigma^*) \wedge (\hat{U}_2^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\qquad\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGL}} (v_m\sigma^*{:}X^{**\prime}\sigma^*) \supset (X^{*\prime})\sigma^{**}
\end{aligned} \tag{10}$$

Applying the Lifting Lemma, Proposition 5.1, to (10), there is some justification term t so that

$$\begin{aligned}&v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(U_1^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_1}{:}(\hat{U}_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(\hat{U}_1^2\sigma^{**}\sigma^*) \wedge \ldots \\ &\wedge v_{i_2}{:}(U_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(U_2^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_2}{:}(\hat{U}_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(\hat{U}_2^2\sigma^{**}\sigma^*) \wedge \ldots \\ &\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGL}} t{:}[(v_m\sigma^*{:}X^{**\prime}\sigma^*) \supset (X^{*\prime})\sigma^{**}]\end{aligned} \tag{11}$$

Now $T\,v_m : X^{**\prime} \in [\![T\,\Box_m X^{**}]\!]$ and σ^* lives away from $\Box_m X^{**}$, so by Theorem 5.4, $T\,(v_m : X^{**\prime})\sigma^* = T\,v_m{:}(X^{**\prime})\sigma^* \in [\![T\,\Box_m X^{**}]\!]$ and so $T\,X^{**\prime}\sigma^* \in [\![T\,X^{**}]\!]$. Similarly $F\,X^{*\prime} \in [\![F\,X^*]\!]$, so $F\,X^{*\prime}\sigma^{**} \in [\![F\,X^*]\!]$. Both X^* and X^{**} are annotation variants of X. It follows, using the forgetful functor, that $(X^{**\prime}\sigma^*)^\circ = (X^{*\prime}\sigma^{**})^\circ$. Also, let $F\,X^t$ be the *trivial* potential quasi-realizer for $F\,X$, as described earlier. Then $(X^t)^\circ = (X^{**\prime}\sigma^*)^\circ = (X^{*\prime}\sigma^{**})^\circ$ too. Thus

$$t{:}[(v_m\sigma^*){:}(X^{**\prime}\sigma^*) \supset (X^{*\prime}\sigma^{**})] \supset [gl(t)]{:}X^t \tag{12}$$

is an instance of the JGL axiom scheme from Section 2, and so from (11) and (12) we have the following.

$$\begin{aligned}&v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(U_1^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_1}{:}(\hat{U}_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(\hat{U}_1^2\sigma^{**}\sigma^*) \wedge \ldots \\ &\wedge v_{i_2}{:}(U_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(U_2^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_2}{:}(\hat{U}_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(\hat{U}_2^2\sigma^{**}\sigma^*) \wedge \ldots \\ &\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGL}} [gl(t)]{:}X^t\end{aligned} \tag{13}$$

We recall our goal: to produce a set of potential quasi-realizers for the tableau block

$$\{T\,\Box_{i_1}U_1, T\,\Box_{i_2}U_2, \ldots, T\,V_1, T\,V_2, \ldots, F\,W_1, F\,W_2, \ldots, F\,\Box_n X\}$$

that is justification sound. We are almost there.

We have that $T\,v_{i_1}{:}U_1^1 \in \langle\!\langle T\,\Box_{i_1}U_1\rangle\!\rangle$, so by item 2 of Theorem 5.5, $T\,(v_{i_1}{:}U_1^1)\sigma^{**}\sigma^* \in \langle\!\langle T\,\Box_{i_1}U_1\rangle\!\rangle$, that is, $T\,v_{i_1} : (U_1^1\sigma^{**}\sigma^*) \in \langle\!\langle T\,\Box_{i_1}U_1\rangle\!\rangle$. Further, $T\,\hat{U}_1^1 \in \langle\!\langle T\,U_1\rangle\!\rangle$, so $T\,(\hat{U}_1^1)\sigma^{**}\sigma^* \in \langle\!\langle T\,U_1\rangle\!\rangle$, and hence $T\,(v_{i_1}{:}\hat{U}_1^1)\sigma^{**}\sigma^* \in \langle\!\langle T\,\Box_{i_1}U_1\rangle\!\rangle$. Similar results hold for $T\,\Box_{i_2}U_2$, and so on.

$F\,X^t$ is the trivial potential quasi-realizer for $F\,X$, so $F\,X^t \in \langle\!\langle F\,X\rangle\!\rangle$, and then $F\,[gl(t)]{:}X^t \in \langle\!\langle F\,\Box_n X\rangle\!\rangle$.

We thus have potential quasi-realizer sets corresponding to all of $T\,\Box_{i_1}U_1, T\,\Box_{i_2}U_2, \ldots$ and to $F\,\Box_n X$. For $T\,V_1, T\,V_2, \ldots$ and $F\,W_1, F\,W_2, \ldots$ *any* potential quasi-realizers will do, and we use trivial ones. These simply add conjuncts to the left of (13) and disjuncts to the right, and so don't affect provability.

We have now constructed a justification sound set of potential quasi-realizers for the premises of (1).

All other steps of the Quasi-Realization algorithm for GL are the same as their counterparts for S4.

7 An Example Continued

In Example 4.2 we gave an annotated block tableau proof in GL. We use the algorithm of Section 6 and convert the tableau to a justification sound quasi-realization tree, from which a provable realization for $\Box(\Box(X \vee Y) \supset (X \wedge Z)) \supset \Box X$ will be produced.

Example 7.1 The GL annotated block tableau from Example 4.1 converts into a Quasi-Realization tree as follows.

$$\{F\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X,\\ F\, v_1{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X\}\ 1.$$

$$\{T\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)),\, T\, v_1{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z))\},\, \{F\, gl(b){:}X\}\ 2.$$

$$\{T\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))\},\, \{T\, gl(a){:}(X \vee Y) \supset (X \wedge Z)\},\, \{F\, X\},\, \{T\, v_4{:}X\}\ 3.$$

$$\{T\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))\},\\ \{F\, gl(a){:}(X \vee Y)\},\, \{F\, X\},\, \{T\, v_4{:}X\}\ 4.$$

$$\{T\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))\},\\ \{T\, v_2{:}(X \vee Y) \supset (X \wedge Z)\},\\ \{F\, X \vee Y\}, \{T\, v_5{:}(X \vee Y)\}, \{T\, v_4{:}X\}, T\, X\ \ 7.$$

$$\{T\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))\},\\ \{T\, v_2{:}(X \vee Y) \supset (X \wedge Z)\},\\ \{F\, X\}, \{F\, Y\}, \{T\, v_5{:}(X \vee Y)\}, \{T\, v_4{:}X\}, \{T\, X\}\ \ 8.$$

$$\{T\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))\},\\ \{T\, X \wedge Z\}, \{F\, X\}, \{T\, v_4{:}X\}\ \ 5.$$

$$\{T\, v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))\},\\ \{T\, X\}, \{T\, Z\}, \{F\, X\},\\ \{T\, v_4{:}X\}\ \ 6.$$

The justification term a is given by the Lifting Lemma (see the explanation below for 4), and is such that

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_4{:}X \vdash_{\mathsf{JGL}} a{:}(v_5{:}(X \vee Y) \supset (X \vee Y))$$

and the justification term b is also from the Lifting Lemma, and is such that

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_1{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z)) \vdash_{\mathsf{JGL}} b{:}(v_4{:}X \supset X).$$

Here is a step-by-step explanation for a few representative parts of this, matching steps in this tree with those from the tree in Example 4.1. We refer to node 8, say, in the tree above as 7.1.8, and to the corresponding node in the original modal tableau as 4.1.8.

8. 7.1.8 is a trivial quasi-realization of 4.1.8, and is justification sound because $X \vdash_{\mathsf{JGL}} X$.

7. 7.1.7 is a potential quasi-realization of 4.1.7 and is justification sound because it simply replaces $F\, X$ and $F\, Y$ from 7.1.8 with $F\, X \vee Y$.

4. 4.1.7 comes from 4.1.4 using the $F\,\Box$ tableau rule, the one peculiar to GL. Since 7.1.7 is justification sound, we have

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_2{:}(X \vee Y) \supset (X \wedge Y), v_5{:}(X \vee Y), v_4{:}X, X \vdash_{\mathsf{JGL}} (X \vee Y)$$

and hence

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_2{:}(X \vee Y) \supset (X \wedge Y), v_4{:}X, X \vdash_{\mathsf{JGL}} v_5{:}(X \vee Y) \supset (X \vee Y).$$

Using the Lifting Lemma, there is a justification term a so that

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Y)), v_4{:}X, v_4{:}X \vdash_{\mathsf{JGL}} a{:}(v_5{:}(X \vee Y) \supset (X \vee Y))$$

or more simply

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_4{:}X \vdash_{\mathsf{JGL}} a{:}(v_5{:}(X \vee Y) \supset (X \vee Y)),$$

Now an instance of the GL Axiom is $a{:}(v_5{:}(X \vee Y) \supset (X \vee Y)) \supset [gl(a)]{:}(X \vee Y)$, and so we have

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_4{:}X \vdash_{\mathsf{JGL}} gl(a){:}(X \vee Y)$$

from which justification soundness of 7.1.4 follows.

This is as far as we fill in reasons.

Example 7.1 tells us that a provable quasi-realization for $\Box_1(\Box_2(X \vee Y) \supset (X \wedge Z)) \supset \Box_3 X$, or unannotated $\Box(\Box(X \vee Y) \supset (X \wedge Z)) \supset \Box X$, is the following.

$$[v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X] \vee [v_1{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X)] \tag{14}$$

Example 7.2 Example 7.1 gave us a quasi-realization. That converts to a realization using the algorithm found in [8, 5]. We, somewhat informally, apply the steps of that algorithm now, to finish the example off.

We have the following.

$\vdash_{\mathsf{JGL}} v_2{:}(X \vee Y) \supset [v_2 + gl(a)]{:}(X \vee Y)$
$\vdash_{\mathsf{JGL}} gl(a){:}(X \vee Y) \supset [v_2 + gl(a)]{:}(X \vee Y)$
and hence
$([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset (v_2{:}(X \vee Y) \supset (X \wedge Z))$
$([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset (gl(a){:}(X \vee Y) \supset (X \wedge Z))$
then using the Lifting Lemma there are c and d so that
$c{:}[([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset (v_2{:}(X \vee Y) \supset (X \wedge Z))]$
$d{:}[([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset (gl(a){:}(X \vee Y) \supset (X \wedge Z))]$
and so
$v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset [c \cdot v_1]{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))$
$v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset [d \cdot v_1]{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z))$
and then finally
$v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset [c \cdot v_1 + d \cdot v_1]{:}(v_2{:}(X \vee Y) \supset (X \wedge Z))$
$v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset [c \cdot v_1 + d \cdot v_1]{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z))$.
From these we get

$$\{[c \cdot v_1 + d \cdot v_1]{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X\} \\ \supset \{v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X\} \tag{15}$$

$$\{[c \cdot v_1 + d \cdot v_1]{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X\} \\ \supset \{v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X\}. \tag{16}$$

Now from (14) we have

$$\vdash_{\mathsf{JGL}} [v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X] \vee [v_1{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X)]$$

and since provability is closed under substitution (though the constant specification may change), we can replace v_1 with $c \cdot v_1 + d \cdot v_1$ getting

$$[[c \cdot v_1 + d \cdot v_1]{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X] \vee [[c \cdot v_1 + d \cdot v_1]{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X)]$$

and from this and (15) and (16) we have

$$\vdash_{\mathsf{JGL}} v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X$$

and we have a provable realization for $\Box(\Box(X \vee Y) \supset (X \wedge Z)) \supset \Box X$.

$$\frac{\mathcal{B}, T\,\Box X}{\mathcal{B}, F\,X} \quad \frac{\mathcal{B}, F\,\Box X}{\mathcal{B}^\sharp, F\,X}$$

where $\mathcal{B}^\sharp = \{T\,Y, T\,\Box Y \mid T\,\Box Y \in \mathcal{B}\}$.

Figure 4: S4 Block Tableau Rules

8 Grzegorczyk Logic

The basic ideas that applied to Gödel-Löb logic turn out to apply to Grzegorczyk logic as well. See [7] for a brief sketch of some history. Once these ideas are formulated, a proof of realization in the style used here is very similar to that connecting GL and JGL. We give details so they can be checked, but much is just taken over quite directly, with apologies for any repetition.

Grzegorczyk logic is most commonly axiomatized by adding to S4 the scheme $\Box(\Box(X \supset \Box X) \supset X) \supset X$. Instead of this scheme we use $\Box(\Box(X \supset \Box X) \supset X) \supset \Box X$, which is easily seen to axiomatize the same logic when S4 is the background. We use this because we feel it corresponds to a more 'natural' justification counterpart. As a tableau system we use a standard block tableau set of S4 rules. These consist of the propositional connective rules from Figure 1, the S4 rules from Figure 4, and the Grzegorczyk rule in Figure 5. This is a variant of the system from [11].

$$\frac{\mathcal{B}, F\,\Box X}{\mathcal{B}^\sharp, F\,X, T\,\Box(X \supset \Box X)}$$

where $\mathcal{B}^\sharp = \{T\,Y, T\,\Box Y \mid T\,\Box Y \in \mathcal{B}\}$.

Figure 5: Grz Block Tableau Rule

The corresponding justification logic is called JGrz and consists of LP, described in [5, 10, 1, 2, 3] and many other places, together with the following, where gz is a new justification function symbol.

Grz Scheme: All formulas of the form $s{:}(t{:}(X_1 \supset u{:}X_2) \supset X_3) \supset [gz(s)]{:}X_4$, where $X_1^\circ = X_2^\circ = X_3^\circ = X_4^\circ$.

And finally, as we did with GL, we use *annotated* block tableaus. Annotations propagate through the Grz rule using the pattern in Figure 6.

$$\frac{\mathcal{B}, F\,\Box_n X}{\mathcal{B}^\sharp, F\,X^*, T\,\Box_m(X^{**} \supset \Box_k X^{***})}$$

where X^*, X^{**}, and X^{***} are like X
but with all annotations replaced by new distinct ones
that have not previously occurred in the tree
and m and k are new and distinct.

Figure 6: Annotated Grz Block Tableau Rule

In a proof of the correctness of a realization algorithm all cases are like those for S4 and LP as in [5] except, of course, the case for the Grz rule which is new. For this, we follow the same pattern we did earlier for GL.

Suppose we have applied the annotated block tableau rule from Figure 6, a counterpart for the consequent has been constructed in the tree of quasi-realizers, and that counterpart is justification sound. We show that it can be converted into a justification sound counterpart of the antecedent. As we did earlier for JGL, let us say that

$$\mathcal{B} = \{T\,\Box_{i_1}U_1, T\,\Box_{i_2}U_2, \ldots, T\,V_1, T\,V_2, \ldots, F\,W_1, F\,W_2, \ldots\}$$

where $V_1, V_2, \ldots$ are not necessitated formulas. Then the rule for this case looks like the following, where X^*, X^{**}, and X^{***} are like X but with all annotations replaced by new distinct ones that do not occur in the antecedent and m and k are new and distinct..

$$\frac{T\,\Box_{i_1}U_1, T\,\Box_{i_2}U_2, \ldots, T\,V_1, T\,V_2, \ldots, F\,W_1, F\,W_2, \ldots, F\,\Box_n X}{T\,\Box_{i_1}U_1, T\,U_1, T\,\Box_{i_2}U_2, T\,U_2, \ldots, F\,X^*, T\,\Box_m(X^{**} \supset \Box_k X^{***})} \tag{17}$$

We assume the consequent is justification sound so for each Z below the line we have a finite subset of $\langle\!\langle Z\rangle\!\rangle$ of potential quasi-realizers for Z. Again as we did with JGL, let us denote by Z^q the finite subset of $\langle\!\langle Z\rangle\!\rangle$. Assume we have the following.

$$\begin{aligned}
(T\,\Box_{i_1}U_1)^q &= \{T\,v_{i_1}{:}U_1^1, T\,v_{i_1}{:}U_1^2, \ldots\} \subseteq \langle\!\langle T\,\Box_{i_1}U_1\rangle\!\rangle \\
(T\,U_1)^q &= \{T\,\hat{U}_1^1, T\,\hat{U}_1^2, \ldots\} \subseteq \langle\!\langle T\,U_1\rangle\!\rangle \\
(T\,\Box_{i_2}U_2)^q &= \{T\,v_{i_2}{:}U_2^1, T\,v_{i_2}{:}U_2^2, \ldots\} \subseteq \langle\!\langle T\,\Box_{i_2}U_2\rangle\!\rangle \\
(T\,U_2)^q &= \{T\,\hat{U}_2^1, T\,\hat{U}_2^2, \ldots\} \subseteq \langle\!\langle T\,U_2\rangle\!\rangle \\
&\vdots \\
(F\,X^*)^q &= \{F\,X^{*1}, F\,X^{*2}, \ldots\} \subseteq \langle\!\langle F\,X^*\rangle\!\rangle \\
(T\,\Box_m(X^{**} \supset \Box_k X^{***}))^q &= \{T\,v_m{:}(X^{**1} \supset v_k{:}X^{***1}), T\,v_m{:}(X^{**2} \supset v_k{:}X^{***2}), \ldots\} \\
&\subseteq \langle\!\langle T\,\Box_m(X^{**} \supset \Box_k X^{***})\rangle\!\rangle
\end{aligned} \tag{18}$$

Note that for each i, $F\,X^{**i} \in \langle\!\langle F\,X^{**}\rangle\!\rangle$ and $T\,X^{***i} \in \langle\!\langle T\,X^{***}\rangle\!\rangle$. Justification soundness of the consequent says the following.

$$\begin{aligned}
v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots \\
\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots \\
\wedge \ldots \\
\wedge v_m{:}(X^{**1} \supset v_k{:}X^{***1}) \wedge v_m{:}(X^{**2} \supset v_m{:}X^{***2}) \wedge \ldots \vdash_{\mathsf{JGrz}} X^{*1} \vee X^{*2} \vee \ldots
\end{aligned} \tag{19}$$

We have Condensing available, Theorem 5.7. By (18) we have $\{F\,X^{*1}, F\,X^{*2}, \ldots\} \subseteq \langle\!\langle F\,X^*\rangle\!\rangle$ so there is a substitution σ^* that lives on X^*, and a formula $X^{*\prime}$ so that

$$F\,X^{*\prime} \in [\![F\,X^*]\!] \text{ and } \vdash_{\mathsf{JGrz}} (X^{*1} \vee X^{*2} \vee \ldots)\sigma^* \supset X^{*\prime}. \tag{20}$$

Likewise, because

$$\{T\,v_m{:}(X^{**1} \supset v_k{:}X^{***1}), T\,v_m{:}(X^{**2} \supset v_k{:}X^{***2}), \ldots\} \subseteq \langle\!\langle T\,\Box_m(X^{**} \supset \Box_k X^{***})\rangle\!\rangle$$

there is a substitution σ^{**} that lives on $\Box_m(X^{**} \supset \Box_k X^{***})$, and a formula $v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime})$ so that

$$\begin{aligned}
&T\,v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime}) \in [\![T\,\Box_m(X^{**} \supset \Box_k X^{***})]\!] \text{ and} \\
&\vdash_{\mathsf{JGrz}} v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime}) \supset [v_m{:}(X^{**1} \supset v_k{:}X^{***1}) \wedge v_m{:}(X^{**2} \supset v_k{:}X^{***2}) \wedge \ldots)]\sigma^{**}.
\end{aligned} \tag{21}$$

Moreover both σ^* and σ^{**} meet the no new variable condition.

We have (19) so we also have the following.

$$\begin{aligned}
&\Big[v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots\\
&\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots\\
&\wedge \ldots\\
&\wedge v_m{:}(X^{**1} \supset v_k{:}X^{***1}) \wedge v_m{:}(X^{**2} \supset v_m{:}X^{***2}) \wedge \ldots\Big]\,\sigma^*\sigma^{**} \vdash_{\mathsf{JGrz}} \big[X^{*1} \vee X^{*2} \vee \ldots\big]\,\sigma^*\sigma^{**}
\end{aligned}\tag{22}$$

Since σ^{**} lives on $\Box_m(X^{**} \supset \Box_k X^{***})$ it must live away from X^*, and since σ^* lives on X^* it must live away from $\Box_m(X^{**} \supset \Box_k X^{***})$. Then σ^* and σ^{**} commute and we have the following.

$$\begin{aligned}
&\Big[v_{i_1}{:}U_1^1 \wedge v_{i_1}{:}U_1^2 \wedge \ldots \wedge \hat{U}_1^1 \wedge \hat{U}_1^2 \wedge \ldots\\
&\wedge v_{i_2}{:}U_2^1 \wedge v_{i_2}{:}U_2^2 \wedge \ldots \wedge \hat{U}_2^1 \wedge \hat{U}_2^2 \wedge \ldots\\
&\wedge \ldots\\
&\wedge v_m{:}(X^{**1} \supset v_k{:}X^{***1}) \wedge v_m{:}(X^{**2} \supset v_m{:}X^{***2}) \wedge \ldots\Big]\,\sigma^{**}\sigma^* \vdash_{\mathsf{JGrz}} \big[X^{*1} \vee X^{*2} \vee \ldots\big]\,\sigma^*\sigma^{**}
\end{aligned}\tag{23}$$

We also have (21) hence

$$\vdash_{\mathsf{JGrz}} v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime})\sigma^* \supset [v_m{:}(X^{**1} \supset v_k{:}X^{***1}) \wedge v_m{:}(X^{**2} \supset v_k{:}X^{***2}) \wedge \ldots)]\sigma^{**}\sigma^*. \tag{24}$$

and we have (20) hence

$$\vdash_{\mathsf{JGrz}} (X^{*1} \vee X^{*2} \vee \ldots)\sigma^*\sigma^{**} \supset X^{*\prime}\sigma^{**}. \tag{25}$$

Combining (23), (24), and (25), we have

$$\begin{aligned}
&(v_{i_1}{:}U_1^1)\sigma^{**}\sigma^* \wedge (v_{i_1}{:}U_1^2)\sigma^{**}\sigma^* \wedge \ldots \wedge (\hat{U}_1^1)\sigma^{**}\sigma^* \wedge (\hat{U}_1^2)\sigma^{**}\sigma^* \wedge \ldots\\
&\wedge(v_{i_2}{:}U_2^1)\sigma^{**}\sigma^* \wedge (v_{i_2}{:}U_2^2)\sigma^{**}\sigma^* \wedge \ldots \wedge (\hat{U}_2^1)\sigma^{**}\sigma^* \wedge (\hat{U}_2^2)\sigma^{**}\sigma^* \wedge \ldots\\
&\wedge \ldots\\
&\wedge v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime})\sigma^* \vdash_{\mathsf{JGrz}} X^{*\prime}\sigma^{**}
\end{aligned}$$

Using the Deduction Theorem, this gives us the following.

$$\begin{aligned}
&(v_{i_1}{:}U_1^1)\sigma^{**}\sigma^* \wedge (v_{i_1}{:}U_1^2)\sigma^{**}\sigma^* \wedge \ldots \wedge (\hat{U}_1^1)\sigma^{**}\sigma^* \wedge (\hat{U}_1^2)\sigma^{**}\sigma^* \wedge \ldots\\
&\wedge(v_{i_2}{:}U_2^1)\sigma^{**}\sigma^* \wedge (v_{i_2}{:}U_2^2)\sigma^{**}\sigma^* \wedge \ldots \wedge (\hat{U}_2^1)\sigma^{**}\sigma^* \wedge (\hat{U}_2^2)\sigma^{**}\sigma^* \wedge \ldots\\
&\wedge \ldots\\
&\vdash_{\mathsf{JGrz}} v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime})\sigma^* \supset X^{*\prime}\sigma^{**}
\end{aligned}\tag{26}$$

Since X^* and $\Box_m(X^{**} \supset \Box_k X^{***})$ were introduced with new annotations, they share no annotations with $\Box_{i_1} U_1$. Since σ^* lives on X^* it lives away from $\Box_{i_1} U_1$, and similarly for σ^{**}. It follows that $v_{i_1}\sigma^{**}\sigma^* = v_{i_1}$. Similarly for v_{i_2}, and so on. Then (26) is equivalent to the following.

$$\begin{aligned}
&v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(U_1^2\sigma^{**}\sigma^*) \wedge \ldots \wedge (\hat{U}_1^1\sigma^{**}\sigma^*) \wedge (\hat{U}_1^2\sigma^{**}\sigma^*) \wedge \ldots\\
&\wedge v_{i_2}{:}(U_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(U_2^2\sigma^{**}\sigma^*) \wedge \ldots \wedge (\hat{U}_2^1\sigma^{**}\sigma^*) \wedge (\hat{U}_2^2\sigma^{**}\sigma^*) \wedge \ldots\\
&\wedge \ldots\\
&\vdash_{\mathsf{JGrz}} v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime})\sigma^* \supset X^{*\prime}\sigma^{**}
\end{aligned}\tag{27}$$

Applying the Lifting Lemma, Proposition 5.1, to (27), there is some justification term t so that

$$\begin{aligned}
&v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(U_1^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_1}{:}(\hat{U}_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(\hat{U}_1^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\wedge v_{i_2}{:}(U_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(U_2^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_2}{:}(\hat{U}_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(\hat{U}_2^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGrz}} t{:}[v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime})\sigma^* \supset X^{*\prime}\sigma^{**}]
\end{aligned} \tag{28}$$

Substitution σ^* lives on X^* which does not share indexes with $\Box_m(X^{**} \supset \Box_k X^{***})$, so σ^* lives away from $\Box_m(X^{**} \supset \Box_k X^{***})$. Then $v_m\sigma^* = v_m$ and $v_k\sigma^* = v_k$. It follows that $t{:}[v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime})\sigma^* \supset X^{*\prime}\sigma^{**}] = t{:}[v_m{:}(X^{**\prime}\sigma^* \supset v_k{:}X^{***\prime}\sigma^*) \supset X^{*\prime}\sigma^{**}]$. Thus we have the following.

$$\begin{aligned}
&v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(U_1^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_1}{:}(\hat{U}_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(\hat{U}_1^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\wedge v_{i_2}{:}(U_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(U_2^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_2}{:}(\hat{U}_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(\hat{U}_2^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGrz}} t{:}[v_m{:}(X^{**\prime}\sigma^* \supset v_k{:}X^{***\prime}\sigma^*) \supset X^{*\prime}\sigma^{**}]
\end{aligned} \tag{29}$$

Now $T\, v_m{:}(X^{**\prime} \supset v_k{:}X^{***\prime}) \in \llbracket T\, \Box_m(X^{**} \supset \Box_k X^{***})\rrbracket$, and it follows that $F\, X^{**\prime} \in \llbracket F\, X^{**}\rrbracket$. Then by Theorem 5.4, $F\, X^{**\prime}\sigma^* \in \llbracket F\, X^{**}\rrbracket$. X^{**} is an annotation variant of X, and so $(X^{**\prime}\sigma^*)^\circ = X$. Similarly for $X^{***\prime}\sigma^*$ and $X^{*\prime}\sigma^{**}$. Now let $F\, X^t$ be the trivial potential quasi-realizer for $F\, X$. Then since $(X^{**\prime}\sigma^*)^\circ = (X^{***\prime}\sigma^*)^\circ = (X^{*\prime}\sigma^{**})^\circ = (X^t)^\circ$,

$$t{:}[v_m{:}(X^{**\prime}\sigma^* \supset v_k{:}X^{***\prime}\sigma^*) \supset X^{*\prime}\sigma^{**}] \supset [gz(t)]{:}X^t \tag{30}$$

is an instance of the JGrz axiom scheme. Now from (29) and (30) we have the following.

$$\begin{aligned}
&v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(U_1^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_1}{:}(\hat{U}_1^1\sigma^{**}\sigma^*) \wedge v_{i_1}{:}(\hat{U}_1^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\wedge v_{i_2}{:}(U_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(U_2^2\sigma^{**}\sigma^*) \wedge \ldots \wedge v_{i_2}{:}(\hat{U}_2^1\sigma^{**}\sigma^*) \wedge v_{i_2}{:}(\hat{U}_2^2\sigma^{**}\sigma^*) \wedge \ldots \\
&\qquad\qquad \wedge \ldots \vdash_{\mathsf{JGrz}} [gz(t){:}X^t]
\end{aligned} \tag{31}$$

We now almost have a set of potential quasi-realizers for

$$T\, \Box_{i_1}U_1, T\, \Box_{i_2}U_2, \ldots, T\, V_1, T\, V_2, \ldots, F\, W_1, F\, W_2, \ldots, F\, \Box_n X$$

that is justification sound.

We have that $T\, v_{i_1}{:}U_1^1 \in \langle\!\langle T\, \Box_{i_1}U_1\rangle\!\rangle$, so by item 2 of Theorem 5.5, $T\,(v_{i_1}{:}U_1^1)\sigma^{**}\sigma^* \in \langle\!\langle T\, \Box_{i_1}U_1\rangle\!\rangle$, that is, $T\, v_{i_1}{:}(U_1^1\sigma^{**}\sigma^*) \in \langle\!\langle T\, \Box_{i_1}U_1\rangle\!\rangle$. Further, $T\, \hat{U}_1^1 \in \langle\!\langle T\, U_1\rangle\!\rangle$, so $T\,(\hat{U}_1^1)\sigma^{**}\sigma^* \in \langle\!\langle T\, U_1\rangle\!\rangle$, and hence $T\,(v_{i_1}{:}\hat{U}_1^1)\sigma^{**}\sigma^* \in \langle\!\langle T\, \Box_{i_1}U_1\rangle\!\rangle$. Similar results hold for $T\, \Box_{i_2}U_2$, and so on. $F\, X^t$ is the trivial potential quasi-realizer for $F\, X$, so $F\, X^t \in \langle\!\langle F\, X\rangle\!\rangle$, and then $F\,[gz(t)]{:}X^t \in \langle\!\langle F\, \Box_n X\rangle\!\rangle$.

We thus have potential quasi-realizer sets corresponding to all of $T\, \Box_{i_1}U_1, T\, \Box_{i_2}U_2, \ldots$ and to $F\, \Box_n X$. For $T\, V_1, T\, V_2, \ldots$ and $F\, W_1, F\, W_2, \ldots$ *any* potential quasi-realizers will do, and we use trivial ones. These simply add conjuncts to the left of (31) and disjuncts to the right, and so don't affect provability.

We have now constructed a justification sound set of potential quasi-realizers for the premises of (17).

9 Comments and Observations

Our comments generally apply to both Gödel-Löb logic and to Grzegorczyk logic. We have stated them primarily for the first case, to keep the wording of sentences simple.

The first comment has to do with a curious aspect of the JGL axiom scheme, $t{:}(u{:}X_1 \supset X_2) \supset [gl(t)]{:}X_3$ where $X_1^\circ = X_2^\circ = X_3^\circ$. The justification term u seems to play a vanishing role, since the function symbol g only makes use of t. (A similar remark can be made about t and u in the Grz scheme.) But in fact u does play a role, though it is a hidden one. The justification term t will, in practice, depend on it. Consider Example 7.2, which continues Examples 7.1 and 4.1, in which a provable realization is produced for $\Box(\Box(X \vee Y) \supset (X \wedge Z)) \supset \Box X$. This realization is $v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X$. In the justification term $gl(b)$ the term b appears, though b does not explicitly occur elsewhere in the formula. But, b comes from an application of the Lifting Lemma, and is such that

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_1{:}(gl(a){:}(X \vee Y) \supset (X \wedge Z)) \vdash_{\mathsf{JGL}} b{:}(v_3{:}X \supset X).$$

Thus b depends on v_1, v_2, v_3, and $gl(a)$, and hence on a. In turn a also comes from the Lifting Lemma as follows.

$$v_1{:}(v_2{:}(X \vee Y) \supset (X \wedge Z)), v_4{:}X \vdash_{\mathsf{JGL}} a{:}(v_5{:}(X \vee Y) \supset (X \vee Y))$$

Then a depends on v_1, v_2, v_4 and v_5.

Thus in our realizer, $v_1{:}([v_2 + gl(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset gl(b){:}X$, the term $gl(b)$ has, in fact, a dependence on both v_4 and v_5. They have disappeared, but the roles they played are, somehow, hidden in the background.

The second comment has to do with what we think the justification terms for JGL might represent. In fact, I don't know. The first justification logic was LP and for it, justifications were to be thought of as representing formal arithmetic proofs—something embodied precisely in Artemov's Arithmetic Completeness Theorem. For JGL this interpretation will not work.

The formula $\Box\neg\Box X \supset \Box X$ is a theorem of GL. There is a provable realization of it in JGL as follows,

$$t{:}\neg u{:}X \supset [gl(a \cdot t)]{:}X$$

where t and u are arbitrary and a justifies $\neg u{:}X \supset (u{:}X \supset X)$. Now if JGL justification logic did embed into arithmetic, we could arrange to map u to the Gödel number of a syntactically correct axiomatic proof that did not have X as one of its lines, independently of whether or not X maps to an arithmetic theorem. This is a simple syntax matter, so there should be a term t that verifies that u is not a proof of X, and hence $t{:}\neg u{:}X$ should map to true. But then there would be a term $gl(a \cdot t)$ that should map to a proof of X. Roughly, every X would have a proof.

The problem arises from the fact that, using the standard translation of GL into arithmetic, $\Box\neg\Box X$ maps into a formula involving existential quantifiers, one of which is negated. Its translation would say that there is a proof that X does not have a proof. If JGL justification terms represented proofs, $t{:}\neg u{:}X$ would say there is a proof that some explicit proof is not a proof of X, but it does not rule out the possibility of something else being a proof.

In short, JGL terms cannot simply be thought of as coding formal arithmetic proofs. I do not, in fact, know how to think of them.

Our third comment is more technical. Generally justification axioms parallel modal axioms quite closely. For instance, $\Box X \supset X$ has as its usual justification counterpart $t{:}X \supset X$, where the two occurrences of X represent the same justification formula. JGL introduces a new pattern since in $t{:}(u{:}X_1 \supset X_2) \supset [gl(t)]{:}X_3$ it is not required that X_1, X_2, and X_3 be the same, but only that $X_1^\circ = X_2^\circ = X_3^\circ$; that is, they 'forget back' to the same thing. This plays a role in two places in the proof of realization. First, it is used in showing that σ^* and σ^{**} commute, giving us formula

(6). Second, it is used to establish that $v_{i_1}, v_{i_2}, \ldots$ are not in the domain of σ^* or of σ^{**}, giving us (10). For the first, it is enough that X_1 and X_2 can be different while $X_1^\circ = X_2^\circ$. The argument still works if X_3 is required to be identical with X_1, and this seems somewhat more natural. In fact, this happens in the proof given for Proposition 2.1. Question: Can the requirement be relaxed to be $X_1^\circ = X_2^\circ$ and $X_1 = X_3$, and still admit realization?

Finally, there are two versions of justification logics corresponding to `GL`, the one here and the one from [13]. The underlying mechanism is quite different between the two versions. It would be very interesting to know if the two justification logics have any natural connections to each other.

References

[1] Sergei N. Artemov. “Explicit Provability and Constructive Semantics”. In: *Bulletin of Symbolic Logic* 7.1 (Mar. 2001), pp. 1–36.

[2] Sergei N. Artemov. “The Logic of Justification”. In: *The Review of Symbolic Logic* 1.4 (2008), pp. 477–513.

[3] Sergei N. Artemov and Melvin C. Fitting. *Justification Logic.* Ed. by Edward N. Zalta. 2011, revised 2015. URL: `http://plato.stanford.edu/entries/logic-justification/`.

[4] Sergei N. Artemov and Melvin C. Fitting. “Justification Logic”. In: *The Stanford Encyclopedia of Philosophy.* Ed. by Edward N. Zalta. First publication in 2011. 2015. Chap. `http://plato.stanford.edu/archives/fall2012/entries/logic-justification/`.

[5] Sergei N. Artemov and Melvin C. Fitting. *Justification Logic: Reasoning with Reasons.* Cambridge Tracts in Mathematics Book 216. Cambridge University Press, 2019.

[6] George Boolos. *The Unprovability of Consistency.* Cambridge University Press, 1979.

[7] Roy Dyckhoff and Sara Negri. “A cut-free sequent system for Grzegorczyk logic, with an application to the Gödel-McKinsey-Tarski embedding”. In: *Journal of Logic and Computation* 26 (2016). Advance Access publication July 18, 2013, pp. 169–187.

[8] Melvin C. Fitting. “Modal Logics, Justification Logics, and Realization”. In: *Annals of Pure and Applied Logic* 167 (2016), pp. 615–648.

[9] Melvin C. Fitting. “Quasi-Realization”. In: *Logic, Language, and Computation.* Ed. by Helle Hvid Hansen et al. Vol. 10148. Lecture Notes in Computer Science. 11th International Tbilisi Symposium, TbiLLC, Tbilisi, Georgia, September 21-26, 2015. Springer, 2016, pp. 313–332.

[10] Roman Kuznets and Thomas Studer. *Logics of Proofs and Justifications.* College Publications, 2019.

[11] Wolfgang Rautenberg. “Modal tableau calculi and interpolation”. In: *Journal of Philosophical Logic* 12.4 (1983), pp. 403–423.

[12] Daniyar S. Shamkanov. “Circular Proofs for the Gödel-Löb Provability Logic”. In: *Mat. Zametki* 96.4 (2014). English translation in *Math. Notes* 96:4 (2014), 575-585, pp. 609–622.

[13] Daniyar S. Shamkanov. “A Realization theorem for the Gödel-Löb provability logic”. In: *Russian Academy of Sciences Sbornik Mathematics* 207.9 (2016), pp. 1–17.