A Family of Strict/Tolerant Logics

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Abstract

Strict/tolerant logic, ST, evaluates the premises and the consequences of its consequence relation differently, with the premises held to stricter standards while consequences are treated more tolerantly. More specifically, ST is a three-valued logic with left sides of sequents understood as if in Kleene's Strong Three Valued Logic, and right sides as if in Priest's Logic of Paradox. Surprisingly, this hybrid validates the same sequents that classical logic does. A version of this result has been extended to meta, metameta , . . . consequence levels in [5].

In an earlier paper, [11], I showed that the original ideas behind ST are, in fact, much more general than first appeared, and an infinite family of many valued logics have Strict/Tolerant counterparts. This family includes both Kleene's and Priest's logic individually, as well as first degree entailment. For instance, for both the Kleene and the Priest logic, the corresponding strict/tolerant logic is six-valued, but with differing sets of strictly and tolerantly designated truth values. In the present paper I extend that generalization in two directions. I examine a reverse notion, of Tolerant/Strict logics, which exist for the same structures that were investigated in [11]. And I show that the generalization extends through the meta, metameta, . . . consequence levels for the same infinite family of many valued logics. Finally I close with remarks on the status of cut and related rules, which can actually be rather nuanced. Throughout, the aim is not the philosophical applications of the Strict/Tolerant idea, but the determination of how general a phenomenon it is.

Keywords: Strict/Tolerant, bilattice, many valued logic, Kleene logic, logic of paradox, first degree entailment

1 Introduction

Sequents are used with a wide variety of logics, $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas, but the basic idea is generally the same: if all members of Γ are true then some member of Δ is true. Or what is usually equivalent, if the conjunction of Γ is true then the disjunction of Δ is true. Of course the meaning of "true" is logic dependent, but whatever its meaning is it applies on both sides of the $\Rightarrow$ symbol. A case has been made, in [6] for instance, that the antecedents (formulas on the left) of a sequent should be held to higher standards than the consequents (formulas on the right). Consider physics as an example. We have some theory, relativity, quantum, Newtonian, whatever, and in deducing things from such a theory we adopt as premises the 'laws' of the theory, which are understood as simply true. We derive consequences, and in testing these we examine the universe using our scientific instruments, which have non-zero margins of error. Premises are taken to be strictly true; consequences are evaluated as true with some leeway allowed. In fact, holding premises and consequences to different standards is commonly done without any explicit mention. Perhaps the utility of

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the practice is clearest when sequents are used backwards (or for the present discussion, when derivations in physics are used backwards). If the consequents turn out to be not true in the sense that they assert things that do not hold in the world around us, within a reasonable margin of error, one of our premises must simply fail.

The idea of evaluating the two sides of sequents under different standards has led to much work on what is called *Strict/Tolerant logic*, ST. Both Kleene's strong three-valued logic and Priest's logic of paradox are three-valued logics, though of course the motivation and intended interpretations differ. We can think of them both as being over a common set of truth values, say $\{0, \frac{1}{2}, 1\}$. With this common carrier, the two logics have the same truth tables for conjunction, disjunction, and negation. They differ in their choices of designated truth values: for Kleene this is $\{1\}$ and for Priest it is $\{\frac{1}{2}, 1\}$. Suppose we work with sequents involving formulas that are built up using conjunction, disjunction, and negation, but we have two versions of truth: a stricter notion following Kleene, and a more tolerant version following Priest. Call a sequent valid in the Strict/Tolerant sense, or just valid in ST, provided that under any assignment of truth values from $\{0, \frac{1}{2}, 1\}$, *strict* truth of all antecedents implies *tolerant* truth of some consequent. Remarkably, the sequents valid in ST are exactly the sequents valid classically. That is, classical propositional logic and this Strict/Tolerant logic have the same consequence relation.

If one identifies a logic with its consequence relation, classical logic and ST would be the same, though differently presented. It has been argued that using a Strict/Tolerant formulation would avoid some of the problems of vagueness while keeping what is essentially a classical framework, see [6] and further references found there. The papers [4, 17] have pointed out that the two logics are not actually the same because consequence alone is not enough to look at. Though ST and classical logic agree at the consequence level, they differ at the metaconsequence level, in particular they differ on cut. On the other hand, [14] argues that cut isn't actually part of classical logic after all. This is not as settled an area as one might think, but the controversy is outside our interests here, which are primarily technical, and no more will be said on the subject.

Recently [5] expanded on the distinctions seen in the classical and ST case by considering metaconsequence, metametaconsequence, and so on. They argue that just as classical and Strict/Tolerant logics differ at the metaconsequence level, one can define related logics that agree with classical logic on the meta, metameta, . . . , to any finite level, but that differ when one more meta is added. Also Chris Scambler, [15], has shown that if one extends the construction from [5] to the limit, and we have agreement at the meta^n level for all n, we still get logics that differ on refutability. Things are complicated indeed.

In [11] I showed that the basic Strict/Tolerant phenomenon is actually quite widespread. There is an infinite family of many-valued logics that have Strict/Tolerant versions, agreeing on consequence, differing on metaconsequence. The present paper is a sequel to that paper, and continues the earlier investigation, showing that all the work of [5], and the results of [15] on antivalidity have a similar broad range. This is closely connected with a notion dual to strict/tolerance, known as Tolerant/Strict, and details of the duality are examined in a general setting. In fact, even the general methodology of the proofs found in [5] and [15] more or less extends to a very general setting. The essential part of what has been added here is the discovery of the right framework in which to apply their ideas. The direct connections with the Kleene and Priest logics are replaced with constructions involving *bilattices*. To keep things relatively self-contained here, we begin with a summary of earlier results from [11], together with a discussion of the bilattice machinery needed to get them. Only after all this do we present what is new to this paper.

2 Previous Work

We briefly summarize our terminology and notation from [11]. We also present a more detailed version of our earlier results, as Proposition 2.4.

Throughout this paper we consider only propositional many valued logics, and we assume their formal languages are built using $\wedge$, $\vee$, $\neg$, but not implication. Much use will be made of sequents, including higher order sequents. For now, a sequent is an ordered pair of finite sets of formulas, and as usual we will write sequents as $\Gamma \Rightarrow \Delta$, rather than as $\langle \Gamma, \Delta \rangle$. We will write $\Gamma, X \Rightarrow \Delta$, where X is a formula, as short for $\Gamma \cup \{X\} \Rightarrow \Delta$, and so on through all the familiar conventions. Throughout we will think of sequents as a representation of a multiple conclusion consequence relation, that is Scott style [16], rather than the single conclusion Tarski style.

We briefly sketch the semantic basics, since this will be used throughout. Many valued logics are specified by giving some space of truth values, V, some interpretation of the connectives in that space, and some subset D of V of designated truth values. This generality is too much for our methods, so we narrow things down. All our spaces of truth values will be De Morgan algebras, where distributivity conditions may or may not hold. We use the ungainly term *non-distributive* De Morgan algebras, though understand that non-distributive does not imply the distributive laws do not hold, but rather that distributive laws are not needed for our purposes, and so may or may not hold.

Definition 2.1 (Non-Distributive De Morgan Algebra) A *De Morgan algebra* is a bounded distributive lattice with a De Morgan involution. We write 0 and 1 for the lower and upper bounds of such an algebra, $\sqcap$ and $\sqcup$ for meet and join, and overbar for the De Morgan involution. The De Morgan involution must meet the conditions that $\overline{a \sqcap b} = \overline{a} \sqcup \overline{b}$ and $\overline{\overline{a}} = a$. ($\overline{a \sqcup b} = \overline{a} \sqcap \overline{b}$ follows easily.) A *non-distributive De Morgan algebra* meets the conditions for a De Morgan algebra except, possibly, for the distributive laws.

Non-distributive De Morgan algebras provide us with natural truth value spaces, and accompanying interpretations for $\wedge$, $\vee$, and $\neg$ as meet, join, and De Morgan involution, and we assume this is how formulas are evaluated from now on. We also need subsets of designated truth values, and we want these with some natural structural properties. Being a prime filter is common. We use the term *logical De Morgan algebra* for the resulting combination. This is a piece of terminology that must be in the literature in some form, but we haven't managed to find it.

Definition 2.2 (Logical De Morgan Algebra) Let L be a non-distributive De Morgan algebra. D is a *prime filter* on L if D is a non-empty proper subset of L that meets the following two conditions:

$$a \sqcap b \in D \text{ if and only if } a \in D \text{ and } b \in D$$
$$a \sqcup b \in D \text{ if and only if } a \in D \text{ or } b \in D.$$

We call the pair $\langle L, D \rangle$ a non-distributive *logical De Morgan algebra*, thinking of it as a many valued logic with D as the set of designated truth values.

A *valuation* in a non-distributive De Morgan algebra L is a mapping from propositional letters to L. Any valuation in L extends (uniquely) to a mapping from all formulas to L using the meet of L to interpret conjunction, the join to interpret disjunction, and the De Morgan involution to interpret negation, and we generally identify a valuation with this extension. A valuation v in L *validates* the sequent $\Gamma \Rightarrow \Delta$ in $\langle L, D \rangle$ provided that, if $v(X) \in D$ for every $X \in \Gamma$ then $v(Y) \in D$ for some $Y \in \Delta$. Equivalently stated, the condition is: $v(X) \notin D$ for some $X \in \Gamma$ or $v(Y) \in D$ for some $Y \in \Delta$. We symbolize this by $\mathsf{C}\langle L, D \rangle \models_v \Gamma \Rightarrow \Delta$, and we say that v is a valuation in $\mathsf{C}\langle L, D \rangle$. The notation involving C is to suggest that this is a many-valued generalization of classical logic. A sequent $\Gamma \Rightarrow \Delta$ is $\mathsf{C}\langle L, D \rangle$ valid if $\mathsf{C}\langle L, D \rangle \models_v \Gamma \Rightarrow \Delta$ for every valuation v in $\mathsf{C}\langle L, D \rangle$. For a single formula X it is easy to check that $\mathsf{C}\langle L, D \rangle \models_v \emptyset \Rightarrow X$ if and only if $v(X) \in D$. We will write this as $\mathsf{C}\langle L, D \rangle \models_v X$.

Two particular De Morgan algebras are of special interest here, since they provided the first example of the Strict/Tolerant phenomenon. $\{0, \frac{1}{2}, 1\}$ is a (distributive) De Morgan algebra where $a \sqcap b$ is the minimum of a and b, $a \sqcup b$ is the maximum of a and b, and $\overline{a} = 1 - a$. Both $\{1\}$ and $\{\frac{1}{2}, 1\}$ are prime filters. Kleene's logic K_3 takes the first of these as designated, Priest's

LP takes the second. Thus both give us logical De Morgan algebras. Non-distributive logical De Morgan algebras also include classical propositional logic and first degree entailment. They are infinite in number.

Next we introduce our generalizations of the logic ST. They are like the many valued logics from Definition 2.2, except that instead of a single designated set of truth values they have two, one stricter than the other.

Definition 2.3 (Strict/Tolerant Logic) Let L be a non-distributive De Morgan algebra, and let T (for *tolerant*) be a proper non-empty subset of L and S (for *strict*) be a proper, non-empty, subset of T; we call $\langle L, S, T\rangle$ a *Strict/Tolerant* structure. If v is a valuation in L, we will also say it is a valuation in $\mathsf{ST}\langle L, S, T\rangle$. We say such a valuation validates sequent $\Gamma \Rightarrow \Delta$ *in the Strict/Tolerant sense* provided that, if v maps every member of Γ to S then v maps some member of Δ to T. We symbolize this by $\mathsf{ST}\langle L, S, T\rangle \models_v \Gamma \Rightarrow \Delta$. A sequent $\Gamma \Rightarrow \Delta$ is $\mathsf{ST}\langle L, S, T\rangle$ valid if $\mathsf{ST}\langle L, S, T\rangle \models_v \Gamma \Rightarrow \Delta$ for every valuation v in $\mathsf{ST}\langle L, S, T\rangle$.

For a formula X we have that $\mathsf{ST}\langle L, S, T\rangle \models_v \emptyset \Rightarrow X$ if and only if $v(X) \in T$, and this will sometimes be written as $\mathsf{ST}\langle L, S, T\rangle \models_v X$.

It should be noted that every non-distributive logical De Morgan algebra has validities. For example, every sequent $X \Rightarrow X$ is a validity, no matter what the algebra. This is an easy consequence of the requirement that every strict designated truth value also be a tolerant one.

The following was proved in [11], but in the earlier paper significant details about the Strict/Tolerant structure that is constructed were left implicit. In the Proposition below they are made explicit. The proof will be sketched in Section 4, since we will build on it.

Proposition 2.4 (Central Result of [11]) *Let $\langle L, D\rangle$ be any non-distributive logical De Morgan algebra. There is an algorithm constructing a Strict/Tolerant structure $\langle L^*, D, D^*\rangle$ from $\langle L, D\rangle$ such that $\mathsf{C}\langle L, D\rangle$ and $\mathsf{ST}\langle L^*, D, D^*\rangle$ validate the same sequents, but differ at the meta-consequence level. The Strict/Tolerant structure $\langle L^*, D, D^*\rangle$ meets the following conditions:*

1. *$\langle L^*, D^*\rangle$ is a non-distributive logical De Morgan algebra;*
2. *L^* properly extends L in the sense that L is a bounded proper sublattice of L^* having the same bounds, and with the operations of L being those of L^* restricted to L;*
3. *D is D^* restricted to members of L.*

We have ended our summary, and elaboration, of previous work. In this paper two new things are added to what was just outlined. First, in response to much prodding from Graham Priest, we examine *Tolerant/Strict* logics. These are like Strict/Tolerant logics except that the roles are reversed. In such a logic a sequent is valid provided whenever all premises evaluate to a tolerantly acceptable truth value, one of the consequents evaluates to a strictly acceptable truth value. It should not be surprising that Tolerant/Strict behavior is, in a sense, dual to that of Strict/Tolerant.

In [5] a hierarchy of logics is constricted, building on ST, and important connections with classical logic are established. We show that this work extends to the same full generality seen in Proposition 2.4. That is, for every non-distributive logical De Morgan algebra there is a hierarchy of logics each of which agrees with with the logical De Morgan algebra on consequence, metaconsequence, metametaconsequence, . . . , up to some particular level, with a limit case that agrees at every level. In fact our constructions are drawn from those of [5], but are generalized using bilattice machinery. Results from [15] concerning *antivalidity* (defined in Section 6) are similarly extended.

3 Bilattices

Bilattices are our principal tool, and this is a summary section of their properties, without proofs. It is not a detailed discussion, but should be enough for present purposes. The primary original references are as follows. Bilattices were introduced by Matt Ginsberg, [12, 13]. Among other things, the bilattice product, discussed in Section 3.2 originates here. Applications to semantical issues, primarily involving fixpoint constructions, can be found in [7, 8, 10], along with a full representation theorem for the bilattice product, but restricted to distributive bilattices. The representation theorem was extended to interlaced bilattices in [3], and fundamental logic issues were beautifully investigated in [1, 2].

3.1 Bilattice Basics

A *bilattice* is an algebraic structure with two lattice orderings, $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$. One ordering, $\leq_t$, is intended to represent degree of truth. The other, $\leq_k$, is intended to represent degree of information (knowledge in an older usage, hence the k subscript). Various conditions are usually added to the basic double lattice structure, in particular conditions connecting the two orderings. The term *bilattice* broadly covers all such structures, with various conditions specified.

The four-valued Dunn-Belnap structure, appropriate for first-degree entailment, gives us the simplest example of a bilattice. As a bilattice it is commonly called $\mathcal{FOUR}$, and plays a role with respect to the family of bilattices analogous to that of the two-valued Boolean algebra in the family of all Boolean algebras. We use $\mathcal{FOUR}$ to illustrate various topics in our summary discussion, and we give a diagram of it in Figure 1. As is customary with bilattices, one ordering is shown horizontally and the other vertically. There is a general construction for bilattices, discussed in Section 3.2, after which it will be clear that there is an infinite family of bilattices. Other double Hasse diagrams for bilattices of interest can be found in [11].

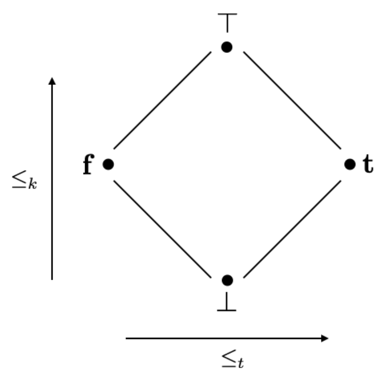


Figure 1: The Bilattice $\mathcal{FOUR}$

A *pre-bilattice*, $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$, is the basic bilattice structure. Each of $\leq_t$ and $\leq_k$ are bounded partial orderings on $\mathcal{B}$. Meet and join operations with respect to $\leq_t$ are denoted by $\wedge$ and $\vee$, and the least and greatest elements are denoted **f** and **t**. Meet and join with respect to $\leq_k$ are denoted by $\otimes$ (consensus) and $\oplus$ (gullability). The least and greatest elements with respect to $\leq_k$ are denoted $\bot$ and $\top$. In the basic example, $\mathcal{FOUR}$, the four extreme elements are all there are. In $\mathcal{FOUR}$, and in most bilattices, their behavior with respect to the meet and join operations is given in Figure 2.

$\wedge$	**f**	**t**	$\bot$	$\top$
f	**f**	**f**	**f**	**f**
t	**f**	**t**	$\bot$	$\top$
$\bot$	**f**	$\bot$	$\bot$	**f**
$\top$	**f**	$\top$	**f**	$\top$

$\vee$	**f**	**t**	$\bot$	$\top$
f	**f**	**t**	$\bot$	$\top$
t	**t**	**t**	**t**	**t**
$\bot$	$\bot$	**t**	$\bot$	**t**
$\top$	$\top$	**t**	**t**	$\top$

$\otimes$	**f**	**t**	$\bot$	$\top$
f	**f**	$\bot$	$\bot$	**f**
t	$\bot$	**t**	$\bot$	**t**
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	**f**	**t**	$\bot$	$\top$

$\oplus$	**f**	**t**	$\bot$	$\top$
f	**f**	$\top$	**f**	$\top$
t	$\top$	**t**	**t**	$\top$
$\bot$	**f**	**t**	$\bot$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Figure 2: Extreme Element Operations on $\mathcal{FOUR}$

A pre-bilattice has a *negation* if there is a mapping $\neg : \mathcal{B} \to \mathcal{B}$, that reverses $\leq_t$, preserves $\leq_k$, and is an involution. That is, $a \leq_t b$ implies $\neg b \leq_t \neg a$; $a \leq_k b$ implies $\neg a \leq_k \neg b$; and $\neg\neg a = a$. Such a negation obeys De Morgan's laws for the truth connectives so, for instance, $\neg(a \wedge b) = (\neg a \vee \neg b)$. It leaves the information connectives alone, for instance $\neg(a \otimes b) = (\neg a \otimes \neg b)$. On the extreme elements, negation switches $\mathbf{f}$ and $\mathbf{t}$, while leaving $\top$ and $\bot$ alone. In $\mathcal{FOUR}$ negation is seen as a left-right flip.

A pre-bilattice has a *conflation* if there is a mapping $- : \mathcal{B} \to \mathcal{B}$, similar to negation, but with the roles of truth and information switched. It reverses $\leq_k$, preserves $\leq_t$, and is an involution. A conflation will obey De Morgan's laws with respect to $\otimes$ and $\oplus$ while leaving $\wedge$ and $\vee$ alone. Also it switches $\top$ and $\bot$, while leaving $\mathbf{f}$ and $\mathbf{t}$ unchanged. In $\mathcal{FOUR}$ conflation is a up-down flip. Generally a conflation is assumed only if there is also a negation, and generally one assumes negation and conflation commute, $-\neg a = \neg - a$. This is the case with $\mathcal{FOUR}$ for example.

Since each of the bilattice orderings is that of a lattice, monotonicity conditions for meet and join with respect to their ordering are immediate. For instance, $a \leq_t b$ implies $a \wedge c \leq_t b \wedge c$ and $a \leq_k b$ implies $a \otimes c \leq_k b \otimes c$. A bilattice is called *interlaced* if such conditions hold across orderings. Explicitly, interlacing requires the following.

$$
\begin{aligned}
a \leq_t b &\text{ implies } a \otimes c \leq_t b \otimes c \\
a \leq_t b &\text{ implies } a \oplus c \leq_t b \oplus c \\
a \leq_k b &\text{ implies } a \wedge c \leq_k b \wedge c \\
a \leq_k b &\text{ implies } a \vee c \leq_k b \vee c
\end{aligned}
$$

Not all pre-bilattices are interlaced, but interlacing is a common minimal condition to require. $\mathcal{FOUR}$ is an example of an interlaced bilattice. In any interlaced bilattice $\mathbf{f} \wedge \mathbf{t} = \bot$, $\mathbf{f} \vee \mathbf{t} = \top$, $\bot \otimes \top = \mathbf{f}$, and $\bot \oplus \top = \mathbf{t}$.

A pre-bilattice is *distributive* if all possible distributive laws hold. For instance, $\wedge$ and $\vee$ should not only distribute over each other, but over $\otimes$ and $\oplus$ as well. Thus $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$ is required, as one example. There are 12 distributive laws altogether. Every distributive bilattice is interlaced. The converse is not true. Once again, $\mathcal{FOUR}$ provides an example, this time of a distributive bilattice.

3.2 Construction and Representation Theorems

There is a simple way of constructing bilattices using what will here be called a *bilattice product*. Today it is sometimes known as a *twist structure*. Suppose $L_1 = \langle L_1, \leq_1 \rangle$ and $L_2 = \langle L_2, \leq_2 \rangle$ are bounded lattices. Their *bilattice product* is $L_1 \odot L_2 = \langle L_1 \times L_2, \leq_t, \leq_k \rangle$ where:

$$
\begin{aligned}
\langle a, b \rangle \leq_k \langle c, d \rangle &\text{ iff } a \leq_1 c \text{ and } b \leq_2 d \\
\langle a, b \rangle \leq_t \langle c, d \rangle &\text{ iff } a \leq_1 c \text{ and } d \leq_2 b
\end{aligned}
$$

Note the reversal of the $\leq_2$ ordering in the definition of $\leq_t$. Informally one can think of members of L_1 as possible evidences for an assertion and members of L_2 as possible evidences against. Then information goes up if both evidence for and evidence against go up, while degree of truth goes up if evidence for goes up while evidence against goes down.

Bilattice products are good tools for bilattice construction because $L_1 \odot L_2$ is always an interlaced bilattice. Further, if L_1 and L_2 are distributive lattices then $L_1 \odot L_2$ is a distributive bilattice. If $L_1 = L_2 = L$ then $L \odot L$ is a bilattice with negation, where $\neg\langle a, b \rangle = \langle b, a \rangle$; and if also L is a non-distributive De Morgan algebra then $L \odot L$ is a bilattice with a conflation that commutes with negation, where $-\langle a, b \rangle = \langle \overline{b}, \overline{a} \rangle$. In all these cases, the extreme elements of the bilattice product are $\bot = \langle 0_1, 0_2 \rangle$, $\top = \langle 1_1, 1_2 \rangle$, $\mathbf{f} = \langle 0_1, 1_2 \rangle$, and $\mathbf{t} = \langle 1_1, 0_2 \rangle$, where 0_1 and 0_2 are the least members of L_1 and L_2, and 1_1 and 1_2 are the greatest. Also in all cases the bilattice operations are the following, where $\sqcup_1$ and $\sqcup_2$ are the respective lattice joins, and $\sqcap_1$

and $\sqcap_2$ are the meets.

$$\begin{aligned}
\langle a,b\rangle \wedge \langle c,d\rangle &= \langle a \sqcap_1 c, b \sqcup_2 d\rangle\\
\langle a,b\rangle \vee \langle c,d\rangle &= \langle a \sqcup_1 c, b \sqcap_2 d\rangle\\
\langle a,b\rangle \otimes \langle c,d\rangle &= \langle a \sqcap_1 c, b \sqcap_2 d\rangle\\
\langle a,b\rangle \oplus \langle c,d\rangle &= \langle a \sqcup_1 c, b \sqcup_2 d\rangle
\end{aligned}$$

Bilattice product as a method of construction is completely general because there is a full family of representation theorems. For instance, if we have an interlaced bilattice then it will be isomorphic to $L_1 \odot L_2$ for some bounded lattices L_1 and L_2, and L_1 and L_2 will be unique up to isomorphism. If we have an interlaced bilattice with negation then L_1 and L_2 are isomorphic (and so can be taken to be identical). And similarly for all the other cases.

3.3 Consistent, AntiConsistent, Exact

For bilattices having a conflation, subsets called *exact* and *consistent* were defined in [7], with *anticonsistent* added in [11]. Here is the simple characterization.

Definition 3.1 For bilattice $\mathcal{B}$ with conflation, $a \in \mathcal{B}$ is *consistent* if $a \leq_k -a$, *anticonsistent* if $-a \leq_k a$, and *exact* if $a = -a$.

In $\mathcal{FOUR}$ the exact values are $\{\mathbf{f}, \mathbf{t}\}$, the consistent values are $\{\mathbf{f}, \mathbf{t}, \bot\}$, and the anticonsistent values are $\{\mathbf{f}, \mathbf{t}, \top\}$. There are examples that show the three-way classification does not always exhaust an entire bilattice. The representation theorems of the previous section provide an alternative characterization, whose equivalence is easily checked. Suppose $\langle x, y\rangle \in L \odot L$; then we have the following.

$$\begin{gathered}
\langle x,y\rangle \text{ is exact if and only if } x = \overline{y}\\
\langle x,y\rangle \text{ is consistent if and only if } x \leq \overline{y}\\
\langle x,y\rangle \text{ is anticonsistent if and only if } \overline{y} \leq x
\end{gathered}$$

The following play a fundamental role here and are, in a direct sense, generalizations of the three-valued logic conditions used in [5, 15].

Proposition 3.2 *Let $\mathcal{B}$ be an interlaced bilattice with negation and conflation.*

1. *The sets of exact values, consistent values, and anticonsistent values each contain* $\mathbf{f}$ *and* $\mathbf{t}$, *while $\bot$ is consistent and $\top$ is anticonsistent.*
2. *Each of the consistent, exact, and anticonsistent sets is closed under $\wedge$, $\vee$, and $\neg$.*
3. *Every consistent value is below some exact value, and every anticonsistent value is above some exact value, in the $\leq_k$ ordering.*
4. *For exact a and b, if $a \leq_k b$ then $a = b$.*

Proof We check the second part of item 3, and leave the rest as exercises. And for this we give two arguments, as illustrations.

Suppose a is anticonsistent. Then $-a \leq_k a$, so $-a \otimes a \leq_k a$. And $-(-a \otimes a) = --a \otimes -a = a \otimes -a$, so $-a \otimes a$ is exact, and below a.

For the second argument we use the representation theorems. Suppose $\langle x, y\rangle \in L \odot L$ is anticonsistent, so $\overline{y} \leq x$. Then $\langle \overline{y}, y\rangle$ is exact, and $\langle \overline{y}, y\rangle \leq_k \langle x, y\rangle$.

■

In [11] the following item was added to the collection of representation theorems mentioned in Section 3.2.

Proposition 3.3 *Suppose L is a non-distributive De Morgan algebra, and $\mathcal{B} = L \odot L$. The set of exact members of $\mathcal{B}$, under the ordering $\leq_t$, is isomorphic to L.*

We refer the reader to the paper just cited for a direct proof, not making use of the representation theorems. In Proposition 3.11 we give an extension, and there we make use of the representation theorems to show it more easily.

3.4 Logical Bilattices

A bilattice is an algebraic structure into which one can map logical formulas. In [2] Arieli and Avron introduced natural machinery to allow bilattices to define many valued logics. Their methodology is analogous to what is done with Boolean algebras and prime filters. For Boolean algebras, all such structures, in fact, characterize the same logic, namely classical logic. For bilattices, all of the structures of Arieli and Avron also define the same logic, first degree entailment, the logic associated with the bilattice $\mathcal{FOUR}$.

Definition 3.4 A *valuation* in bilattice $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$, with negation, is a mapping v from the set of propositional letters to members of $\mathcal{B}$. Valuations extend uniquely to the set of all logical formulas in the familiar way, where the symbols $\wedge$, $\vee$, and $\neg$ on the left are part of the syntax of logic formulas, and those on the right are bilattice operations associated with the truth ordering.

$$\begin{aligned} v(X \wedge Y) &= v(X) \wedge v(Y) \\ v(X \vee Y) &= v(X) \vee v(Y) \\ v(\neg X) &= \neg v(X) \end{aligned}$$

We will use the same symbol v for both a valuation and its extension to all formulas.

We observed in Proposition 3.2 that consistent truth values in $\mathcal{B}$ are closed under the truth operations, and similarly for anticonsistent and exact. This gives us the following.

Proposition 3.5 *If a valuation v in a bilattice with negation and conflation maps every propositional letter to a consistent truth value, it maps every logical formula to a consistent truth value. Similarly for the exact truth values, and for the anticonsistent truth values.*

Valuations have an important monotonicity property that is fundamental to Kripke-style theories of truth, [7, 9, 10]. Though formal work on self-reference and truth does not concern us here, monotonicity retains its importance. The following is an easy consequence of the interlacing conditions and negation conditions.

Proposition 3.6 *Let u and v be valuations in $\mathcal{B}$, an interlaced bilattice having negation and conflation. If $u(P) \leq_k v(P)$ for every propositional letter P then $u(X) \leq_k v(X)$ for every logical formula X.*

Next we prove a simple result that appears in a more restricted version in earlier work on strict/tolerant logics. First a few definitions.

Definition 3.7 For valuations u, v in bilattice $\mathcal{B}$ that is interlaced and has negation and conflation, we write $u \leq_k v$ if $u(P) \leq_k v(P)$ for every propositional letter P. We say u is exact if $u(P)$ is exact for every propositional letter P, and similarly for consistent and anticonsistent.

It follows from Proposition 3.6 that if $u \leq_k v$ in an interlaced bilattice with negation and conflation, then $u(X) \leq_k v(X)$ for every formula X. Likewise by Proposition 3.5, if u is exact, then $u(X)$ is exact for every formula X, and similarly for consistent and anticonsistent.

We next turn to the notion of *sharpening*, which was introduced in [15] specifically for the original ST case, and is now broadened.

Definition 3.8 Let u and v be anticonsistent valuations in bilattice $\mathcal{B}$. We say u *sharpens* v if $u \leq_k v$ and u is exact.

We have the following rather simple proposition, which plays a surprisingly important role.

Proposition 3.9 *Let $\mathcal{B}$ be an interlaced bilattice with negation and conflation.*

1. *Every anticonsistent valuation v in $\mathcal{B}$ has a sharpening.*
2. *If u sharpens v in $\mathcal{B}$ then u and v agree on exact values. That is, if $u(P)$ and $v(P)$ are both exact, then $u(P) = v(P)$.*

Proof For part 1, let v be an anticonsistent valuation. By Proposition 3.2, every anticonsistent member of $\mathcal{B}$ has an exact value below it in the $\leq_k$ ordering. For each propositional letter P choose an exact $a \leq_k v(P)$, and let $u(P) = a$. Then u is exact and $u \leq_k v$, so u is a sharpening of v.

For part 2, suppose u sharpens v and $u(P)$ and $v(P)$ are both exact. By definition of sharpening, $u(P) \leq_k v(P)$ so by Proposition 3.2 part 4, $u(P) = v(P)$. ■

Now we turn to an important and fundamental notion, originating in [2].

Definition 3.10 A *prime bifilter* on $\mathcal{B}$ is a proper, non-empty subset $\mathcal{F} \subseteq \mathcal{B}$, that meets the following conditions.

(PB-1) $(a \wedge b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ and $b \in \mathcal{F}$

(PB-2) $(a \otimes b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ and $b \in \mathcal{F}$

(PB-3) $(a \vee b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ or $b \in \mathcal{F}$

(PB-4) $(a \oplus b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ or $b \in \mathcal{F}$

A *logical bilattice* is a pair $\langle \mathcal{B}, \mathcal{F} \rangle$ where $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

It is easy to show that a prime bifilter is upward closed in both bilattice orderings. $\mathcal{FOUR}$ has exactly one prime bifilter, and thus the only logical bilattice built on $\mathcal{FOUR}$ is the one in Figure 3, with the prime bifilter shown circled. Other examples can be found in [2, 11]. A logical bilattice characterizes a many valued logic by taking the members of the bilattice as truth values, with members of the prime bifilter as designated. The logical bilattice $\mathcal{FOUR}$ is one of the common ways of specifying first degree entailment. In [2] a very nice result is shown: the valid sequents of any logical bilattice are the same as they are for $\mathcal{FOUR}$ using the prime bifilter $\{\mathbf{t}, \top\}$. This hardly means that the general family of logical bilattices looses its interest. We will see some reasons why later on.

Finally, Proposition 3.3 has the following extension to logical bilattices and it is central to our work.

Proposition 3.11 *Let $\langle L, D \rangle$ be a non-distributive logical De Morgan algebra, where D is a prime filter on L. The following hold.*

1. *$D \times L$ is a prime bifilter on the bilattice $L \odot L$, so $\langle L \odot L, D \times L \rangle$ is a logical bilattice.*
2. *The set of exact members of the bilattice $L \odot L$, under $\leq_t$, is isomorphic to L.*
3. *Under that isomorphism, the set of exact members of $D \times L$ is isomorphic to D.*

Briefly stated, the logical De Morgan algebra $\langle L, D \rangle$ is isomorphic to the exact part of the logical bilattice $\langle L \odot L, D \times L \rangle$.

Proof Assume $\langle L, D \rangle$ is a non-distributive logical De Morgan algebra, and D is a prime filter on L.

1. Four conditions must be checked; we do one, (PB-3), the others are similar. Assume $\langle x, y \rangle, \langle z, w \rangle \in L \odot L$. Suppose first that $\langle x, y \rangle \vee \langle z, w \rangle \in D \times L$, that is, $\langle x \sqcup z, y \sqcap w \rangle \in D \times L$. Then $x \sqcup z \in D$, which is a prime filter, so either $x \in D$ or $z \in D$, say the first. Then $\langle x, y \rangle \in D \times L$. Conversely, suppose that $\langle x, y \rangle \in D \times L$. Then $x \in D$ so $x \sqcup z \in D$ since it is a prime filter. But then $\langle x \sqcup z, y \sqcap w \rangle \in D \times L$, that is, $\langle x, y \rangle \vee \langle z, w \rangle \in D \times L$.

2. Let $\mathcal{E}$ be the set of exact members of $L \odot L$. As we noted in Section 3.3, $\mathcal{E}$ consists of those members of $L \odot L$ of the form $\langle x, \overline{x} \rangle$. Let $\theta : \mathcal{E} \to L$ be defined by $\theta(\langle x, \overline{x} \rangle) = x$. It is obvious that θ is 1-1 and onto. Finally, for $x, y \in L$, $x \leq y$ if and only if $\overline{y} \leq \overline{x}$. So $x \leq y$ if and only if $x \leq y$ and $\overline{y} \leq \overline{x}$ if and only if $\langle x, \overline{x} \rangle \leq_t \langle y, \overline{y} \rangle$. And thus $\theta(\langle x, \overline{x} \rangle) \leq \theta(\langle y, \overline{y} \rangle)$ if and only if $\langle x, \overline{x} \rangle \leq_t \langle y, \overline{y} \rangle$.
3. The mapping θ, defined above, when restricted to $\mathcal{E} \cap (D \times L)$, the exact members of $D \times L$, is still 1-1, and order preserving. It is onto D because, for each $x \in D$, $\langle x, \overline{x} \rangle \in \mathcal{E} \cap (D \times L)$ and $\theta(\langle x, \overline{x} \rangle) = x$.

■

4 Our Basic Construction

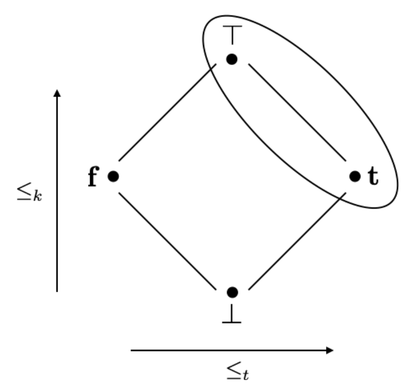


Figure 3: First Degree Entailment, $\mathcal{FOUR}$

In [11] we proved a simpler version of Proposition 2.4 using a construction involving bilattices. We begin this section by repeating some notation from that paper and giving an outline of the steps of the construction.

Definition 4.1 Let $\mathcal{B}$ be an interlaced bilattice with negation and conflation, and let $\mathcal{F}$ be a prime bifilter on $\mathcal{B}$, so that $\langle \mathcal{B}, \mathcal{F} \rangle$ is a logical bilattice. Let $\mathcal{A}$ be the set of anticonsistent members of $\mathcal{B}$, and let $\mathcal{E}$ be the set of exact members. We define the following sets, structures, and notations for them.

Tolerant Truth Values: $\mathcal{D}_T\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{A}$.

Strict Truth Values: $\mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{E}$.

Strict/Tolerant: $\langle \langle \mathcal{A}, \leq_t \rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle, \mathcal{D}_T\langle \mathcal{B}, \mathcal{F} \rangle \rangle$ is a Strict/Tolerant structure, where the underlying space of truth values is the anticonsistent part of the bilattice $\mathcal{B}$ using $\leq_t$ as lattice ordering, the strict truth values are the exact members of the prime bifilter $\mathcal{F}$, and the tolerant values are the anticonsistent members of $\mathcal{F}$. We abbreviate Strict/Tolerant $\mathsf{ST}\langle \langle \mathcal{A}, \leq_t \rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle, \mathcal{D}_T\langle \mathcal{B}, \mathcal{F} \rangle \rangle$ by $\mathsf{ST}\langle \mathcal{B}, \mathcal{F} \rangle$, and thus Definition 2.3 provides a meaning for $\mathsf{ST}\langle \mathcal{B}, \mathcal{F} \rangle \models_v \Gamma \Rightarrow \Delta$, and for $\mathsf{ST}\langle \mathcal{B}, \mathcal{F} \rangle$ validity. This provides our generalization of the standard Strict/Tolerantlogic ST.

Classical: $\langle \langle \mathcal{E}, \leq_t \rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle \rangle$ is a many valued logic structure, whose truth value space is the exact part of $\mathcal{B}$ ordered by $\leq_t$, and having those exact members that are in the prime bifilter $\mathcal{F}$ as the designated set. We write $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$ as shorthand for $\mathsf{C}\langle \langle \mathcal{E}, \leq_t \rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle \rangle$, so Definition 2.2 provides a meaning for $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \models_v \Gamma \Rightarrow \Delta$ and for $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$ validity. This provides our generalization of classical logic.

Next we sketch our construction for establishing Proposition 2.4. As we go along we illustrate the steps by applying them to classical logic, which was the original setting for Strict/Tolerant investigations.

Construction 4.2 (Establishing Proposition 2.4)

(Step 1) Start with a non-distributive logical De Morgan algebra, $\langle L, D\rangle$. We say how to construct a Strict/Tolerant logic agreeing with $\langle L, D\rangle$ at the consequence level, but differing at the metaconsequence level.

Continuing Example: We follow an example through the steps of the construction. In this example we take for L the simplest non-trivial De Morgan algebra whose lattice is $\{0, 1\}$ with $0 < 1$, and whose involution is $\overline{0} = 1$, and $\overline{1} = 0$. (It is distributive, but we make no use of that.) The subset $D = \{1\}$ is a prime filter, and the corresponding logical De Morgan algebra is simply the standard truth value space for classical logic.

(Step 2) Construct the bilattice product $L \odot L$ as described in Section 3.2. It will be interlaced, with a negation and a conflation. Note that by Proposition 3.3 the exact part, ordered by $\leq_t$, will be isomorphic to L.

Continuing Example: For the De Morgan algebra of our classical logic example, the resulting bilattice is $\mathcal{FOUR}$, shown in Figure 1. Using notation from that diagram the exact part is $\{\mathbf{f}, \mathbf{t}\}$, which is isomorphic to the domain $\{0, 1\}$ of the De Morgan algebra that we began with.

(Step 3) The subset $D \times L$ of $L \times L$ is a prime bifilter of $L \odot L$, so $\langle L \odot L, D \times L\rangle$ is a logical bilattice as described in Section 3.4.

Continuing Example: In our running example, we get what is shown in Figure 3 with the prime bifilter circled. In fact this is the structure for first degree entailment.

(Step 4) The exact part of $\langle L \odot L, D \times L\rangle$ is isomorphic to the logical De Morgan algebra $\langle L, D\rangle$ with which we began, by Proposition 3.11. In other words, $\mathsf{C}\langle L \odot L, D \times L\rangle$ and $\langle L, D\rangle$ are isomorphic.

Continuing Example: Continuing our example, we have a logical De Morgan algebra with truth values $\{0, 1\}$ and with $\{1\}$ designated. The exact part of the bilattice $L \odot L$ is $\{\mathbf{f}, \mathbf{t}\}$ with $\{\mathbf{t}\}$ designated.

(Step 5) $\mathsf{ST}\langle L \odot L, D \times L\rangle$ and $\mathsf{C}\langle L \odot L, D \times L\rangle$ validate the same sequents, shown as Proposition 4.3 below. There is a metaconsequence scheme every instance of which is locally valid in $\mathsf{C}\langle L \odot L, D \times L\rangle$ but with an instance that is not locally valid in $\mathsf{ST}\langle L \odot L, D \times L\rangle$, shown as Proposition 4.4. Finally, there is an isomorphic copy $\langle L^*, D, D^*\rangle$ of $\mathsf{ST}\langle L \odot L, D \times L\rangle$ that meets the structural conditions of Proposition 2.4. This is shown as Proposition 4.5.

Continuing Example: The anticonsistent part of $\mathcal{FOUR}$ is $\{\mathbf{f}, \top, \mathbf{t}\}$ with $\mathbf{f} \leq_t \top \leq_t \mathbf{t}$. $T = \{\top, \mathbf{t}\}$ and $S = \{\mathbf{t}\}$. This is, in fact, a presentation of the semantic structure for the standard strict/classical logic ST. When applied to this example, the proofs for Propositions 4.3 and 4.4 are clearly versions of those for the standard classical and ST connection, as found in the literature.

In (Step 5) above, we simply stated results; the details are found in the Propositions below. Some of this repeats what is in [11].

Proposition 4.3 *Using the Construction above, $\mathsf{ST}\langle L \odot L, D \times L\rangle$ and $\mathsf{C}\langle L \odot L, D \times L\rangle$ validate the same sequents.*

Proof First assume $\Gamma \Rightarrow \Delta$ is valid in $\mathsf{ST}\langle L \odot L, D \times L\rangle$. Let v be an arbitrary valuation in $\mathsf{C}\langle L \odot L, D \times L\rangle$ mapping every formula in Γ to $\mathcal{D}_S\langle L \odot L, D \times L\rangle$. Since exact values are also anticonsistent, v is a valuation in $\mathsf{ST}\langle L \odot L, D \times L\rangle$ too. Since v maps all of Γ to $\mathcal{D}_S\langle L \odot L, D \times L\rangle$ and $\Gamma \Rightarrow \Delta$ is valid in $\mathsf{ST}\langle L \odot L, D \times L\rangle$, then for some $Y \in \Delta$, $v(Y) \in \mathcal{D}_T\langle L \odot L, D \times L\rangle$. But by Proposition 3.5, $v(Y)$ will be exact, and so is in the set $\mathcal{D}_S\langle L \odot L, D \times L\rangle$. It follows that $\mathsf{C}\langle L \odot L, D \times L\rangle \models_v \Gamma \Rightarrow \Delta$. Since v was arbitrary, $\Gamma \Rightarrow \Delta$ is validated in $\mathsf{C}\langle L \odot L, D \times L\rangle$.

Next assume $\Gamma \Rightarrow \Delta$ is not valid in $\mathsf{ST}\langle L \odot L, D \times L\rangle$. Then there is a valuation v in $\mathsf{ST}\langle L \odot L, D \times L\rangle$ mapping every X in Γ to $\mathcal{D}_S\langle L \odot L, D \times L\rangle$, but for some $Y \in \Delta$, $v(Y) \notin \mathcal{D}_T\langle L \odot L, D \times L\rangle$. Then $v(Y)$ must be anticonsistent but not in the prime bifilter $D \times L$.

By Proposition 3.9, v has a sharpening, choose one and call it u. Then u is a valuation in $\mathsf{C}\langle L \odot L, D \times L\rangle$. We show that $\mathsf{C}\langle L \odot L, D \times L\rangle \not\models_u \Gamma \Rightarrow \Delta$. Since u is exact, then for every logical formula X, $u(X)$ is exact. Also v maps members of Γ to exact values. Then by Proposition 3.9 part 2, u and v must agree on members of Γ, and so u maps every member of Γ to $\mathcal{D}_S\langle L \odot L, D \times L\rangle$.

There is a formula $Y \in \Delta$ such that $v(Y) \notin \mathcal{D}_T\langle L \odot L, D \times L\rangle$, so $v(Y)$ is anticonsistent but not in $D \times L$. If we had that $u(Y) \in \mathcal{D}_S\langle L \odot L, D \times L\rangle$ then $u(Y)$ would be both exact and in the prime bifilter $D \times L$. Being in this prime bifilter we would also have $v(Y)$ in the prime bifilter because $u(Y) \leq_k v(Y)$ and prime bifilters are upward closed in both bilattice orderings. But $v(Y)$ is not in $D \times L$, and hence $u(Y) \notin \mathcal{D}\langle L \odot L, D \times L\rangle$. It follows that $\mathsf{C}\langle L \odot L, D \times L\rangle \not\models_u \Gamma \Rightarrow \Delta$. ∎

Proposition 4.4 *Again using the Construction above, the metaconsequence scheme*

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}$$

is locally valid in $\mathsf{C}\langle L \odot L, D \times L\rangle$*, but an instance is not locally valid in* $\mathsf{ST}\langle L \odot L, D \times L\rangle$*. (Local validity for a logic means that each valuation that validates all the premises of a schema also validates the conclusion.)*

Proof First we show local validity in $\mathsf{C}\langle L \odot L, D \times L\rangle$, by showing the contrapositive. Suppose v is a valuation in $\mathsf{C}\langle L \odot L, D \times L\rangle$ but $\mathsf{C}\langle L \odot L, D \times L\rangle \not\models_v \Gamma \Rightarrow \Delta$. Then v maps formulas to exact members of $L \odot L$, for every $X \in \Gamma$, $v(X) \in \mathcal{D}_S\langle L \odot L, D \times L\rangle$, and for every $Y \in \Delta$, $v(Y) \notin \mathcal{D}_S\langle L \odot L, D \times L\rangle$. Either $v(A) \in \mathcal{D}_S\langle L \odot L, D \times L\rangle$ or $v(A) \notin \mathcal{D}_S\langle L \odot L, D \times L\rangle$. If the first, then $v(X) \in \mathcal{D}_S\langle L \odot L, D \times L\rangle$ for every X in Γ, A, so $\mathsf{C}\langle L \odot L, D \times L\rangle \not\models_v \Gamma, A \Rightarrow \Delta$. If the second, then $v(Y) \notin \mathcal{D}_S\langle L \odot L, D \times L\rangle$ for every Y in Δ, A, so $\mathsf{C}\langle L \odot L, D \times L\rangle \not\models_v \Gamma \Rightarrow \Delta, A$. Either way, v does not validate one of the premises of the metaconsequence.

Second we show local non-validity in $\mathsf{ST}\langle L \odot L, D \times L\rangle$. Pick any sequent $\Gamma \Rightarrow \Delta$ that is not valid in $\mathsf{ST}\langle L \odot L, D \times L\rangle$ ($\emptyset \Rightarrow \emptyset$ will do, for instance) and let v be a valuation that does not validate it. That is, v maps to anticonsistent members of $L \odot L$, $v(X) \in \mathcal{D}_S\langle L \odot L, D \times L\rangle$ for every $X \in \Gamma$, and $v(Y) \notin \mathcal{D}_T\langle L \odot L, D \times L\rangle$ for every $Y \in \Gamma$.

Let P be a propositional letter that does not occur in Γ or in Δ and redefine v so that $v(P) = \top$, where $\top$ is the largest member of $L \odot L$ in the $\leq_k$ ordering. This can be done without changing the behavior of v on Γ or Δ. We know that $\top$ is in the prime bifilter $D \times L$ because prime bifilters are upward closed in both orderings. Also $\top$ is anticonsistent, but not exact. Then $\mathsf{ST}\langle L \odot L, D \times L\rangle \models_v \Gamma, P \Rightarrow \Delta$ because $v(P) \notin \mathcal{D}_S\langle L \odot L, D \times L\rangle$, since $v(P)$ is not exact. Also $\mathsf{ST}\langle L \odot L, D \times L\rangle \models_v \Gamma \Rightarrow \Delta, P$ because $v(P) = \top \in \mathcal{D}_T\langle L \odot L, D \times L\rangle$, since $\top$ is anticonsistent. Then v is a counterexample to the local validity of the metaconsequence scheme, taking A to be the propositional letter P. ∎

The proper status of cut has nuances that might not be obvious on first sight. This will be discussed in Section 8.

Proposition 4.5 *There is a Strict/Tolerant structure* $\langle L^*, D, D^*\rangle$ *that meets the structural conditions of Proposition 2.4 and is isomorphic to the Strict/Tolerant structure of* $\mathsf{ST}\langle L \odot L, D \times L\rangle$*.*

Proof Recall that $\mathsf{ST}\langle L\odot L, D\times L\rangle$ abbreviates $\mathsf{ST}\langle\langle\mathcal{A},\leq_t\rangle, \mathcal{D}_S\langle L\odot L, D\times L\rangle, \mathcal{D}_T\langle L\odot L, D\times L\rangle\rangle$, where $\mathcal{A}$ is the anticonsistent part of $L\odot L$. Making use of Definition 4.1, the Strict/Tolerant structure here is actually $\langle\langle\mathcal{A},\leq_t\rangle, (D\times L)\cap\mathcal{E}, (D\times L)\cap\mathcal{A}\rangle$, where $\mathcal{E}$ is the exact part of $L\odot L$. By Proposition 3.11 we know that $\langle\mathcal{E},\leq_t\rangle$ (where $\leq_t$ is understood as being restricted to $\mathcal{E}$) is isomorphic to L, under an isomorphism θ that pairs up $(D\times L)\cap\mathcal{E}$ and D. Extend θ to the entire of $\mathcal{A}$ by making it the identity on anticonsistent but not exact members. Now let L^* be the image of $\langle\mathcal{A},\leq_t\rangle$ under θ. Let D^* be the image of $(D\times L)\cap\mathcal{A}$ under θ, and note that D is the image of $(D\times L)\cap\mathcal{E}$. The structure $\langle L^*, D, D^*\rangle$ is easily seen to meet the conditions of Proposition 2.4. ■

5 Examples

We present several familiar examples of logical De Morgan algebras, and of the logical bilattices constructed from them.

Example 5.1 (Classical Logic) The truth value space of classical logic is, of course, the logical De Morgan algebra $\{0,1\}$ with the usual numerical ordering as its lattice ordering, and with $\{1\}$ as the set of designated truth values. The bilattice product of this with itself is $\mathcal{FOUR}$, and the prime bifilter we get from (Step 3) of Construction 4.2 is shown in Figure 3.

Example 5.2 (Kleene Strong Three-Valued Logic, K_3) As truth values take $\{0,\frac{1}{2},1\}$ with the numerical ordering as lattice ordering, and with the designated set being the prime filter $\{1\}$. It is a simple but good exercise for this and the next example, to construct the corresponding logical bilattice.

Example 5.3 (Priest's Logic of Paradox, LP) Use the same De Morgan algebra as in the previous example, but now $\{\frac{1}{2},1\}$ is the prime filter of designated values.

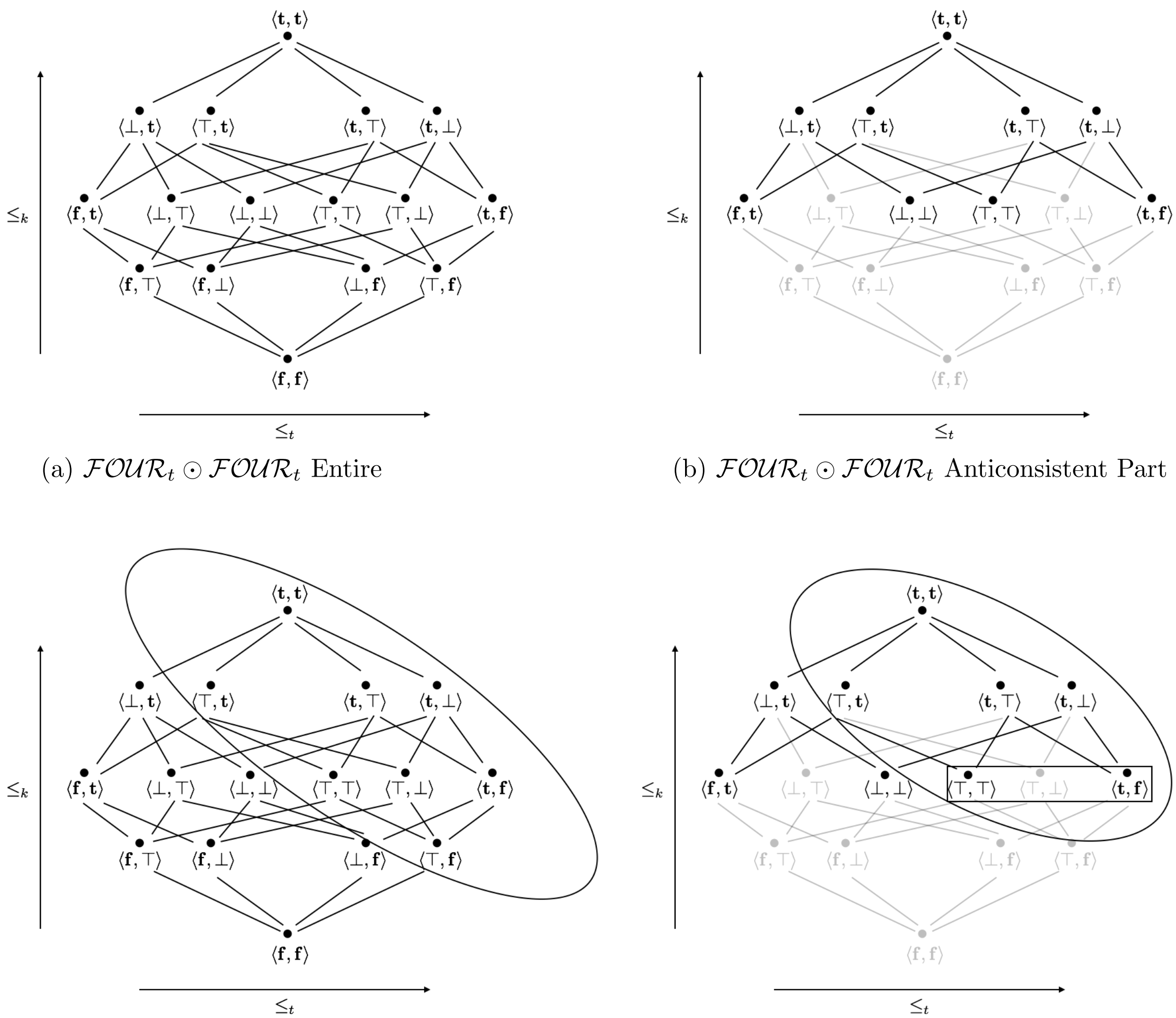
Example 5.4 (First Degree Entailment, FDE) In [11] a Strict/Tolerant example of some complexity was presented, Example 10.5 in the numbering of that paper. Since it is somewhat complex, it should be informative to present a detailed picture, which we do in Figure 4. Start with the bilattice $\mathcal{FOUR}$ as shown in Figure 1, and consider the lattice we get by only considering the $\leq_t$ ordering, call it $\mathcal{FOUR}_t$. This is a De Morgan algebra, using the bilattice negation as the De Morgan involution. The bilattice product $\mathcal{FOUR}_t\odot\mathcal{FOUR}_t$ is shown in Figure 4a; it was called $\mathcal{SIXTEEN}$ in [11].

In the bilattice $\mathcal{FOUR}_t\odot\mathcal{FOUR}_t$, $-\langle\bot,\top\rangle = \langle\neg\top,\neg\bot\rangle = \langle\top,\bot\rangle$, using the negation of $\mathcal{FOUR}$. Likewise $-\langle\top,\bot\rangle = \langle\bot,\top\rangle$. Thus neither $\langle\bot,\top\rangle$ nor $\langle\top,\bot\rangle$ is exact. Likewise, neither is consistent or anticonsistent. Our three-way classification is not exhaustive here. Figure 4b highlights the anticonsistent values.

The only prime bifilter for $\mathcal{FOUR}$ is shown in Figure 3. It is a prime filter when considered just on $\mathcal{FOUR}_t$; let us call this prime filter $\mathcal{P}_t$. Then $\langle\mathcal{FOUR}_t,\mathcal{P}_t\rangle$ is a logical De Morgan algebra. This induces a prime bifilter in $\mathcal{FOUR}_t\odot\mathcal{FOUR}_t$, namely $\mathcal{P}_t\times\mathcal{FOUR}_t$, consisting of the members of the bilattice product whose first component is $\mathbf{t}$ or $\top$, and this prime bifilter is shown in Figure 4c.

We now construct the Strict/Tolerant/strict logic $\mathsf{ST}\langle\mathcal{FOUR}_t\odot\mathcal{FOUR}_t,\mathcal{P}_t\times\mathcal{FOUR}_t\rangle$. The truth values are the anticonsistent members of $\mathcal{FOUR}_t\odot\mathcal{FOUR}_t$. The tolerantly designated truth values are the anticonsistent members of the prime bifilter, $\mathcal{D}_T\langle\mathcal{FOUR}_t\odot\mathcal{FOUR}_t,\mathcal{P}_t\rangle = \{\langle\mathbf{t},\mathbf{t}\rangle,\langle\top,\mathbf{t}\rangle,\langle\mathbf{t},\top\rangle, \langle\mathbf{t},\bot\rangle,\langle\top,\top\rangle,\langle\mathbf{t},\mathbf{f}\rangle\}$. The strictly designated truth values are the exact members of the prime bifilter, $\mathcal{D}_S\langle\mathcal{FOUR}_t\odot\mathcal{FOUR}_t,\mathcal{P}_t\rangle = \{\langle\top,\top\rangle,\langle\mathbf{t},\mathbf{f}\rangle\}$. All this is shown in Figure 4d, with the anticonsistent values highlighted, the tolerant designated values in the oval , and the strictly designated values in the rectangle.

The classical analog, $\mathsf{C}\langle\mathcal{FOUR}_t\odot\mathcal{FOUR}_t,\mathcal{P}_t\times\mathcal{FOUR}_t\rangle$, has as its truth values the set of exact values, that is, $\{\langle\mathbf{f},\mathbf{t}\rangle,\langle\bot,\bot\rangle,\langle\top,\top,\rangle,\langle\mathbf{t},\mathbf{f}\rangle$, with $\langle\top,\top\rangle,\langle\mathbf{t},\mathbf{f}\rangle\}$ as designated values. This is isomorphic to the logical De Morgan algebra $\langle\mathcal{FOUR}_t,\mathcal{P}_t\rangle$ with which we began.



(a) $\mathcal{FOUR}_t \odot \mathcal{FOUR}_t$ Entire

(b) $\mathcal{FOUR}_t \odot \mathcal{FOUR}_t$ Anticonsistent Part

(c) $\mathcal{FOUR}_t \odot \mathcal{FOUR}_t$ with Prime Bifilter

(d) $\mathcal{FOUR}_t \odot \mathcal{FOUR}_t$ with Strict/Tolerant Structure

Figure 4: $\mathcal{FOUR}_t \odot \mathcal{FOUR}_t$ and Subsystems

6 Tolerant/Strict Logics

We now start on new material. We have been looking at Strict/Tolerant logics; now we dualize to Tolerant/Strict logics, in which the conclusion of the consequence relation is held to stricter standards than the premise. For Tolerant/Strict logics definitions are almost the same as in the Strict/Tolerant version we have been discussing, but with the roles of strict and tolerant interchanged.

Definition 6.1 (Tolerant/Strict Logic) Let $\langle L, S, T\rangle$ be a Strict/Tolerant logic structure, as specified in Definition 2.3. A valuation v in L *validates* sequent $\Gamma \Rightarrow \Delta$ in the *Tolerant/Strict* sense provided that, if v maps every member of Γ to T then v maps some member of Δ to S. We symbolize this by $\mathsf{TS}\langle L, S, T\rangle \models_v \Gamma \Rightarrow \Delta$ (and call v a valuation in $\mathsf{TS}\langle L, S, T\rangle$). For a single formula X we have that $\mathsf{TS}\langle L, S, T\rangle \models_v \emptyset \Rightarrow X$ if and only if $v(X) \in S$, which we may write as $\mathsf{TS}\langle L, S, T\rangle \models_v X$.

1. $\Gamma \Rightarrow \Delta$ is a $\mathsf{TS}\langle L, S, T\rangle$ *validity* if $\mathsf{TS}\langle L, S, T\rangle \models_v \Gamma \Rightarrow \Delta$ for every valuation v in $\mathsf{TS}\langle L, S, T\rangle$.

2. $\Gamma \Rightarrow \Delta$ is a $\mathsf{TS}\langle L, S, T\rangle$ *antivalidity* if $\mathsf{TS}\langle L, S, T\rangle \models_v \Gamma \Rightarrow \Delta$ for no valuation v in $\mathsf{TS}\langle L, S, T\rangle$.

Apart from the reversal of the roles of tolerant and strict designated values, the main new item here is *antivalidity*. The emphasis on it derives from [15], where it turned out to play a fundamental role, as it does here. In fact, one should not expect TS results about validity that are in any way analogous to the ST results in Proposition 2.4, because *the* TS *validities of any Strict/Tolerant structure are never the same set as the set of validities of a non-distributive logical De Morgan algebra.* The reason is simple. In any non-distributive logical De Morgan algebra $X \Rightarrow X$ is validated for every formula X because it says that, for any valuation v, if $v(X)$ is designated then $v(X)$ is designated. But if $\langle L, S, T\rangle$ is any Strict/Tolerant structure then S will be a proper non-empty subset of T. So if we take X to be atomic and let $v(X)$ be any member of T that is not in S, then v does not TS validate $X \Rightarrow X$ in $\langle L, S, T\rangle$.

Antivalidity for any logic is defined in a similar way. We will need it for C, but omit the obvious definition. Connections now turn out to be between antivalidity in non-distributive logical De Morgan algebras and TS antivalidity in Strict/Tolerant structures. We should note that every non-distributive logical De Morgan algebra has antivalidities since $\emptyset \Rightarrow \emptyset$ can never be validated by any valuation.

Definition 6.2 (Continuing Definition 4.1) Assume the same conditions and notation as in Definition 4.1: $\mathcal{B}$ is an interlaced bilattice with negation and conflation, $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$ and so $\langle \mathcal{B}, \mathcal{F}\rangle$ is a logical bilattice, and $\mathcal{A}$ is the set of anticonsistent members of $\mathcal{B}$.

Tolerant/Strict: Recall that $\langle\langle \mathcal{A}, \leq_t\rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle, \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle\rangle$ is a Strict/Tolerant structure, with truth values being the anticonsistent part of bilattice $\mathcal{B}$ with $\leq_t$ as ordering, the tolerant values being the anticonsistent members of $\mathcal{F}$, and the strict values being the exact members of $\mathcal{F}$. We abbreviate $\mathsf{TS}\langle\langle \mathcal{A}, \leq_t\rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle, \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle\rangle$ by $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$, thus providing a meaning for $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle \models_v \Gamma \Rightarrow \Delta$, and for $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$ validity and antivalidity.

As described in Construction 4.2, starting with a non-distributive logical De Morgan algebra $\langle L, D\rangle$ we construct a logical bilattice $\langle L \odot L, D \times L\rangle$ following (Step 1) through (Step 4). But now in place of (Step 5), we switch the roles of tolerant and strict, and of valid and antivalid.

($\mathsf{Alt\ Step}$ 5) $\mathsf{TS}\langle L \odot L, D \times L\rangle$ and $\mathsf{C}\langle L \odot L, D \times L\rangle$ have the same set of antivalid sequents, but differ on antivalidity at the metaconsequence level.

The following supply the proof for this.

Proposition 6.3 *Let $\langle \mathcal{B}, \mathcal{F}\rangle$ be a logical bilattice, where $\mathcal{B}$ is interlaced, with negation and conflation. The logics $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$ (Definition 6.1) and $\mathsf{C}\langle \mathcal{B}, \mathcal{F}\rangle$ (Definition 4.1) have the same antivalid sequents.*

Proof Recall that the strictly designated values, $\mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle$, in $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$ are the same as the designated values in $\mathsf{C}\langle \mathcal{B}, \mathcal{F}\rangle$, and also that $\mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle$ is the exact subset of the set of tolerantly designated values $\mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle$.

Left to Right: We show that if $\Gamma \Rightarrow \Delta$ is antivalid in $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$ then it is antivalid in $\mathsf{C}\langle \mathcal{B}, \mathcal{F}\rangle$, and we argue conversely. Assume $\Gamma \Rightarrow \Delta$ is not antivalid in $\mathsf{C}\langle \mathcal{B}, \mathcal{F}\rangle$. Then there is some valuation v that maps to $\mathcal{E}$, the exact part of $\mathcal{B}$, and $\mathsf{C}\langle \mathcal{B}, \mathcal{F}\rangle \models_v \Gamma \Rightarrow \Delta$. Either $v(X) \notin \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle$ for some $X \in \Gamma$ or $v(Y) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle$ for some $Y \in \Delta$. Since every exact value is anticonsistent, v is also a valuation in $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$. If $v(X) \in \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle$, since v maps to exact values we would have $v(X) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle$. Hence either $v(X) \notin \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle$ for some $X \in \Gamma$, or $v(Y) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle$ for some $Y \in \Delta$, so $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle \models_v \Gamma \Rightarrow \Delta$, and thus $\Gamma \Rightarrow \Delta$ is not antivalid in $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$.

Right to Left: We show that if $\Gamma \Rightarrow \Delta$ is antivalid in $\mathsf{C}\langle \mathcal{B}, \mathcal{F}\rangle$ then it is antivalid in $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$, and again we argue conversely. Assume $\Gamma \Rightarrow \Delta$ is not antivalid in $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle$. Then there is some valuation v mapping to $\mathcal{A}$, the anticonsistent part of $\mathcal{B}$, such that $\mathsf{TS}\langle \mathcal{B}, \mathcal{F}\rangle \models_v \Gamma \Rightarrow \Delta$. Either $v(X) \notin \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle$ for some $X \in \Gamma$, or $v(Y) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle$ for some $Y \in \Delta$.

Define a new valuation v' as follows. For every propositional letter P, $v(P)$ is anticonsistent, and so there is some exact value x with $x \leq_k v(P)$. Choose one, and let it be $v'(P)$. Then v' maps to exact values and hence is a valuation in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$. By Proposition 3.6, $v'(Z) \leq_k v(Z)$ for every formula Z.

Suppose we have that $v(X) \notin \mathcal{D}_T\langle\mathcal{B},\mathcal{F}\rangle$ for some $X \in \Gamma$. If $v'(X) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ then $v'(X) \in \mathcal{D}_T\langle\mathcal{B},\mathcal{F}\rangle$, since the set of strict values is a subset of the set of tolerant ones. But this set is $\mathcal{F} \cap \mathcal{A}$, the intersection of the prime bifilter and the anticonsistent values. Since $v'(X) \leq_k v(X)$ then we would have $v(X) \in \mathcal{D}_T\langle\mathcal{B},\mathcal{F}\rangle$ since bifilters are closed upward in both orderings, and the anticonsistent part of $\mathcal{B}$ is also closed upward in the $\leq_k$ ordering. This contradicts the supposition, so $v'(X) \notin \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$.

Suppose we have that $v(Y) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ for some $Y \in \Delta$. Then $v(Y)$ is exact, since members of $\mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ are. Since v' maps to exact values, $v'(Y)$ is exact. But also $v'(Y) \leq_k v(Y)$, so it follows that $v'(Y) = v(Y)$. Thus $v'(Y) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$.

Thus either $v'(X) \notin \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ for some $X \in \Gamma$ or $v'(Y) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ for some $Y \in \Delta$, so $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_{v'} \Gamma \Rightarrow \Delta$, and $\Gamma \Rightarrow \Delta$ is not antivalid in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$.

■

Proposition 6.4 *Under the same conditions as in Proposition 6.3, the metaconsequence scheme*

$$\frac{X \Rightarrow X}{\emptyset \Rightarrow \emptyset}$$

is locally antivalid in $\mathsf{C}\langle L \odot L, D \times L\rangle$*, but there is an instance that is not in* $\mathsf{TS}\langle L \odot L, D \times L\rangle$*. (Local antivalidity means that every valuation validates the premises but does not validate the conclusion.)*

Proof The first assertion is simply the consequence of the observations that $X \Rightarrow X$ is a validity in every logical De Morgan algebra, while $\emptyset \Rightarrow \emptyset$ is never validated by any valuation. For the second, let P be a propositional letter and let v be a valuation in $\mathsf{TS}\langle L \odot L, D \times L\rangle$ mapping P to $\top$, the largest member of $L \odot L$ under the $\leq_k$ ordering. Then $\mathsf{TS}\langle L \odot L, D \times L\rangle \not\models_v P \Rightarrow P$ because $\top \in \mathcal{D}_T\langle\mathcal{B},\mathcal{F}\rangle$ but $\top \notin \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$. ■

7 The Strict/Tolerant Hierarchy

The main result of [5] is that the results relating the original ST and classical logic can be "pushed upwards" by moving to metaconsequence, past it to metametaconsequence, and beyond. Let us say this in a bit more detail.

We have been thinking of a sequent, $\Gamma \Rightarrow \Delta$, as a concrete representation of a multiconclusion consequence relation. Ordinarily cut is represented as

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}$$

which is a kind of metaconsequence statement, Read it as: from the two premise consequences the conclusion consequence follows. But this is only one level up, and one can go further. Of course this gets hard to read after a while, not to mention hard to type. In [5] the representation selected is a version of higher type sequents. The cut rule becomes something like the following, where parentheses have been used to avoid any confusion, and a 'meta-arrow' $\Rightarrow^*$ is employed.

$$(\Gamma, A \Rightarrow \Delta), (\Gamma \Rightarrow \Delta, A) \Rightarrow^* (\Gamma \Rightarrow \Delta)$$

A metaconsequence expression, then, would have as premises some consequence statements, and have some consequence statements as conclusion. But then one can introduce a metameta-consequence expression, which would have metaconsequence statements, and metaconsequence conclusions. And so on. Following [5] we use integer indexes on our arrows, thinking of them as representing the level of 'meta' we are at. Formulas themselves are not really sequents, but it is handy to think of them as sequents in a degenerate sense—sequents of level 0. Above this ground level we have the hierarchy of proper sequent levels as we have loosely described them, and which we now properly characterize.

Definition 7.1 (The Sequent Hierarchy)

$$\begin{aligned} Seq^0 &= \text{the set of all formulas} \\ Seq^{n+1} &= \{\Gamma \Rightarrow_{n+1} \Delta \mid \Gamma, \Delta \text{ finite subsets of } Seq^n\} \end{aligned}$$

So far this has all been syntax. Semantics can be introduced, and for logical De Morgan algebras at least, this is almost straightforward. We say "almost" because there are actually two ways of doing it: local validity and global validity. Consider, for example, the familiar rule of *disjunctive syllogism*, in classical logic.

$$\frac{\neg X \qquad X \vee Y}{Y}$$

Using the classical notion of two-valued valuations in the space $\{\mathbf{f}, \mathbf{t}\}$, this could be taken to mean either of the following. First, *local* validity: each valuation that maps both premises $\neg X$ and $X \vee Y$ to $\mathbf{t}$ also maps the conclusion, Y, to $\mathbf{t}$. Second, *global* validity: if both premises are valid then the conclusion is valid, where validity for a formula means that every valuation maps that formula to $\mathbf{t}$. Commonly when discussing classical axiom systems we would justify disjunctive syllogism as a rule by saying that it preserves validity, and we (probably) have global validity in mind when we say this. But when we verify this preservation assertion, it is typically local validity that we prove. In fact local validity is stronger; it is easy to see that it implies global validity, at least for single conclusion sequents. On the other hand, in a modal setting we commonly have a Rule of Necessitation, and this is globally valid without being locally valid.

Actually, what was just said is an oversimplification, but it is enough to motivate what immediately follows. A fuller discussion will be found in Section 8, after some preliminaries have been considered that will make our discussion more fruitful.

In the discussion of Strict/Tolerant logic and the hierarchy based on it, found in [5], *local* validity is used throughout, and here we follow them in this. We begin by extending the notion of validity in logical De Morgan algebras (Definition 2.2) to higher level sequents.

Definition 7.2 Let $\langle L, D\rangle$ be a non-distributive logical De Morgan algebra, and let v be a valuation mapping propositional letters to L. We know that v extends to a mapping from all formulas to L in the standard way.

Seq^0: Let $X \in Seq^0$, that is, X is a formula. Then $\mathsf{C}\langle L, D\rangle \models_v X$ provided $v(X) \in D$.

Seq^{n+1}: Let $\Gamma \Rightarrow_{n+1} \Delta \in Seq^{n+1}$. Then $\mathsf{C}\langle L, D\rangle \models_v \Gamma \Rightarrow_{n+1} \Delta$ provided that $\mathsf{C}\langle L, D\rangle \models_v \gamma$ for each $\gamma \in \Gamma$ implies $\mathsf{C}\langle L, D\rangle \models_v \delta$ for some $\delta \in \Delta$.

A sequent $\Gamma \Rightarrow_{n+1} \Delta \in Seq^{n+1}$ is $\mathsf{C}\langle L, D\rangle$ *valid* if $\mathsf{C}\langle L, D\rangle \models_v \Gamma \Rightarrow_{n+1} \Delta$ for every valuation v in L.

We look at a couple of examples. Suppose that $n = 0$, so $\Gamma \Rightarrow_{n+1} \Delta$ is $\Gamma \Rightarrow_1 \Delta$, and Γ and Δ are subsets of Seq_0 and hence are sets of formulas. The definition of $\Gamma \Rightarrow_1 \Delta$ coincides with that for $\Gamma \Rightarrow \Delta$ from Definition 2.2, and so $\Rightarrow_1$ is simply alternate notation for $\Rightarrow$. Then the metaconsequence scheme appearing in the statement of Proposition 4.4 can also be written as

$$\frac{\Gamma, A \Rightarrow_1 \Delta \qquad \Gamma \Rightarrow_1 \Delta, A}{\Gamma \Rightarrow_1 \Delta}$$

and this in turn can be written as the following higher level sequent.

$$\{(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A)\} \Rightarrow_2 \{\Gamma \Rightarrow_1 \Delta\}$$

Proposition 4.4 really amounts to showing local validity for this in every non-distributive logical De Morgan algebra, and non validity in the corresponding Strict/Tolerant logic..

The sets Seq^0, Seq^1, Seq^2, ... do not overlap, but there is a natural sense in which each contains copies of its predecessors. Suppose, for instance, that $\Gamma \Rightarrow_3 \Delta \in Seq^3$. While $\Gamma \Rightarrow_3 \Delta$ is not in Seq^4, the sequent $\emptyset \Rightarrow_4 \{\Gamma \Rightarrow_3 \Delta\}$ is, and it is easy to see that for any valuation v in logical De Morgan algebra $\langle L, D\rangle$, we have that $\mathsf{C}\langle L, D\rangle \models_v \Gamma \Rightarrow_3 \Delta$ if and only if $\mathsf{C}\langle L, D\rangle \models_v \emptyset \Rightarrow_4 \{\Gamma \Rightarrow_3 \Delta\}$. Indeed we have already seen the special case of $n = 0$, where we identified $\mathsf{C}\langle L, D\rangle \models_v \emptyset \Rightarrow X$ with $\mathsf{C}\langle L, D\rangle \models_v X$. In Gentzen sequent calculus proof systems this is used to define provability of a formula, since technically only sequents are provable. We note that similar remarks apply to higher level sequents, and to the corresponding higher level Strict/Tolerant and Tolerant/Strict systems we are about to discuss.

In [5], the Strict/Tolerant and Tolerant/Strict counterparts of classical logic were extended to higher level sequents, with interesting results, which were further extended in [15]. We now set about showing that the results are more general than the classical setting, extending to all non-distributive logical De Morgan algebras.

Definition 7.3 (Strict/Tolerant Hierarchy) Let $\langle L, S, T\rangle$ be a Strict/Tolerant structure, and let v be a valuation in L. For each $n = 0, 1, 2, \ldots$ we define $\mathsf{ST}_n\langle L, S, T\rangle$ and $\mathsf{TS}_n\langle L, S, T\rangle$ local validity by simultaneous recursion.

ST_0: Let $X \in Seq^0$. Then $\mathsf{ST}_0\langle L, S, T\rangle \models_v X$ provided $v(X) \in T$.

TS_0: Let $X \in Seq^0$. Then $\mathsf{TS}_0\langle L, S, T\rangle \models_v X$ provided $v(X) \in S$.

ST_{n+1}: Let $\Gamma \Rightarrow_{n+1} \Delta \in Seq^{n+1}$. Then $\mathsf{ST}_{n+1}\langle L, S, T\rangle \models_v (\Gamma \Rightarrow_{n+1} \Delta)$ provided that $\mathsf{TS}_n\langle L, S, T\rangle \models_v \gamma$ for all $\gamma \in \Gamma$ implies $\mathsf{ST}_n\langle L, S, T\rangle \models_v \delta$ for some $\delta \in \Delta$.

TS_{n+1}: Let $\Gamma \Rightarrow_{n+1} \Delta \in Seq^{n+1}$. Then $\mathsf{TS}_{n+1}\langle L, S, T\rangle \models_v (\Gamma \Rightarrow_{n+1} \Delta)$ provided that $\mathsf{ST}_n\langle L, S, T\rangle \models_v \gamma$ for all $\gamma \in \Gamma$ implies $\mathsf{TS}_n\langle L, S, T\rangle \models_v \delta$ for some $\delta \in \Delta$.

It should be pointed out that notation here is not yet standardized. In [15] what we denote by ST_n is denoted by T_n, and TS_n by S_n. In [5] L_n is used for our ST_n, and $\overline{L_n}$ for our TS_n. Following [5, 15], one often sees the $n + 1$ level conditions above symbolized as the following (using our notation).

$$\mathsf{ST}_{n+1}\langle L, S, T\rangle = \mathsf{TS}_n\langle L, S, T\rangle/\mathsf{ST}_n\langle L, S, T\rangle$$
$$\mathsf{TS}_{n+1}\langle L, S, T\rangle = \mathsf{ST}_n\langle L, S, T\rangle/\mathsf{TS}_n\langle L, S, T\rangle$$

Putting various definitions together, the following items are straightforward to show.

(Seq-1) For a formula X, $\mathsf{ST}_0\langle L, S, T\rangle \models_v X$ if and only if $\mathsf{ST}\langle L, S, T\rangle \models_v X$ if and only if $\mathsf{C}\langle L, T\rangle \models_v X$ (Definition 2.3)

(Seq-2) For a formula X, $\mathsf{TS}_0\langle L, S, T\rangle \models_v X$ if and only if $\mathsf{TS}\langle L, S, T\rangle \models_v X$ if and only if $\mathsf{C}\langle L, S\rangle \models_v X$ (Definition 6.1)

(Seq-3) For sets of formulas Γ and Δ, $\mathsf{ST}_1\langle L, S, T\rangle \models_v \Gamma \Rightarrow_1 \Delta$ if and only if $\mathsf{ST}\langle L, S, T\rangle \models_v \Gamma \Rightarrow \Delta$ (Definition 2.3)

(Seq-4) For sets of formulas Γ and Δ, $\mathsf{TS}_1\langle L, S, T\rangle \models_v \Gamma \Rightarrow_1 \Delta$ if and only if $\mathsf{TS}\langle L, S, T\rangle \models_v \Gamma \Rightarrow \Delta$ (Definition 6.1)

Now we extend Definitions 4.1 and 6.2 to the hierarchical setting.

Definition 7.4 Let $\langle \mathcal{B}, \mathcal{F}\rangle$ be a logical bilattice, where $\mathcal{B}$ is an interlaced bilattice with negation and conflation and $\mathcal{F}$ is a prime bifilter on it. As before, $\langle\langle \mathcal{A}, \leq_t\rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle, \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle\rangle$ is a Strict/Tolerant structure, where $\mathcal{A}$ is the anticonsistent part of $\mathcal{B}$. We write $\mathsf{ST}_n\langle \mathcal{B}, \mathcal{F}\rangle$ as short for $\mathsf{ST}_n\langle\langle \mathcal{A}, \leq_t\rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle, \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle\rangle$, and $\mathsf{TS}_n\langle \mathcal{B}, \mathcal{F}\rangle$ for $\mathsf{TS}_n\langle\langle \mathcal{A}, \leq_t\rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle, \mathcal{D}_T\langle \mathcal{B}, \mathcal{F}\rangle\rangle$. Also recall that $\mathsf{C}\langle \mathcal{B}, \mathcal{F}\rangle$ abbreviates $\mathsf{C}\langle\langle \mathcal{E}, \leq_t\rangle, \mathcal{D}_S\langle \mathcal{B}, \mathcal{F}\rangle\rangle$, where $\mathcal{E}$ is the exact part of $\mathcal{B}$.

The original version of the heirarchy above was introduced in [5], where only the case of classical logic and the original notion of ST based on the Kleene and Priest logics were considered. This work was extended in [15], but still only the classical setting was examined. In fact, the results in the papers just cited extend to the entire infinite family of non-distributive logical De Morgan algebras, with classical logic as the simplest example. We are seeing a very general phenomenon.

Proposition 7.5 *Let $\langle \mathcal{B}, \mathcal{F} \rangle$ be a logical bilattice, where $\mathcal{B}$ is interlaced and has negation and conflation. For $n = 1, 2, \ldots$, the many valued $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$ and the Strict/Tolerant $\mathsf{ST}_n\langle \mathcal{B}, \mathcal{F} \rangle$ validate the same members of Seq^n, but differ on Seq^{n+1}. Likewise for $n = 1, 2, \ldots$, $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$ and the Tolerant/Strict setting $\mathsf{TS}_n\langle \mathcal{B}, \mathcal{F} \rangle$ antivalidate the same members of Seq^n but differ on Seq^{n+1}.*

This Proposition is an easy consequence of a sequence of technical lemmas given below and we omit the simple argument. By combining Proposition 7.5 with the construction outlined in Section 4, we have the following.

Corollary 7.6 *Let $\langle L, D \rangle$ be a non-distributive logical De Morgan algebra. The Strict/Tolerant structure $\langle L^*, D, D^* \rangle$ is such that for each $n = 1, 2, \ldots$ we have $\mathsf{C}\langle L, D \rangle$ and $\mathsf{ST}_n\langle L^*, D, D^* \rangle$ validate the same members of Seq^n, and $\mathsf{C}\langle L, D \rangle$ and $\mathsf{TS}_n\langle L^*, D, D^* \rangle$ antivalidate the same members of Seq^n.*

The Lemmas below generalize similar results from [5, 15]. To keep clutter down, throughout all of them $\langle \mathcal{B}, \mathcal{F} \rangle$ is a logical bilattice where $\mathcal{B}$ is interlaced, with negation and conflation. Recall that $\mathcal{D}_T\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{A}$ and $\mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{E}$, where $\mathcal{A}$ and $\mathcal{E}$ are the anticonsistent and exact parts of $\mathcal{B}$. We systematically use $\mathcal{A}$ and $\mathcal{E}$ in these roles below. For the first two of the lemmas the ground cases have appeared earlier in this paper but in a different form, when discussing strict/tolerant and tolerant/strict logics. They are repeated here for convenience. There is similar repetition involved in the third Lemma.

Lemma 7.7 *Let v be a valuation mapping to the anticonsistent part of $\mathcal{B}$, and let v^s be any sharpening of v. Then for each $n = 1, 2, 3, \ldots$ we have the following.*

1. $\mathsf{TS}_n\langle \mathcal{B}, \mathcal{F} \rangle \models_v \Gamma \Rightarrow_n \Delta$ *implies* $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \models_{v^s} \Gamma \Rightarrow_n \Delta$.
2. $\mathsf{ST}_n\langle \mathcal{B}, \mathcal{F} \rangle \not\models_v \Gamma \Rightarrow_n \Delta$ *implies* $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \not\models_{v^s} \Gamma \Rightarrow_n \Delta$

Proof We show 1 and 2 together. Let v be a valuation mapping to anticonsistent values, and let v^s be a sharpening of v.

Basis Ground step, $n = 1$. By (Seq-3) and (Seq-4) we can replace $\Rightarrow_1$ with $\Rightarrow$, TS_1 with TS, and ST_1 with ST.

Case 1. Assume $\mathsf{TS}\langle \mathcal{B}, \mathcal{F} \rangle \models_v \Gamma \Rightarrow \Delta$, where Γ and Δ are sets of formulas. Suppose that for each $X \in \Gamma$, $v^s(X) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{E}$. Since $v^s \leq_k v$, by upward closure of prime bifilters $v(X) \in \mathcal{F}$, and so $v(X) \in \mathcal{D}_T\langle \mathcal{B}, \mathcal{F} \rangle = \mathcal{F} \cap \mathcal{A}$. Then by our assumption, for some $Y \in \Delta$ we have $v(Y) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle$, in particular, $v(Y) \in \mathcal{E}$. Since $v^s \leq_k v$ and both $v^s(Y)$ and $v(Y)$ are exact, then $v^s(Y) = v(Y)$, so $v^s(Y) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle$. We have shown that $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \models_{v^s} \Gamma \Rightarrow \Delta$.

Case 2. Assume that $\mathsf{ST}\langle \mathcal{B}, \mathcal{F} \rangle \not\models_v \Gamma \Rightarrow \Delta$. Γ and Δ are sets of formulas and, for each $X \in \Gamma$, $v(X) \in \mathcal{F} \cap \mathcal{E}$ while for each $Y \in \Delta$, $v(Y) \notin \mathcal{F} \cap \mathcal{A}$. Since v maps to $\mathcal{A}$, it must be that $v(Y) \notin \mathcal{F}$ for each $Y \in \Delta$.
Let $X \in \Gamma$. We have that $v^s \leq_k v$ and v^s maps to $\mathcal{E}$. Since both $v^s(X)$ and $v(X)$ are exact, $v^s(X) = v(X)$. Then $v^s(X) \in \mathcal{F}$, and hence $v^s(X) \in \mathcal{D}_S\langle \mathcal{B}, \mathcal{F} \rangle$.
Let $Y \in \Delta$. If $v^s(Y) \in \mathcal{F}$, by upward closure of prime bifilters $v(Y) \in \mathcal{F}$ which is not the case, hence $v^s(Y) \notin \mathcal{F}$ and so $v^s(Y) \notin \mathcal{D}_S\langle \mathcal{D}, \mathcal{F} \rangle$.
We have shown that $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \not\models_{v^s} \Gamma \Rightarrow \Delta$.

Induction Step Assume that both implications 1 and 2 hold for n.

Case 1. Suppose $\mathsf{ST}_{n+1}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_{n+1} \Delta$. Then $\mathsf{TS}_n\langle\mathcal{B},\mathcal{F}\rangle \models_v \gamma$ for every $\gamma \in \Gamma$ and $\mathsf{ST}_n\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \delta$, for every $\delta \in \Delta$. By the induction hypothesis, $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_{v^s} \gamma$ for every $\gamma \in \Gamma$, and $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_{v^s} \delta$ for every $\delta \in \Delta$. These are the conditions for $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_{v^s} \Gamma \Rightarrow_{n+1} \Delta$.

Case 2. Suppose $\mathsf{TS}_{n+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_{n+1} \Delta$. Then $\mathsf{ST}_n\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \gamma$ for some $\gamma \in \Gamma$ or $\mathsf{TS}_n\langle\mathcal{B},\mathcal{F}\rangle \models \delta$ for some $\delta \in \Delta$. By the induction hypothesis, $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_{v^s} \gamma$ for some $\gamma \in \Gamma$ or $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_{v^s} \delta$ for some $\delta \in \Delta$. But then $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_{v^s} \Gamma \Rightarrow_{n+1} \Delta$.

■

Lemma 7.8 *Let v be a valuation mapping to the exact part $\mathcal{E}$ of $\mathcal{B}$. Then for each $n = 1, 2, 3, \ldots$ we have the following.*

1. $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_n \Delta$ *implies* $\mathsf{TS}_n\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_n \Delta$.
2. $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_n \Delta$ *implies* $\mathsf{ST}_n\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_n \Delta$

Proof Once again we show 1 and 2 together, by induction.

Basis Ground case, replacing $\Rightarrow_1$ with $\Rightarrow$, TS_1 with TS, and ST_1 with ST, as in the previous proof.

Case 1 Assume Γ and Δ are sets of formulas and $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow \Delta$, for valuation v mapping to $\mathcal{E}$. Suppose that $v(X) \in \mathcal{D}_T\langle\mathcal{B},\mathcal{F}\rangle$ for each $X \in \Gamma$. Since v maps to $\mathcal{E}$, we have the stronger fact that $v(X) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$. But then since $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow \Delta$ we have $v(Y) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ for every $Y \in \Delta$. We have shown that $\mathsf{TS}\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow \Delta$.

Case 2 Assume $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow \Delta$, for valuation v mapping to $\mathcal{E}$. Then for each $X \in \Gamma$, $v(X) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle = \mathcal{F} \cap \mathcal{E}$ while for each $Y \in \Delta$, $v(Y) \notin \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$. Since v maps to $\mathcal{E}$, if $v(Y) \notin \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ it must be that $v(Y) \notin \mathcal{F}$, and so $v(Y) \notin \mathcal{D}_T\langle\mathcal{B},\mathcal{F}\rangle$ for each $Y \in \Delta$. We have shown that $\mathsf{ST}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow \Delta$.

Induction Step Assume that both implications 1 and 2 hold for n.

Case 1 Suppose $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_{n+1} \Delta$. Then $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \gamma$ for some $\gamma \in \Gamma$ or $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models \delta$ for some $\delta \in \Delta$, where γ and δ are level n sequents. By the induction hypothesis, $\mathsf{ST}_n\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \gamma$ or $\mathsf{TS}_n\langle\mathcal{B},\mathcal{F}\rangle \models_v \delta$. But then $\mathsf{TS}_{n+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_{n+1} \Delta$.

Case 2 Suppose $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_{n+1} \Delta$. Then $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_v \gamma$ for every $\gamma \in \Gamma$ and $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \delta$, for every $\delta \in \Delta$. By the induction hypothesis, $\mathsf{TS}_n\langle\mathcal{B},\mathcal{F}\rangle \models_v \gamma$ for every $\gamma \in \Gamma$, and $\mathsf{ST}_n\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \delta$ for every $\delta \in \Delta$. These are the conditions for $\mathsf{ST}_{n+1}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_{n+1} \Delta$.

■

Lemma 7.9 *For each $n = 1, 2, 3, \ldots$ there is an* Seq^{n+1} *schema for which each instance is a validity of* $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$ *but some instance is not a validity of* $\mathsf{ST}_n\langle\mathcal{B},\mathcal{F}\rangle$.

Proof First suppose $n = 1$, and consider the following schema in Seq^2, where A is a formula and Γ, Δ are sets of formulas.

$$(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A) \Rightarrow_2 (\Gamma \Rightarrow_1 \Delta)$$

Assume we have a valuation v in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$. To show

$$\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A) \Rightarrow_2 (\Gamma \Rightarrow_1 \Delta)$$

we argue contrapositively. Suppose $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_1 \Delta$. Then for every $X \in \Gamma$, $v(X) \in \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$, and for every $Y \in \Delta$ is $v(Y) \notin \mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$. Either $v(A)$ is in $\mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ or it isn't. If it is, then $v(X)$ is in $\mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ for every X in Γ, A, so $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v (\Gamma, A \Rightarrow_1 \Delta)$. If it isn't, then $v(Y)$ is not in $\mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$ for every $Y \in \Delta, A$, so $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_1 \Delta, A$.

Next we show the negative result concerning $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle$. Pick any sequent $\Gamma \Rightarrow_1 \Delta$ that is not valid in $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle$, and let v be a valuation such that $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \Gamma \Rightarrow_1 \Delta$. That is, v maps every member of Γ to $\mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$, and maps no member of Δ to a member of $\mathcal{D}_T\langle\mathcal{B},\mathcal{F}\rangle$. Take for A a propositional letter that does not occur in Γ or in Δ and redefine v so that $v(A) = \top$, where $\top$ is the largest member of $\mathcal{B}$ in the $\leq_k$ ordering. This can be done without changing the behavior of v on Γ or Δ. We know that $\top$ is in the prime bifilter $D \times L$ because prime bifilters are upward closed in both orderings. Also $\top$ is anticonsistent, but not exact. Then $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma, A \Rightarrow_1 \Delta)$ because v does not map every member of Γ, A to $\mathcal{D}_S\langle\mathcal{B},\mathcal{F}\rangle$, since $v(A)$ is not exact. Also $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow \Delta, A)$ because $v(A) = \top \in \mathcal{D}_T\langle\mathcal{B}, D \times L\rangle$, since $\top$ is anticonsistent. Then

$$\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \not\models_v (\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A) \Rightarrow_2 (\Gamma \Rightarrow_1 \Delta).$$

For values of $n \geq 2$, properly the argument is by induction but the first few cases will suffice to give the idea without all the formal details. For $n = 2$, use the following Seq^3 schema.

$$\emptyset \Rightarrow_3 [(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A) \Rightarrow_2 (\Gamma \Rightarrow_1 \Delta)]$$

It is easy to see that $\mathsf{ST}_2\langle\mathcal{B},\mathcal{F}\rangle \models_v [\emptyset \Rightarrow_3 [(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A) \Rightarrow_2 (\Gamma \Rightarrow_1 \Delta)]]$ if and only if $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v [(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A) \Rightarrow_2 (\Gamma \Rightarrow_1 \Delta)]$, and similarly for $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$, and this reduces things to the previous case.

Likewise, for the $n = 3$ case use the following

$$\emptyset \Rightarrow_4 [\emptyset \Rightarrow_3 [(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A) \Rightarrow_2 (\Gamma \Rightarrow_1 \Delta)]]$$

and so on. ■

Lemma 7.10 *For each $n = 1, 2, 3, \ldots$ there is a* Seq^{n+1} *schema for which every instance is an antivalidity of $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$ but some instance is not an antivalidity of $\mathsf{TS}_n\langle\mathcal{B},\mathcal{F}\rangle$.*

Proof For the $n = 1$ case we use the following schema in Seq^2, where A is any formula.

$$(A \Rightarrow_1 A) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset)$$

For every valuation v in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$ it is clear that $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_v A \Rightarrow_1 A$ but $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v \emptyset \Rightarrow_1 \emptyset$, so

$$\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v (A \Rightarrow_1 A) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset)$$

and thus we have an antivalidity of $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$.

Next we consider $\mathsf{TS}\langle\mathcal{B},\mathcal{F}\rangle$. Take A to be a propositional letter and let v be a valuation such that $v(A) = \top$. Since $\top \in \mathcal{F}$ and is anticonsistent but not exact in $L \odot L$, $v(A)$ is tolerantly designated but not strictly designated. Then $\mathsf{TS}\langle\mathcal{B},\mathcal{F}\rangle \not\models_v A \Rightarrow A$ so

$$\mathsf{TS}_2\langle\mathcal{B},\mathcal{F}\rangle \models_v (A \Rightarrow_1 A) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset)$$

and thus $(A \Rightarrow_1 A) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset)$ is not an antivalidity of $\mathsf{TS}_2\langle\mathcal{B},\mathcal{F}\rangle$.

For values of $n \geq 2$ we proceed exactly as in the proof of the previous Lemma, using the following metasequents.

$$\begin{gathered}\emptyset \Rightarrow_3 [(A \Rightarrow_1 A) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset)] \\ \emptyset \Rightarrow_4 [\emptyset \Rightarrow_3 [(A \Rightarrow_1 A) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset)]] \\ \vdots\end{gathered}$$

■

8 The Status of Cut

Despite the title, this section is really about the general status of rules, metarules, ..., sequents, metasequents, ... Cut simply serves as a significant case study. For the rest of this section assume $\langle \mathcal{B}, \mathcal{F} \rangle$ is a logical bilattice, where $\mathcal{B}$ is interlaced and has negation and conflation. Recall Definition 7.4 of $\mathsf{ST}_n\langle \mathcal{B}, \mathcal{F} \rangle$, $\mathsf{TS}_n\langle \mathcal{B}, \mathcal{F} \rangle$, and $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$.

In Section 7 we formulated cut as a rule, and we repeat it here for convenience. We use notation that has $\Rightarrow_1$ rather than $\Rightarrow$. This is allowed since $\Rightarrow$ and $\Rightarrow_1$ have the same behavior. It is a minor point about which we say no more.

$$\frac{\Gamma, A \Rightarrow_1 \Delta \qquad \Gamma \Rightarrow_1 \Delta, A}{\Gamma \Rightarrow_1 \Delta} \tag{$\star$}$$

We also discussed how to represent cut as a metasequence, and we repeat this here too.

$$\{(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A)\} \Rightarrow_2 \{\Gamma \Rightarrow_1 \Delta\} \tag{$\star\star$}$$

We said validity for these could be understood locally or globally, and we said enough about these notions to get things going. It is time to clear up some remaining obscurity about the two validity notions, and establish the relationships they have to each other.

For the moment, let us assume we are just discussing classical logic. Then a particular classical valuation v *locally* validates the sequent $(\star\star)$ provided, if it validates both $\Gamma, A \Rightarrow_1 \Delta$ and $\Gamma \Rightarrow_1 \Delta, A$ then it validates $\Gamma \Rightarrow_1 \Delta$. The point is, the *same* valuation v is applied to each of the three level 1 sequents throughout. In brief, local validation requires that for each valuation, if *it* valuates the premises then *it* valuates the conclusion. Then the metasequent $(\star\star)$ is locally valid if each valuation locally validates it.

The same sequent, $(\star\star)$, is *globally* valid provided, if each of the two premises is valid then the conclusion is valid, where a level 1 sequent is valid if every valuation valuates it in the usual way. Or, restating, if *every* valuation validates each of the premises, then *every* valuation validates the conclusion. Rather than the same valuation being applied throughout, all valuations are applied to all three sequents, independently.

The rule version, $(\star)$, can also be understood either locally or globally, of course. But somehow, as things get more complex the sequent version seems easier to work with. Perhaps that is just personal prejudice, however.

We move from classical logic to its generalization $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$, where local and global validity are understood in same way as classically. There is an easy general result: if $(\star\star)$ is locally valid then $(\star\star)$ is globally valid, and similarly for $(\star)$ of course. Here is the easy argument. Assume $(\star\star)$ is locally valid; each valuation v individually validates it, meaning that if v validates the two premise sequents, v validates the consequent sequent. And now suppose that the two level 1 premise sequents of $(\star\star)$ are valid, that is, each is validated by every valuation. Let v be an arbitrary valuation in $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$. Then $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \models_v \Gamma, A \Rightarrow_1 \Delta$ and $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \models_v \Gamma \Rightarrow_1 \Delta, A$, since all valuations validate these sequents. Since $(\star\star)$ is locally valid, and v validates each premise, it must validate the consequent, that is, $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle \models_v \Gamma \Rightarrow_1 \Delta$. And since v was arbitrary, every valuation must validate $\Gamma \Rightarrow_1 \Delta$, so we have shown validity of $\Gamma \Rightarrow_1 \Delta$ in $\mathsf{C}\langle \mathcal{B}, \mathcal{F} \rangle$ under the assumption of the validity of the two premise sequents.

The argument we just gave does not work if the logic is not an ordinary many-valued logic, but instead is a strict/tolerant logic. Let us assume $(\star\star)$ is locally valid in $\mathsf{ST}_2\langle \mathcal{B}, \mathcal{F} \rangle$. That is, each valuation v individually validates it. Symbolically we assume that for each v, $\mathsf{ST}_2\langle \mathcal{B}, \mathcal{F} \rangle \models_v \{(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A)\} \Rightarrow_2 \{\Gamma \Rightarrow_1 \Delta\}$, and using Definition 7.3, this says that if $\mathsf{TS}_1\langle \mathcal{B}, \mathcal{F} \rangle \models_v (\Gamma, A \Rightarrow_1 \Delta)$ and $\mathsf{TS}_1\langle \mathcal{B}, \mathcal{F} \rangle \models_v (\Gamma \Rightarrow_1 \Delta, A)$ then $\mathsf{ST}_1\langle \mathcal{B}, \mathcal{F} \rangle \models_v (\Gamma \Rightarrow_1 \Delta)$. And now suppose the two premises of $(\star\star)$ are valid in $\mathsf{ST}_1\langle \mathcal{B}, \mathcal{F} \rangle$; we would like to show the consequent is also valid in $\mathsf{ST}_1\langle \mathcal{B}, \mathcal{F} \rangle$. But there is a mismatch here: we are supposing that v validates the premises in $\mathsf{ST}_1\langle \mathcal{B}, \mathcal{F} \rangle$, but we actually need this in $\mathsf{TS}_1\langle \mathcal{B}, \mathcal{F} \rangle$. The attempt to derive global validity does not go through. Still, despite the failure of the argument just discussed, we actually do have global cut, but by a more roundabout route.

Proposition 8.1 *The cut schema has the following behavior.*

1. *($\star\star$) is not locally valid in $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle$ in the sense that there is an instance of the schema, and a valuation v, such that $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma, A \Rightarrow_1 \Delta)$ and $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow_1 \Delta, A)$ but $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \not\models_v (\Gamma \Rightarrow_1 \Delta)$.*
2. *($\star\star$) is globally valid in $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle$ in the sense that for every instance of the schema, if $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma, A \Rightarrow_1 \Delta)$ for every v and $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow_1 \Delta, A)$ for every v then $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow_1 \Delta)$ for every v.*
3. *As a level 2 sequent ($\star\star$) is locally valid in $\mathsf{ST}_2\langle\mathcal{B},\mathcal{F}\rangle$, in the sense that for every instance of the schema and for every v, $\mathsf{ST}_2\langle\mathcal{B},\mathcal{F}\rangle \models_v \{(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A)\} \Rightarrow_2 \{\Gamma \Rightarrow_1 \Delta\}$.*

Proof The arguments are as follows.

1. This is Proposition 4.4.
2. Suppose that both $\Gamma, A \Rightarrow_1 \Delta$ and $\Gamma \Rightarrow_1 \Delta, A$ are valid in $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle$. Then by Corollary 7.6 we have that both $\Gamma, A \Rightarrow_1 \Delta$ and $\Gamma \Rightarrow_1 \Delta, A$ are valid in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$. By Proposition 4.4, cut is locally valid in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$ and then, as we showed above, globally valid as well. We conclude that $\Gamma \Rightarrow_1 \Delta$ is valid in $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle$, and hence it is valid in $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle$ by Proposition 4.3.
3. Finally we show that $\mathsf{ST}_2\langle\mathcal{B},\mathcal{F}\rangle \models_v \{(\Gamma, A \Rightarrow_1 \Delta), (\Gamma \Rightarrow_1 \Delta, A)\} \Rightarrow_2 \{\Gamma \Rightarrow_1 \Delta\}$ for an arbitrary v. Assume $\mathsf{TS}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma, A \Rightarrow_1 \Delta$ and $\mathsf{TS}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_1 \Delta, A$; we show $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_1 \Delta$.

   From the assumptions and part 1 of Lemma 7.7 we have both $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_{v^s} \Gamma, A \Rightarrow_1 \Delta$ and $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_{v^s} \Gamma \Rightarrow_1 \Delta, A$, where v^s is any sharpening of v. It follows by Proposition 4.4 that $\mathsf{C}\langle\mathcal{B},\mathcal{F}\rangle \models_{v^s} \Gamma \Rightarrow_1 \Delta$. Then $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v \Gamma \Rightarrow_1 \Delta$ by part 2 of Lemma 7.7.

■

At the end of Section 7 we made use of a kind of ‘lifting’ of level n sequents to level $n+1$. We now need this more systematically. We use notation from [15].

Definition 8.2 Let $\gamma \in \mathit{Seq}^n$. We define a sequence $\gamma^0, \gamma^1, \gamma^2, \ldots$, where $\gamma^k \in \mathit{Seq}^{n+k}$.

$$\begin{aligned}\gamma^0 &= \gamma \\ \gamma^{k+1} &= (\emptyset \Rightarrow^{n+(k+1)} \gamma^k)\end{aligned}$$

It is easy to check that for any valuation v, $\mathsf{ST}_n\langle\mathcal{B},\mathcal{F}\rangle \models_v \gamma^0$ if and only if $\mathsf{ST}_{n+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v \gamma^1$ if and only if $\mathsf{ST}_{n+2}\langle\mathcal{B},\mathcal{F}\rangle \models_v \gamma^2$ if and only if ..., and similarly for TS. This immediately gives us the following metacut generalization of Proposition 8.1.

Corollary 8.3 *For $k = 0, 1, 2, \ldots$, $\{(\Gamma, A \Rightarrow_1 \Delta)^k, (\Gamma \Rightarrow_1 \Delta, A)^k\} \Rightarrow_{k+2} (\Gamma \Rightarrow_1 \Delta)^k$ is a* Seq^{k+2} *schema with the following behavior.*

1. *The sequent is not locally valid in $\mathsf{ST}_{k+1}\langle\mathcal{B},\mathcal{F}\rangle$. That is, $\mathsf{ST}_{k+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma, A \Rightarrow_1 \Delta)^k$ and $\mathsf{ST}_{k+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow_1 \Delta, A)^k$ does not always imply $\mathsf{ST}_{k+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow_1 \Delta)^k$.*
2. *The sequent is globally valid in ST_{k+1}. That is, $\mathsf{ST}_{k+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma, A \Rightarrow_1 \Delta)^k$ for all v, and $\mathsf{ST}_{k+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow_1 \Delta, A)^k$ for all v, always implies $\mathsf{ST}_{k+1}\langle\mathcal{B},\mathcal{F}\rangle \models_v (\Gamma \Rightarrow_1 \Delta)^k$ for all v.*
3. *The sequent is locally valid in $\mathsf{ST}_{k+2}\langle\mathcal{B},\mathcal{F}\rangle$, when understood as a level $k+2$ sequent. That is, for each v, $\mathsf{ST}_{k+2}\langle\mathcal{B},\mathcal{F}\rangle \models_{k+2} \{(\Gamma, A \Rightarrow_1 \Delta)^k, (\Gamma \Rightarrow_1 \Delta, A)^k\} \Rightarrow_{k+2} (\Gamma \Rightarrow_1 \Delta)^k$.*

Analogous results obtain for antivalidity and TS_n. Recall the rule examined in Proposition 6.4, and restated below.

$$\frac{X \Rightarrow_1 X}{\emptyset \Rightarrow_1 \emptyset} \qquad (\dagger)$$

The sequent formulation of this is the following.

$$(X \Rightarrow_1 X) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset). \tag{††}$$

Proposition 8.4 *Antivalidity has the following behavior.*

1. *The rule (†) has an instance that is not locally antivalid in* $\mathsf{TS}_1\langle\mathcal{B},\mathcal{F}\rangle$.
2. *The sequent (††) is locally antivalid in* $\mathsf{TS}_2\langle\mathcal{B},\mathcal{F}\rangle$.

Proof The arguments are as follows.

1. This appeared earlier, in Proposition 6.4.
2. The claim is that for every valuation v, $\mathsf{TS}_2\langle\mathcal{B},\mathcal{F}\rangle \not\models_v (X \Rightarrow_1 X) \Rightarrow_2 (\emptyset \Rightarrow_1 \emptyset)$. This in turn is equivalent to having both $\mathsf{ST}_1\langle\mathcal{B},\mathcal{F}\rangle \models_v (X \Rightarrow_1 X)$ and $\mathsf{TS}_1\langle\mathcal{B},\mathcal{F}\rangle \not\models_v (\emptyset \Rightarrow_1 \emptyset)$. Both are easily checked.

■

Again things lift to higher levels, and we omit details.

9 Conclusion

Strict/tolerant logic, ST, was introduced as a kind of mashup of Kleene's Strong Three Valued Logic and Priest's Logic of Paradox, and was seen to be of interest because of its connections with classical logic. For some purposes it can serve as a replacement for classical logic, since it is missing some of the features that can make classical logic problematic at times. More fundamentally, it has led to a multifaceted discussion concerning just what classical logic is and, more generally, what makes different logics different. One might think that a logic can be identified with its consequence relation, but here we have a counter-example. We can go to higher order consequence, meta, metameta, and so on, but no additional level is sufficient to fully characterize classical logic. When the whole sequence of levels has been accumulated, as in [5], we have full consequence agreement between ST and classical logic but [15] argues that even so, ST and classical logic aren't the same because the two differ on *antivalidity*. The discussion continues, as it should.

In [11] and here I have avoided entirely the issue of what makes logics the same, or different. Instead I have been investigating the extent to which the ideas behind ST, and its connections to classical logic are, in fact, general. It turned out that there is an infinite family of many valued logics that have Strict/Tolerant counterparts, that this extends from consequence to meta consequence, metameta consequence, and so on, and further that these many valued logics also have Tolerant/Strict counterparts, with all that that brings. What is happening is not rare.

The machinery we have been using for our investigation is that of bilattices. This, so far, has been a good fit, but there is no reason to suppose that it is the only machinery that might work. And further, what about logics that aren't many valued? Is there such a thing as a Strict/Tolerant counterpart of intuitionistic logic, for instance? Indeed, what might this mean? Or again, what might be said about those sequents for which, if all antecedents are intuitionistically valid then some consequent is classically valid? Once the door has been opened into a world in which different sides of sequents meet different standards, might it be possible that further interesting things will be seen?

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