Possible World Semantics for First Order LP

Melvin Fitting

April 7, 2012

Abstract

In the tech report [3] an elegant formulation of the first-order logic of proofs was given, FOLP. This logic plays a fundamental role in providing an arithmetic semantics for first-order intuitionistic logic, as was shown. In particular, the tech report proved an arithmetic completeness theorem, and a realization theorem for the FOLP. In this paper we provide a possible-world semantics for FOLP, based on the propositional semantics of [4]. We also give an Mkrtychev semantics. Motivation and intuition for FOLP can be found in [3], and are not discussed here.

1 Introduction

Propositional Justification Logics are modal-like logics in which the usual necessity operator is split into a family of more complex terms called *justifications*. Instead of $\Box A$ one finds $t{:}A$, which can be read "t is a justification for A." The structure of t embodies, in a straightforward way, how we come to know A or verify A. Many standard propositional modal logics have justification logic counterparts, where the notion of *counterpart* has a precise definition via what are called *Realization Theorems*. One can think of justification logics as *explicit* versions of modal logics, with conventional modal operators embodying justifications in an *implicit* way, but we do not go further into this point here. The first propositional justification logic was LP, the Logic of Proofs, an explicit version of propositional S4. It was introduced by Artemov as part of a project to provide an arithmetic semantics for propositional intuitionistic logic, [1]. Briefly, propositional intuitionistic logic embeds into propositional S4 via the well-known Gödel translation. Propositional S4 in turn embeds into LP via a Realization Theorem. Propositional LP embeds into arithmetic, Artemov's *Arithmetical Completeness Theorem*.

Since this initial work there has been much study of propositional justification logics, thinking of them as explicit logics of knowledge or belief. A general survey of the subject can be found in [2]. But to reiterate, all this was at the propositional level. Recently Artemov and Yavorskaya [3] defined a first-order extension of the logic of proofs, FOLP. The original results on an arithmetic semantics for propositional intuitionistic logic were shown to extend to the first-order case as well. This completed the arithmetic semantics project for intuitionistic logic, but it also introduced a new family of interesting explicit logics to study.

In [4] a possible world semantics was introduced for LP, and for a few other propositional justification logics. On the one hand this semantics elaborates the familiar Kripke semantics for modal logics by adding machinery to model the behavior of explicit reasons, and on the other hand it extends, in a direct way, an earlier LP semantics of Mkrtychev, [6]. The purpose of the present paper is to extend this propositional work to a first-order setting. The resulting possible world semantics obeys a *monotonicity* condition, familiar from propositional modal logics. This is natural because of the intended application to intuitionistic logic. We postpone to future work the study of constant domain versions. The work here is specifically for the first-order version of LP. Simple modifications adapt the results to several other logics, and we will discuss this briefly at the end of the paper.

The contents of this paper first appeared, in a somewhat different form, in [5].

2 FOLP, The Language and the Axioms

Let us think proof-theoretically for a bit. In a first-order proof free variables play two different but easily confused roles. One role is simply that of a formal symbol. The Universal Generalization rule allows us to claim a proof of $(\forall x)A(x)$ given a proof of $A(x)$. In this x is a syntactic object, with no inherent meaning. The other role of variables is that of a place-holder that can be substituted for. Suppose we have a proof of $A(x)$, and say 3 is a constant symbol of our language. We can turn the proof of $A(x)$ into a proof of $A(3)$ by going through it and replacing all free occurrences of x with occurrences of 3 (assuming universal generalization on x was not used). Similarly for $A(4)$ and so on. We can think of the proof as more like a proof template from which we can stamp out many concrete proofs. Note that we had to put in a *caveat* about non-use of universal generalization—the two roles of variables are not compatible.

In propositional LP, if t is a proof term (justification term) and A is a formula then $t{:}A$ is a formula, and can be thought of as asserting that A is so, with t as its proof (justification). Indeed, under an arithmetic interpretation, proof terms are interpreted as Gödel numbers for proofs. In the first-order language of [3] this is modified into $t{:}_X A$, where X is a finite set of variables. This can be thought of as asserting that t represents a proof of A in which the variables in X are the ones that can be substituted for, and hence are not allowed in applications of universal generalization. Our possible world semantics directly incorporates the idea of two roles for variables, as will be seen later on, and the axioms in this section should be read with the double role in mind.

This following definition is taken from the technical report of Artemov and Yavorskaya, [3].

The language of FOLP has a countable set of predicate symbols of any arity, but no function symbols or equality. There are also countably many individual variables, and an atomic formula is $Q(x_1, x_2, \ldots, x_n)$, where Q is a predicate symbol of arity n and each x_i is an individual variable. (Typically we write x, y, $\ldots$, with or without subscripts, for individual variables.) Formulas are built up as usual using Boolean connectives and quantifiers over individual variables, and one additional construction described below.

The language of FOLP also has a family of *proof terms*, more generally called *justification terms* when logics not directly connected with intuitionistic logic are considered. These are built up from a countable family of proof variables (typically p or p_i) and a countable family of proof constants (typically c or c_i). More complex proof terms are built up using special function and operation symbols, as follows. If t and s are proof terms, then $t \cdot s$, $!t$, and $t + s$ are proof terms. This much is inherited from the propositional logic LP, and we do not discuss their intended meanings here. In addition, if x is an individual variable then $\mathsf{gen}_x(t)$ is a proof term. It should be noted that in $\mathsf{gen}_x(t)$ the operation symbol is $\mathsf{gen_x}$; we are assuming an infinite family of such operation symbols, one for each x. The individual variable x does not have a free occurrence in $\mathsf{gen}_x(t)$—indeed, proof terms do not contain occurrences of individual variables.

In addition to the usual rules of formation for formulas, we have the following. If t is a proof term, X is a finite set of individual variables, and A is a formula, then $t{:}_X A$ is a formula.

Free individual variable occurrences in formulas are defined as usual, with the addition of the following. The free individual variable occurrences in $t{:}_X A$ are the free individual variable occurrences in A, provided the variables also occur in X, together with the occurrences in X itself.

We need the standard notion of an individual variable y being free for x in a formula—then occurrences of y can be substituted for free occurrences of x in the formula. This is defined as usual, with one more case. An individual variable y is free for x in $t{:}_X A$ if two conditions are met: first y is free for x in A and second, if y occurs free in A then $y \in X$.

Finally we have the following axiom system, taken from [3] and presented with the original numbering. Throughout X, Y, etc. denote finite sets of individual variables. If y is an individual variable, then Xy is short for $X \cup \{y\}$. In addition, when writing Xy it is assumed that $y \notin X$.

The following are axiom schemes, in which A and B are formulas, s and t are proof terms, X is a set of individual variables, and y is an individual variable.

A1 classical axioms of first order logic

A2 $t{:}_{Xy}A \rightarrow t{:}_X A$, provided y does not occur free in A

A3 $t{:}_X A \rightarrow t{:}_{Xy}A$

B1 $t{:}_X A \rightarrow A$

B2 $s{:}_X(A \rightarrow B) \rightarrow (t{:}_X A \rightarrow (s \cdot t){:}_X B)$

B3 $t{:}_X A \rightarrow (t + s){:}_X A, \quad s{:}_X A \rightarrow (t + s){:}_X A$

B4 $t{:}_X A \rightarrow !t{:}_X t{:}_X A$

B5 $t{:}_X A \rightarrow \mathsf{gen}_x(t){:}_X \forall x A$, provided $x \notin X$

R1 $\vdash A, A \rightarrow B \;\Rightarrow\; \vdash B$

R2 $\vdash A \;\Rightarrow\; \vdash \forall x A$

R3 $\vdash c{:}_\emptyset A$, where A is an axiom and c is a proof constant

We generally assume that universal instantiation is one of the axiom schemes under **A1** above, $\forall x A(x) \rightarrow A(y)$, where $A(y)$ is like $A(x)$ except that all free occurrences of x have been replaced with occurrences of y. This axiom scheme has the usual restriction that y must be free for x in $A(x)$.

Remark Recall that the familiar notion of y being free for x in a formula was modified and extended above to handle cases involving proof terms. Here is an example of what the modification saves us from. First, $t{:}_{\{x,z\}}A(x,y) \rightarrow t{:}_{\{x\}}A(x,y)$ is an instance of **A2**, where z is distinct from x and y. Then by universal generalization we get $(\forall z)[t{:}_{\{x,z\}}A(x,y) \rightarrow t{:}_{\{x\}}A(x,y)]$. Using universal instantiation we have $(\forall z)[t{:}_{\{x,z\}}A(x,y) \rightarrow t{:}_{\{x\}}A(x,y)] \rightarrow [t{:}_{\{x,y\}}A(x,y) \rightarrow t{:}_{\{x\}}A(x,y)]$ and hence $t{:}_{\{x,y\}}A(x,y) \rightarrow t{:}_{\{x\}}A(x,y)$ by *modus ponens*. But we don't want this formula to be provable, because it says the *proviso* in axiom scheme **A2** can be dropped, and we will see in Example 11.2 that doing so is unsound with respect to our semantics. Fortunately the derivation just presented is not correct, because the use of universal instantiation is not permitted since y is not free for z in $[t{:}_{\{x,z\}}A(x,y) \rightarrow t{:}_{\{x\}}A(x,y)]$.

The final rule, **R3**, is called *Axiom Necessitation*. The idea is that every axiom is justified without additional analysis, and a constant can be introduced during the course of a proof to represent such a justification. Often more control over constants is desirable.

Definition 2.1 (Constant Specification) A *constant specification* is a set $\mathcal{C}$ of FOLP formulas of the form $c{:}_{\emptyset}A$. It is assumed that A is an axiom (or at least a formula that is valid in the semantics given in the next section). A proof meets constant specification $\mathcal{C}$ provided that whenever rule **R3** is used to introduce $c{:}_{\emptyset}A$ then $c{:}_{\emptyset}A$ is a member of $\mathcal{C}$.

Every proof using the axiom system above generates a constant specification—put into $\mathcal{C}$ just the formulas introduced by **R3** during the course of the proof. For the time being we need no special restrictions on constant specifications. When we come to establishing completeness we will need such restrictions, and we discuss them in Section 6.

3 Semantics

We begin with informal motivation, before getting to the details. Models are essentially those for monotonic first-order $\mathsf{S4}$ with some extras, so we have have possible worlds and a transitive, reflexive accessibility relation. Models also have domains of quantification associated with each possible world, meeting a monotonicity condition. Why monotonicity? Recall, the motivation is to provide a semantics for intuitionistic logic. One thinks of classical mathematics as essentially Platonic. Mathematical structures are simply there, changeless and timeless. They are discovered, not made. But constructive mathematics is different. Brouwer spoke of the creative subject constructing mathematical objects, and making use of free choices in the process. Even classically we can distinguish between the 'real' universe of Platonic mathematics, and the 'known' universe. At one time, in the known mathematical universe there were no complex numbers. That structure, historically, came into the scope of our knowledge. Whether it was there all along is not epistemically important. What is important is that in the realm of what we know, complex numbers once were not, and then were. It is reasonable to assume that after creation a structure continues to exist—we do not forget. Whether we consider things constructively or epistemically, monotonicity and not constant domains is the norm.

Propositional connectives behave truth functionally at each world, as usual in modal logic. The key issue is how proof terms behave, and we discuss this informally now. Consider, as a representative example, the formula $t{:}_{\{x,y\}}Q(x,y,z,w)$, and a possible world Γ. In this formula occurrences of x and y are free, but not those of z or w. We could use the machinery of valuations to assign values to free variables, but let us simply allow members of domains to appear directly in formulas. Say a and b are in the quantification domain associated with Γ; what will it mean for $t{:}_{\{a,b\}}Q(a,b,z,w)$ to be true at Γ? For this two conditions must be met, one syntactic, one semantic.

That there is a syntactic condition might be a little surprising at first glance, but after further thought it should not be. Modal semantics works with propositions and not with formulas—equivalent formulas evaluate the same at each possible world. But different equivalent formulas might have quite different justifications—a proof of $A \wedge B$ is different from a proof of $\neg(\neg A \vee \neg B)$ after all. Syntactic matters matter. To handle this we use machinery first introduced for propositional justification logics, an *evidence function*, $\mathcal{E}$. Informally, for each proof term t and each formula A, $\mathcal{E}(t,A)$ is the set of possible worlds at which t can serve as meaningful evidence for A. (Closure conditions will be imposed on evidence functions in Definition 3.5, but need not concern us just now.) Meaningful evidence is not conclusive evidence, it is merely evidence that is, in some way, relevant.

Propositionally we take $t{:}A$ to be true at a possible world if A is true at all accessible worlds (the usual Kripkean condition) and also t serves as meaningful evidence for A at that world. With first-order machinery added, this idea becomes more complicated because of the distinction between the two roles that variables can play in proofs. Recall that in $t{:}_{\{x,y\}}Q(x,y,z,w)$ the variables in $\{x,y\}$ are supposed to be those that can be substituted for, but the remaining variables, z and w, are the ones to which universal generalization can be applied. To take the first item into account, the role of the variables in $\{x,y\}$, we will only talk about truth at possible world Γ of instances like $t{:}_{\{a,b\}}Q(a,b,z,w)$, where a and b are in the domain of Γ—in effect, we only talk about what proof term t says about the results of substitution for those variables that are subject to substitution. We still need to capture the idea that z and w are universally quantifiable. We do this by saying $Q(a,b,c,d)$ is true at every possible world Δ accessible from Γ, for every c,d in the quantificational domain of Δ. Roughly, the z and w play universal roles in $t{:}_{\{a,b\}}Q(a,b,z,w)$ because, no matter what future work we might carry out (the move from Γ to Δ), and no matter what mathematical objects we might encounter (any c and d available at Δ) we will have $Q(a,b,c,d)$.

Putting all this together, we will take $t{:}_{\{a,b\}}Q(a,b,z,w)$ to be true at Γ provided t is meaningful evidence for $Q(a,b,z,w)$ at Γ, that is, $\Gamma \in \mathcal{E}(t, Q(a,b,z,w))$, and for every Δ accessible from Γ, and for every c,d in the quantificational domain of Δ, $Q(a,b,c,d)$ is true at Δ.

Now we present the formal details.

Definition 3.1 (Skeleton) An FOLP *skeleton* is a structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}\rangle$ where $\mathcal{G}$ is a non-empty set (of states or possible worlds), $\mathcal{R}$ is a binary reflexive and transitive *accessibility relation* on $\mathcal{G}$, and $\mathcal{D}$ is a *domain function* mapping each member of $\mathcal{G}$ to a non-empty set and subject to a monotonicity condition, $\Gamma\mathcal{R}\Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$ for $\Gamma, \Delta \in \mathcal{G}$. We write $\mathcal{D}^*$ for $\cup\{\mathcal{D}(\Gamma) \mid \Gamma \in \mathcal{G}\}$, and call $\mathcal{D}^*$ the *domain of the skeleton*.

Before we expand skeletons into models, we make some remarks about languages. It is common in model theory to use formulas in which members of the domain of the model appear as if they were individual constant symbols of the language itself. This is mathematically justifiable because a formula is a sequence of symbols, and members of the domain are as much entitled to be symbols as anything else. Doing this simplifies notation considerably—valuation functions are not needed, for instance. In our case working with formulas of the model yields more than just simplification—it brings us closer to the intuitions the logic FOLP is intended to capture. In fact, we have already made use of formulas of a model in our informal discussion above.

Definition 3.2 (D-formulas) Let D be a non-empty set. A D-formula is the result of replacing some (possibly all) free occurrences of individual variables in an FOLP formula with members of D. Members of D act like individual constant symbols—not free variables—in the resulting formula and we refer to them as *individual constants*, or D *constants*, or just *domain constants*. We call a D-formula *closed* if it contains no free occurrences of individual variables, though there may be domain constants present.

Suppose a, b are domain constants, x is an individual variable, and Q is a two-place relation symbol. The formula $t{:}_{\{a\}}Q(a,x)$ is well-formed but $t{:}_{\{a\}}Q(a,b)$ is not. In the first, the occurrence of x is not free since it does not occur in the subscript set. Then in the second, the occurrence of b would also count as bound, but domain constants cannot be bound. The rules of substitution are such that non-well-formed formulas like the second cannot occur in the process of evaluating truth at possible worlds of models.

When working with D-formulas we systematically use $\vec{x}$, $\vec{y}$, etc. as sequences of individual variables, and x, y, etc. as single individual variables. Likewise we use $\vec{a}$, $\vec{b}$, etc. as sequences of D constants, and a, b, etc. as single D constants. Whenever possible we are informal about substitution notation. If we write $A(\vec{x})$, and later we write $A(\vec{a})$, we mean that free occurrences of the individual variables in $\vec{x}$ (if any) have been replaced with corresponding occurrences of domain constants in $\vec{a}$. In more complicated circumstances we will use notation like $\{\vec{x}/\vec{a}, y/b\}$ to indicate the substitution that replaces free occurrences of variables in $\vec{x}$ with D constants in $\vec{a}$, and replaces free occurrences of y with occurrences of b. We assume D constants can always be substituted for free occurrences of individual variables—in effect the notion of an D constant being free for a variable in a formula is taken to be always true.

Now we say how to expand skeletons to models. We characterized the domain of a skeleton in Definition 3.1. Models will be built on skeletons, and we will refer to the domain of a *model*, meaning the domain of its underlying skeleton. We will be working with formulas *of the model*, $\mathcal{D}^*$-formulas where $\mathcal{D}^*$ is the domain of the model, or with $\mathcal{D}(\Gamma)$-formulas, where $\mathcal{D}(\Gamma)$ is the domain associated with possible world Γ. This allows us to simplify the definition of evidence function, as compared to the version in [5].

Definition 3.3 (Models) Let $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be an FOLP skeleton. A model for FOLP based on $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ where:

1. $\mathcal{I}$ is an *interpretation function*—for each n-place relation symbol Q and each $\Gamma \in \mathcal{G}$, $\mathcal{I}(Q, \Gamma)$ is an n-place relation on $\mathcal{D}(\Gamma)$.

2. $\mathcal{E}$ is an *evidence function*—for each proof term t and each $\mathcal{D}^*$-formula A, $\mathcal{E}(t, A)$ is some set of possible worlds meeting the condition that, if $\Gamma \in \mathcal{E}(t, A)$ then all domain constants in A are from $\mathcal{D}(\Gamma)$.

As noted earlier, the idea behind the evidence function is this. If $\Gamma \in \mathcal{E}(t, A)$, then informally Γ is a possible world in which t serves as relevant evidence for the formula A.

Definition 3.4 (Lives In) Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ be an FOLP model, and let A be a $\mathcal{D}^*$-formula. We say A *lives in* Γ, where $\Gamma \in \mathcal{G}$, if all members of $\mathcal{D}^*$ that occur in A are in $\mathcal{D}(\Gamma)$.

Using the terminology just introduced, the condition in part 2 of Definition 3.3 can be rephrased as: if $\Gamma \in \mathcal{E}(t, A)$ then A must live in Γ.

Special conditions are imposed on evidence functions. Most come from LP; two are new to FOLP. *From now on we assume all FOLP models meet these Evidence Function Conditions.*

Definition 3.5 (Evidence Function Conditions) Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ be an FOLP model. We require the evidence function to meet the following conditions.

$\cdot$ **Condition** $\mathcal{E}(s, A \to B) \cap \mathcal{E}(t, A) \subseteq \mathcal{E}((s \cdot t), B)$.

$+$ **Condition** $\mathcal{E}(s, A) \cup \mathcal{E}(t, A) \subseteq \mathcal{E}((s + t), A)$.

! **Condition** $\mathcal{E}(t, A) \subseteq \mathcal{E}(!t, t{:}_X A)$, where X is the set of domain constants in A.

$\mathcal{R}$ **Closure Condition** $\Gamma \in \mathcal{E}(t, A)$ and $\Gamma \mathcal{R} \Delta$ imply $\Delta \in \mathcal{E}(t, A)$.

Instantiation Condition $\Gamma \in \mathcal{E}(t, A(x))$ and $a \in \mathcal{D}(\Gamma)$ imply $\Gamma \in \mathcal{E}(t, A(a))$.

$\mathbf{gen}_x$ **Condition** $\mathcal{E}(t, A) \subseteq \mathcal{E}(\mathsf{gen}_x(t), \forall x A)$.

Note that the monotonicity condition on domains figures into the $\mathcal{R}$ Closure Condition above. If $\Gamma \in \mathcal{E}(t, A)$, A must live in Γ. If also $\Gamma \mathcal{R} \Delta$ then using monotonicity, $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$, so A lives in Δ too, and thus part of the requirement for $\Delta \in \mathcal{E}(t, A)$ is automatic. The Instantiation Condition is a formal version of the idea that a proof of, say, $\varphi(x)$ is a template from we can generate proofs of $\varphi(a)$, $\varphi(b)$, etc., with all proofs having the same structure.

Definition 3.6 (Constant Specifications) An evidence function *meets constant specification $\mathcal{C}$* provided, if $c{:}_{\emptyset}A \in \mathcal{C}$ then for each $\Gamma \in \mathcal{G}$ such that A lives in Γ, $\Gamma \in \mathcal{E}(c, A)$. A model meets a constant specification if its evidence function does.

Now we define truth at possible worlds of models. Truth is only defined directly *for formulas having no free individual variables*, though they can contain domain constants, what we called closed $\mathcal{D}^*$ formulas earlier. The definition is extended to cover validity in the more general case afterward.

Definition 3.7 (Truth At Worlds) Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ be an FOLP model. We write $\mathcal{M}, \Gamma \Vdash A$ to symbolize that the closed $\mathcal{D}^*$ formula A is true at world Γ of $\mathcal{G}$ in model $\mathcal{M}$. The conditions that must be met are as follows.

1. Let Q be an n-place predicate symbol.
   $\mathcal{M}, \Gamma \Vdash Q(\vec{a}) \Longleftrightarrow \langle \vec{a} \rangle \in \mathcal{I}(\Gamma, Q)$;
2. $\mathcal{M}, \Gamma \nVdash \bot$;
3. $\mathcal{M}, \Gamma \Vdash A \to B \Longleftrightarrow \mathcal{M}, \Gamma \nVdash A$ or $\mathcal{M}, \Gamma \Vdash B$, and similarly for other propositional connectives;
4. $\mathcal{M}, \Gamma \Vdash \forall x A(x) \Longleftrightarrow \mathcal{M}, \Gamma \Vdash A(a)$ for every $a \in \mathcal{D}(\Gamma)$;
5. Assume $t{:}_X A(\vec{x})$ is closed and $\vec{x}$ are all the free variables of A. (Note that since the formula is closed, members of X must be individual constants.) $\mathcal{M}, \Gamma \Vdash t{:}_X A(\vec{x}) \Longleftrightarrow$
   (a) $\Gamma \in \mathcal{E}(t, A(\vec{x}))$ and
   (b) $\mathcal{M}, \Delta \Vdash A(\vec{a})$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$ and for every $\vec{a}$ in $\mathcal{D}(\Delta)$.

Definition 3.8 (Validity) Let A be a closed formula in the language FOLP (that is, with no domain constants). We say A is *valid in the FOLP model* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ provided for every $\Gamma \in \mathcal{G}$, $\mathcal{M}, \Gamma \Vdash A$. An FOLP formula with free individual variables is valid if its universal closure is.

The definition of validity for FOLP formulas with free individual variables needs a few small comments. Consider $A(x)$ as an example, where only x has a free occurrence. To show validity of $A(x)$ we must show validity of $\forall x A(x)$. To show $\forall x A(x)$ is true at a possible world of an FOLP model we must show the truth of $A(a)$ for each a in the domain of that possible world. In effect, showing validity of an FOLP formula with free individual variables amounts to showing the truth, at each possible world Γ, of all instances of that formula that live in Γ.

4 Non-Validity Examples

We present two examples of non-validity. Validity examples are in the next section, where axiomatic soundness is discussed. As noted earlier, models have both a semantic and a syntactic component. It often happens that non-validity can be shown by appropriate use of only one of these—indeed, this is at the heart of the Mkrtychev semantics discussed in Section 11. In the examples below we concentrate on the semantic, possible world, side, and trivialize the evidence function. The following terminology is useful here.

Definition 4.1 Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ be a model. We say the evidence function $\mathcal{E}$ is *universal* provided $\Gamma \in \mathcal{E}(t, A)$ whenever A lives in Γ, for every justification term t. (It is easy to check that a universal evidence function meets all the conditions of Definition 3.5.)

We sometimes give diagrams representing models. In these we systematically omit the display of arrows representing reflexivity, but they may play a significant role and their implicit presence should be remembered.

Example 4.2 Axiom **A2** asserts $t{:}_{Xy}A \to t{:}_X A$, provided y does not occur free in A. The proviso is necessary. We show the non-validity of $t{:}_{\{x,y\}}Q(x,y) \to t{:}_{\{x\}}Q(x,y)$, where $Q(x,y)$ is atomic and the individual variables are the ones displayed.

Let $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be a skeleton given by: $\mathcal{G} = \{\Gamma, \Delta\}$; $\mathcal{R}$ is reflexive on $\mathcal{G}$ and also $\Gamma \mathcal{R} \Delta$; $\mathcal{D}(\Gamma) = \{a, b\}$ and $\mathcal{D}(\Delta) = \{a, b, c\}$. We build a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ on this. First we set $\mathcal{I}(\Gamma, Q) = \mathcal{I}(\Delta, Q) = \{\langle a, b \rangle\}$. Second we set $\mathcal{E}(u, A)$ to be a universal evidence function. Here is the model schematically. Since the evidence function is universal, it is not shown. Also, recall we do not indicate reflexivity of the accessibility relation.

$$\begin{tikzcd} \Gamma \arrow[d] & \boxed{a, b} \Vdash Q(a,b) \\ \Delta & \boxed{a, b, c} \Vdash Q(a,b) \end{tikzcd}$$

Consider the instance of $t{:}_{\{x,y\}}Q(x,y) \to t{:}_{\{x\}}Q(x,y)$ resulting from the substitution $\{x/a, y/b\}$, which is a formula that lives in Γ. We show this instance is not true at Γ. More precisely, we show the following.

$$\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}}Q(a,b) \text{ but } \mathcal{M}, \Gamma \not\Vdash t{:}_{\{a\}}Q(a,y)$$

We have $\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}}Q(a,b)$ because $\Gamma \in \mathcal{E}(t, Q(a,b))$, and also $\mathcal{M}, \Gamma \Vdash Q(a,b)$ and $\mathcal{M}, \Delta \Vdash Q(a,b)$. We have $\mathcal{M}, \Gamma \not\Vdash t{:}_{\{a\}}Q(a,y)$ because, although $\Gamma \in \mathcal{E}(t, Q(a,y))$, we do not have that $\mathcal{M}, \Delta \Vdash Q(a,c)$, violating part 5b of Definition 3.7.

Remark We showed non-validity by constructing a two-world model in which the evidence function is universal, and hence trivial. All the work is done by the modal structure. The reader might try constructing a one-world counter model in which all the work is done by the evidence function.

Example 4.3 In the next section we show the validity of axiom scheme **B5**, $t{:}_X A \to \mathsf{gen}_x(t){:}_X \forall x A$, where $x \notin X$. Here we construct a model to show validity does not hold if the proviso is not met; we give a counter-model to $t{:}_{\{x\}}Q(x) \to \mathsf{gen}_x(t){:}_{\{x\}}\forall x Q(x)$ (where $Q(x)$ is atomic). To do this we construct a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ with $\Gamma \in \mathcal{G}$ and $a \in \mathcal{D}(\Gamma)$, so that $\mathcal{M}, \Gamma \not\Vdash t{:}_{\{a\}}Q(a) \to \mathsf{gen}_x(t){:}_{\{a\}}\forall x Q(x)$, where this formula results from the substitution of a for free occurrences of x.

Let $\mathcal{G} = \{\Gamma\}$ with $\Gamma \mathcal{R} \Gamma$. Let $\mathcal{D}(\Gamma) = \{a, b\}$. Let $\mathcal{I}(Q, \Gamma) = \{a\}$. Finally, let $\mathcal{E}$ be a universal evidence function. Here is the model schematically.

$$\Gamma \quad \boxed{a, b} \Vdash Q(a)$$

We have $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}Q(a)$ because $\Gamma \in \mathcal{E}(t, Q(a))$ (since $\mathcal{E}$ is universal), and we have $\mathcal{M}, \Gamma \Vdash Q(a)$ (reflexivity comes in here). If we had $\mathcal{M}, \Gamma \Vdash \mathsf{gen}_x(t){:}_{\{a\}}\forall x Q(x)$ we should also have $\mathcal{M}, \Gamma \Vdash \forall x Q(x)$ (reflexivity again), but $\mathcal{M}, \Gamma \not\Vdash Q(b)$ and $b \in \mathcal{D}(\Gamma)$.

5 Soundness

Each of the FOLP axioms is valid in all FOLP models, and the rules preserve validity, hence each theorem is valid. We show this for a few of the axioms, and omit details for the rest. An axiom of FOLP may contain free individual variables, in which case we must show validity closed substitution instances.

In what follows, assume $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{E}, \mathcal{I} \rangle$ is an FOLP model. We show validity of four representative axioms.

A2 $t{:}_{Xy}A \rightarrow t{:}_X A$, where y does not occur free in A. For simplicity let us say $X = \{x\}$ and $A = A(x, z)$. By assumption, $y \neq x$ and $y \neq z$. We show validity of $t{:}_{\{x,y\}}A(x, z) \rightarrow t{:}_{\{x\}}A(x, z)$. Let $\Gamma \in \mathcal{G}$ and consider the (arbitrary) $\mathcal{D}(\Gamma)$ instantiation resulting from $\{x/a, y/b\}$. We must show $\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}}A(a, z) \rightarrow t{:}_{\{a\}}A(a, z)$ (recall, the occurrences of z here are not free). The reasoning is quite simple.

The Evidence Function condition that must be met for $\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}}A(a, z)$ is that $\Gamma \in \mathcal{E}(t, A(a, z))$, and this is also the condition for $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}A(a, z)$. The modal condition for $\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}}A(a, z)$ is that $\mathcal{M}, \Delta \Vdash A(a, d)$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ and every $d \in \mathcal{D}(\Delta)$, and this is the same modal condition for $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}A(a, z)$.

A3 $t{:}_X A \rightarrow t{:}_{Xy}A$. Recall the convention from Section 2 that when Xy is written it is understood that $y \notin X$. Again for simplicity, aasume $X = \{x\}$ and $A = A(x, y, z)$, so the formula is $t{:}_{\{x\}}A(x, y, z) \rightarrow t{:}_{\{x,y\}}A(x, y, z)$. Let $\Gamma \in \mathcal{G}$ and consider the $\mathcal{D}(\Gamma)$ instantiation resulting from $\{x/a, y/b\}$ where $a, b \in \mathcal{D}(\Gamma)$. We show $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}A(a, y, z) \rightarrow t{:}_{\{a,b\}}A(a, b, z)$.

Assume $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}A(a, y, z)$. By part 5 of Definition 3.7 this has two consequences, which we consider separately.

First, $\Gamma \in \mathcal{E}(t, A(a, y, z))$. It follows from the Instantiation Condition of Definition 3.5 that $\Gamma \in \mathcal{E}(t, A(a, b, z))$.

Second, for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash A(a, y, z))$ for all substitutions of members of $\mathcal{D}(\Delta)$ for y and z. Since $b \in \mathcal{D}(\Gamma)$, by monotonicity $b \in \mathcal{D}(\Delta)$. Then $\mathcal{M}, \Delta \Vdash A(a, b, z)$ for all substitutions of members of $\mathcal{D}(\Delta)$ for z.

By part 5 of Definition 3.7 again, it follows that $\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}}A(a, b, z)$.

B4 $t{:}_X A \rightarrow !t{:}_X t{:}_X A$. As above, we consider a representative special case. Assume $X = \{x\}$ and $A = A(x, y)$, so the formula is $t{:}_{\{x\}}A(x, y) \rightarrow !t{:}_{\{x\}}t{:}_{\{x\}}A(x, y)$. Let $\Gamma \in \mathcal{G}$ and consider the $\mathcal{D}(\Gamma)$ instantiation resulting from the substitution $\{x/a\}$ where $a \in \mathcal{D}(\Gamma)$. We show $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}A(a, y) \rightarrow !t{:}_{\{a\}}t{:}_{\{a\}}A(a, y)$. Suppose $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}A(a, y)$.

First, $\Gamma \in \mathcal{E}(t, A(a, y))$ so by the **!** Condition of Definition 3.5, $\Gamma \in \mathcal{E}(!t, t{:}_{\{a\}}A(a, y))$.

Next, suppose $\Gamma\mathcal{R}\Delta$ and $\Delta\mathcal{R}\Omega$. Since $\mathcal{R}$ is transitive, $\Gamma\mathcal{R}\Omega$ and since $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}}A(a, y)$ then $\mathcal{M}, \Omega \Vdash A(a, y)$ for every instance of y from $\mathcal{D}(\Omega)$. Also since $\Gamma \in \mathcal{E}(t, A(a, y))$ then $\Delta \in \mathcal{E}(t, A(a, y))$ by the $\mathcal{R}$ Closure Condition of Definition 3.5. Since Ω is arbitrary, $\mathcal{M}, \Delta \Vdash t{:}_{\{a\}}A(a, y)$. And since Δ is arbitrary, $\mathcal{M}, \Gamma \Vdash !t{:}_{\{a\}}t{:}_{\{a\}}A(a, y)$.

B5 $t{:}_X A \rightarrow \mathsf{gen}_x(t){:}_X \forall x A$, where $x \notin X$. Again we consider a representative simple case. Assume $X = \{y\}$ and $A = A(x, y, z)$, so the formula is $t{:}_{\{y\}} A(x, y, z) \rightarrow \mathsf{gen}_x(t){:}_{\{y\}} \forall x A(x, y, z)$, where $x \neq y$. Let $\Gamma \in \mathcal{G}$ and consider the $\mathcal{D}(\Gamma)$ instantiation resulting from the substitution $\{y/b\}$ where $b \in \mathcal{D}(\Gamma)$. We show that $\mathcal{M}, \Gamma \Vdash t{:}_{\{b\}} A(x, b, z) \rightarrow \mathsf{gen}_x(t){:}_{\{b\}} \forall x A(x, b, z)$. Assume $\mathcal{M}, \Gamma \Vdash t{:}_{\{b\}} A(x, b, z)$.

We have that $\Gamma \in \mathcal{E}(t, A(x, b, z))$. By the gen_x Condition of Definition 3.5, it follows that $\Gamma \in \mathcal{E}(\mathsf{gen}_x(t), (\forall x) A(x, b, z))$.

Next, for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$, $\mathcal{M}, \Delta \Vdash A(a, b, c)$ for all a, c in $\mathcal{D}(\Delta)$. Then by part 4 of Definition 3.7, $\mathcal{M}, \Delta \Vdash \forall x A(x, b, c)$ for all c in $\mathcal{D}(\Delta)$.

Now by part 5 of Definition 3.7, we have $\mathcal{M}, \Gamma \Vdash \mathsf{gen}_x(t){:}_{\{b\}} \forall x A(x, b, z)$.

Note that Example 4.3 shows the restriction $x \notin X$ is essential for validity of instances of **B5**.

The other axioms are valid, and the rules preserve validity—results left to the reader. It follows that the axiom system is sound with respect to the semantics. This allows for taking constant specifications into account. More specifically, we have the following.

Theorem 5.1 (Soundness) *Let $\mathcal{C}$ be a constant specification. If the* FOLP *formula A is provable using constant specification $\mathcal{C}$ then A is valid in every* FOLP *model meeting constant specification $\mathcal{C}$.*

6 More About Constant Specifications

While soundness holds with respect to any constant specification, completeness requires something more restricted. The first item below is familiar from propositional justification logics; the second is new.

Definition 6.1 A constant specification $\mathcal{C}$ is *axiomatically appropriate* if, for every axiom A there is a proof constant c such that $c{:}_\emptyset A \in \mathcal{C}$.

An *internalization theorem* can be proved for FOLP provided an axiomatically appropriate constant specification is assumed—see [3].

Theorem 6.2 (Internalization) *Let $p_0, \ldots, p_k$ be proof variables, $X_0, \ldots, X_k$ be sets of individual variables, and $X = X_0 \cup \ldots \cup X_k$. Suppose that in* FOLP *the following is provable: $p_0{:}_{X_0} A_0 \rightarrow \ldots \rightarrow p_k{:}_{X_k} A_k \rightarrow F$. Then there exists a proof term $t = t(p_0, \ldots, p_k)$ such that $p_0{:}_{X_0} A_0 \rightarrow \ldots \rightarrow p_k{:}_{X_k} A_k \rightarrow t{:}_X F$ is provable.*

Our completeness proof uses a Henkin construction, and so we must extend the basic language by the addition of 'witnesses.' A constant specification is for the original FOLP language, so we will need a way of extending it usefully to larger languages. The condition of being variant closed makes it straightforward to do this. The idea is that it is the pattern of individual variable usage that matters, and not what we call the particular variables that are involved.

Definition 6.3 Two formulas are *variable variants* provided each can be turned into the other by a renaming of free and bound individual variables.

A constant specification $\mathcal{C}$ is *variant closed* provided that whenever A and B are variable variants, $c{:}_\emptyset A \in \mathcal{C}$ if and only if $c{:}_\emptyset B \in \mathcal{C}$.

Our completeness proof will work for constant specifications that are axiomatically appropriate and variant closed. One restriction on constant specifications that is familiar for propositional justification logics is *schematic*. A constant specification is schematic if all instances of an axiom scheme are assigned the same constants. Every schematic constant specification is automatically variant closed, but not conversely.

7 Language Extensions

Throughout the rest of this paper we need a systematic way of referring to multiple related languages. We use L for the language of FOLP itself. We extend this language, Henkin-style, for purposes of proving completeness, and we do this by adding individual variables to L. Let $\mathbf{V}$ be some countable set of symbols not used in L; $\mathbf{V}$ is fixed for the rest of this paper and its members will be used as additional variables—we call them *witness variables*. We will need multiple extensions of L, since we have multiple possible worlds in models. The following provides us with flexibility.

Definition 7.1 Let $W \subseteq \mathbf{V}$. $L(W)$ is the language defined like L except that it also allows individual variables from W, as well as justification operators gen_x for $x \in W$.

The axiomatization of FOLP in Section 2 is by schemes, and these make sense for formulas from the language $L(W)$ too. When working with the language $L(W)$ we will allow such formulas as axioms without further comment. But we do need to discuss extending constant specifications from L to $L(W)$.

Definition 7.2 Let $\mathcal{C}$ be a variant closed constant specification for the language L, Definition 6.3. We define a constant specification $\mathcal{C}(W)$ for $L(W)$, where $W \subseteq \mathbf{V}$, as follows. Let A be a formula of $L(W)$ in which the individual variables $v_1, \ldots, v_n$ from W occur, free or bound. Let $x_1, \ldots, x_n$ be individual variables from the base language L that do not occur in A, and let A' be the result of replacing each v_i with x_i throughout. If $c{:}_\emptyset A \in \mathcal{C}$, put $c{:}_\emptyset A' \in \mathcal{C}(W)$.

Some observations. First, since $\mathcal{C}$ is variant closed the actual choice of individual variables x_1, $\ldots$, x_n doesn't matter. Second, if A contains no occurrences of members of W then replacement for members of W, in the definition above, doesn't change anything, and it follows that $\mathcal{C}(W)$ extends $\mathcal{C}$ conservatively. Third, it is simple to check that $\mathcal{C}(W)$ is also variant closed. And finally, since FOLP is axiomatized using axiom schemes, it is easy to see that if $\mathcal{C}$ is axiomatically appropriate with respect to L, then $\mathcal{C}(W)$ will be axiomatically appropriate with respect to $L(W)$ using the same schemes but with the larger language. It follows that the Internalization Theorem still applies, and with the same proof. We omit the verification.

We take a set D to be inconsistent if there is a finite subset $\{A_1, \ldots, A_n\}$ of D such that $(A_1 \rightarrow \ldots \rightarrow (A_n \rightarrow \perp) \ldots)$ is provable. D is consistent if it is not inconsistent. If D is a consistent set and an existential formula $(\exists x)A(x)$ is in D, a *witness* for the formula is an individual variable v such that $A(v)$ is also in D, and similarly for negated universal formulas. A consistent set D is *E-complete* if every existential or negated universal formula in D has a witness. The usual Henkin/Lindenbaum construction can be carried out and we have the following, whose proof we omit.

Proposition 7.3 *Let $\emptyset \subseteq W_1 \subset W_2 \subseteq \mathbf{V}$, where W_2 is a countable extension of W_1. Assume $\mathcal{C}$ is a variant closed constant specification for L, and $\mathcal{C}(W_1)$ and $\mathcal{C}(W_2)$ are its extensions according* *to Definition 7.2. Let F be a consistent set of formulas in the language $L(W_1)$, using constant specification $\mathcal{C}(W_1)$. Then F extends to a set F' in the language $L(W_2)$ that is consistent using constant specification $\mathcal{C}(W_2)$, maximally so with respect to the language $L(W_2)$, and E-complete with members of W_2 as witnesses.*

8 Canonical Models

In this section we construct what we call a *canonical* model, which will be used to establish completeness for FOLP. We assume a constant specification $\mathcal{C}$ for L that is variant closed and axiomatically appropriate. The language L is extended with new individual variables from the set $\mathbf{V}$, as discussed in the previous section. Members of $\mathbf{V}$ play a fundamental role in creating the model, by providing the domains, but ultimately it is only formulas from the original language L that we are concerned with. We will generally refer to individual variables from the base language L as *L-variables*, and members of $\mathbf{V}$ as *witness variables*.

We work with formulas from $L(\mathbf{V})$ so that we can use members of $\mathbf{V}$ to make up the domain of a model. We also need to quantify over members of $\mathbf{V}$ because we deal with consistent sets, consistency is defined as non-derivability of $\perp$, and derivations allow free individual variables to be quantified, including members of $\mathbf{V}$. But of course we do not quantify over members of the domain of a model, so the members of $\mathbf{V}$ play two rather different roles. We introduce some special terminology for this.

The canonical model will be defined in such a way that the domain $\mathcal{D}^*$ of the model is $\mathbf{V}$. Adapting terminology from Section 3, from here on when we refer to a *$\mathcal{D}^*$-formula* we mean a formula in which members of $\mathbf{V}$ play the role of constants of the model, and so in $\mathcal{D}^*$-formulas members of $\mathbf{V}$ can occur but *can only occur free*. We stretch terminology a bit and say a $\mathcal{D}^*$-formula is *closed* if no L variable occurrences are free (but individual variables from $\mathbf{V}$ may occur free, though not bound since it is a $\mathcal{D}^*$-formula). In particular, if the formula $t{:}_{\{x,y\}}A(x,y,z,w)$ is a $\mathcal{D}^*$-formula then z and w cannot be from $\mathcal{D}^*$ ($= \mathbf{V}$) since they occur bound, and if it a closed $\mathcal{D}^*$-formula then x and y must be from $\mathcal{D}^*$ since they occur free.

The Truth Lemma, in the next section, is specifically for closed $\mathcal{D}^*$-formulas. But to understand the overall behavior of the canonical model we need properties of consistent and maximal consistent sets, and for these formal derivations come in. In formal derivations restrictions on formulas are dropped—we allow any $L(\mathbf{V})$-formula, any variable might occur free or bound, including members of $\mathbf{V}$.

Definition 8.1 (Canonical Model) A *canonical model* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$, using constant specification $\mathcal{C}$, is specified as follows.

Specification of $\mathcal{G}$ Call $\Gamma = \langle \mathsf{form}(\Gamma), \mathsf{var}(\Gamma) \rangle$ an *HL-world* (standing for Henkin/Lindenbaum) where:

1. $\mathsf{var}(\Gamma)$ is a countable subset of $\mathbf{V}$ that omits countably many members of $\mathbf{V}$.
2. $\mathsf{form}(\Gamma)$ is a set of formulas in the language $L(\mathsf{var}(\Gamma))$.
3. $\mathsf{form}(\Gamma)$ is consistent, maximally so among sets of formulas in the language $L(\mathsf{var}(\Gamma))$, and E-complete, with members of $\mathsf{var}(\Gamma)$ as witnesses. Note that the definition of consistency brings constant specification $\mathcal{C}$ into things.

$\mathcal{G}$ is the collection of all *HL*-worlds. For each $\Gamma \in \mathcal{G}$, $\mathsf{form}(\Gamma)$ and $\mathsf{var}(\Gamma)$ are the sets such that $\Gamma = \langle \mathsf{form}(\Gamma), \mathsf{var}(\Gamma) \rangle$.

Specification of $\mathcal{R}$ For a set S of formulas of $L(\mathsf{var}(\Gamma))$, let $S^\sharp$ be the set of all formulas $(\forall\vec{y})A$ such that $t{:}_X A \in S$, where $t{:}_X A$ is a closed $\mathcal{D}^*$-formula with X the set of witness variables in A, and $\vec{y}$ are the free L-variables of A. (Note that $(\forall\vec{y})A$ is also a closed $\mathcal{D}^*$-formula.) For $\Gamma, \Delta \in \mathcal{G}$, set $\Gamma\mathcal{R}\Delta$ provided:

1. $\mathsf{var}(\Gamma) \subseteq \mathsf{var}(\Delta)$
2. $(\mathsf{form}(\Gamma))^\sharp \subseteq \mathsf{form}(\Delta)$

Specification of $\mathcal{D}$ For $\Gamma \in \mathcal{G}$, set $\mathcal{D}(\Gamma) = \mathsf{var}(\Gamma)$.

Specification of $\mathcal{I}$ For an n-place relation symbol Q and for $\Gamma \in \mathcal{G}$, let $\mathcal{I}(Q, \Gamma)$ be the set of all $\langle v_1, \ldots, v_n\rangle$ where each $v_i \in \mathcal{D}(\Gamma)(= \mathsf{var}(\Gamma))$, and $Q(v_1, \ldots, v_n) \in \mathsf{form}(\Gamma)$.

Specification of $\mathcal{E}$ For $\Gamma \in \mathcal{G}$, set $\Gamma \in \mathcal{E}(t, A)$ provided $t{:}_X A \in \mathsf{form}(\Gamma)$, where $t{:}_X A$ is a closed $\mathcal{D}^*$-formula with X the set of witness variables in A.

We have finished the definition of a canonical model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}\rangle$. It must be checked that it is an FOLP model. We conclude the section by going through some of the details.

First we verify that $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}\rangle$ is an FOLP skeleton, Definition 3.1. Monotonicity is immediate: if $\Gamma\mathcal{R}\Delta$ then by definition of $\mathcal{R}$, $\mathcal{D}(\Gamma) = \mathsf{var}(\Gamma) \subseteq \mathsf{var}(\Delta) = \mathcal{D}(\Delta)$.

Reflexivity of $\mathcal{R}$ requires two things. The first is trivial, $\mathsf{var}(\Gamma) \subseteq \mathsf{var}(\Gamma)$. For the second, $(\mathsf{form}(\Gamma))^\sharp \subseteq \mathsf{form}(\Gamma)$, we have the following argument. Suppose $F \in (\mathsf{form}(\Gamma))^\sharp$. Then $F = (\forall\vec{y})A$ where $t{:}_X A \in \mathsf{form}(\Gamma)$, $t{:}_X A$ is a closed $\mathcal{D}^*$-formula, X is the set of witness variables in A, and $\vec{y}$ are the free L-variables of A. Let us say $\vec{y}$ is $y_1, \ldots, y_n$. No y_i can occur in X. Then by repeated use of axiom **B5** the following is provable

$$t{:}_X A \to \mathsf{gen}_{y_1}(\mathsf{gen}_{y_2}(\cdots \mathsf{gen}_{y_n}(t))){:}_X(\forall\vec{y})A$$

so by maximal consistency of $\mathsf{form}(\Gamma)$, $\mathsf{gen}_{y_1}(\mathsf{gen}_{y_2}(\cdots \mathsf{gen}_{y_n}(t))) :_X (\forall\vec{y})A \in \mathsf{form}(\Gamma)$. Then by axiom **B1** and maximal consistency again, $(\forall\vec{y})A \in \mathsf{form}(\Gamma)$, that is, $F \in \mathsf{form}(\Gamma)$.

Transitivity of $\mathcal{R}$ is by a similar argument, but axiom **B4** also comes in. We omit the proof.

It is straightforward to check that the conditions of Definition 3.3 are met.

Finally we need to verify that the evidence function $\mathcal{E}$ meets the conditions of Definition 3.5. We check four of the six cases.

$\mathcal{R}$ Closure Condition. Suppose $\Gamma, \Delta \in \mathcal{G}$ and $\Gamma\mathcal{R}\Delta$. Suppose also that $\Gamma \in \mathcal{E}(t, A)$. We show $\Delta \in \mathcal{E}(t, A)$. Since $\Gamma \in \mathcal{E}(t, A)$ we have $t{:}_X A \in \mathsf{form}(\Gamma)$ where $t{:}_X A$ is a closed $\mathcal{D}^*$-formula with X the set of witness variables in A—we show that $t{:}_X A \in \mathsf{form}(\Delta)$. Using Axiom **B4** and maximal consistency, $!t{:}_X t{:}_X A \in \mathsf{form}(\Gamma)$, and this is also a closed $\mathcal{D}^*$-formula with X the set of witness variables in $t{:}_X A$. Then $(\forall\vec{y})t{:}_X A \in \mathsf{form}(\Delta)$ by definition of $\mathcal{R}$, where $\vec{y}$ are the free L-variables of $t{:}_X A$. But there are no free L-variables in $t{:}_X A$ since it is closed. That is, $t{:}_X A \in \mathsf{form}(\Delta)$, and hence $\Delta \in \mathcal{E}(t, A)$.

! Condition. Suppose $\Gamma \in \mathcal{E}(t, A)$. We show $\Gamma \in \mathcal{E}(!t, t{:}_X A)$, where X is the set of witness variables in A. Since $\Gamma \in \mathcal{E}(t, A)$ then $t{:}_X A \in \mathsf{form}(\Gamma)$, where $t{:}_X A$ is a closed $\mathcal{D}^*$-formula and X is the set of witness variables in A. By Axiom **B4** and maximal consistency, $!t{:}_X t{:}_X A \in \mathsf{form}(\Gamma)$. Since X is also the set of witness variables in $t{:}_X A$, it follows that $\Gamma \in \mathcal{E}(!t, t{:}_X A)$.

Instantiation Condition. In the present context we read this as, if $\Gamma \in \mathcal{E}(t, A(x))$ where x is an L-variable and $a \in \mathcal{D}(\Gamma)$, then $\Gamma \in \mathcal{E}(t, A(a))$. Assume $\Gamma \in \mathcal{E}(t, A(x))$. Then $t{:}_X A(x) \in \mathsf{form}(\Gamma)$ where $t{:}_X A(x) \in \mathsf{form}(\Gamma)$ is a closed $\mathcal{D}^*$-formula and X is the set of witness variables in $A(x)$.

Now by Axiom **A3**, universal generalization (**R2**), and the maximal consistency of $\mathsf{form}(\Gamma)$, $(\forall x)[t{:}_X A(x) \rightarrow t{:}_{X\cup\{x\}}A(x)] \in \mathsf{form}(\Gamma)$. Then $t{:}_{X\cup\{a\}}A(a)] \in \mathsf{form}(\Gamma)$, and hence $\Gamma \in \mathcal{E}(t, A(a))$ by definition of $\mathcal{E}$.

gen_x Condition. Suppose $\Gamma \in \mathcal{E}(t, A)$. We show $\Gamma \in \mathcal{E}(\mathsf{gen}_x(t), \forall x A)$, where x is an L-variable. This is rather simple. By definition, $t{:}_X A \in \Gamma$ where X is the set of witness variables in A, a set that cannot contain x. It follows that $\mathsf{gen}_x(t){:}_X \forall x A \in \Gamma$ using maximal consistency of Γ and Axiom **B5**, which gives us what we need.

9 Completeness

The main item we need is the familiar Truth Lemma. Once this has been proved, completeness is simple.

Theorem 9.1 (Truth Lemma) *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}\rangle$ be a canonical model. For each $\Gamma \in \mathcal{G}$ and for each closed formula $\mathcal{D}^*$-A that lives in Γ,*

$$\mathcal{M}, \Gamma \Vdash A \Longleftrightarrow A \in \mathit{form}(\Gamma).$$

Proof The proof is by induction on formula degree. Much of this is familiar, so we only give the most significant cases.

Justification Formulas Assume $\Gamma \in \mathcal{G}$, $t{:}_X A$ is a closed $\mathcal{D}^*$-formula that lives in Γ, and the result is known for simpler formulas.

One direction is very simple, just as it is propositionally. Suppose $t{:}_X A \notin \mathsf{form}(\Gamma)$. Since this is a closed $\mathcal{D}^*$-formula, X consists entirely of witness variables, and includes all those that occur in A. Let X' be the subset of X containing exactly the witness variables that occur in A. Then $t{:}_{X'} A \notin \mathsf{form}(\Gamma)$ because otherwise, by repeated use of Axiom **A3** and the maximal consistency of $\mathsf{form}(\Gamma)$ we would have that $t{:}_X A \in \mathsf{form}(\Gamma)$. Then by definition of $\mathcal{E}$, $\Gamma \notin \mathcal{E}(t, A)$ and it follows that $\mathcal{M}, \Gamma \nVdash t{:}_X A$.

Next suppose $t{:}_X A \in \mathsf{form}(\Gamma)$. As above, let X' be the subset of X containing exactly the witness variables that occur in A. Then $t{:}_{X'} A \in \mathsf{form}(\Gamma)$, by repeated use of Axiom **A2** and maximal consistency of $\mathsf{form}(\Gamma)$, and hence $\Gamma \in \mathcal{E}(t, A)$. Further, if $\Gamma \mathcal{R} \Delta$ we have $\forall \vec{y} A \in \Delta$, by definition of $\mathcal{R}$, and hence (maximal consistency again), every instance of A, replacing the variables in $\vec{y}$ with members of $\mathcal{D}(\Delta)$, belongs to Δ. By the induction hypothesis, for each such instance $A_{\vec{a}}$ say, $\mathcal{M}, \Delta \Vdash A_{\vec{a}}$. We now have the conditions needed to conclude $\mathcal{M}, \Gamma \Vdash t{:}_X A$.

Quantified Formulas Assume that $\forall x A(x)$ is a formula in the language L and the result is known for simpler formulas.

Suppose first that $\forall x A(x) \in \mathsf{form}(\Gamma)$. Let a be an arbitrary member of $\mathcal{D}(\Gamma)$. Then a is also a variable of the language $L(\mathsf{var}(\Gamma))$, and $\forall x A(x) \rightarrow A(a)$ is a provable formula. By maximal consistency of $\mathsf{form}(\Gamma)$, $A(a) \in \mathsf{form}(\Gamma)$. By the induction hypothesis, $\mathcal{M}, \Gamma \Vdash A(a)$. Since a was arbitrary, $\mathcal{M}, \Gamma \Vdash \forall x A(x)$.

Finally, suppose that $\forall x A(x) \notin \mathsf{form}(\Gamma)$. Since $\mathsf{form}(\Gamma)$ is E-complete, for some witness variable a in $\mathsf{var}(\Gamma)$, $A(a) \notin \mathsf{form}(\Gamma)$. By the induction hypothesis, $\mathcal{M}, \Gamma \nVdash A(a)$, and it follows that $\mathcal{M}, \Gamma \nVdash \forall x A(x)$.

■

Completeness now follows in the usual way. Suppose closed formula A of L is not provable. Then $\{\neg A\}$ is consistent. Let $V \subseteq \mathbf{V}$ contain countably many members of $\mathbf{V}$, while also omitting countably many. Extend $\{\neg A\}$ to a set M that is maximally consistent and E-complete with respect to the language $L(V)$, with members of V serving as witnesses. Let $\Gamma = \langle M, V\rangle$. This is a possible world in the canonical model, and A will be false in it.

10 Fully Explanatory Models

For propositional justification logics a subclass of possible world models, called *fully explanatory*, was defined in [4]. Such models meet a strong, but intuitively appealing, special condition which not all models do. Still, completeness can be established relative to the family of fully explanatory models. The notion of fully explanatory extends to the first-order setting quite naturally, with a proof and with consequences similar to those in the propositional setting.

Definition 10.1 (Fully Explanatory) A model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{E}, \mathcal{I}\rangle$ is *fully explanatory* if it meets the following condition. Assume A is a formula with no free individual variables, but with constants from the domain of the model, as in Section 3. Let $\Gamma \in \mathcal{G}$ and suppose that A lives in Γ. If $\mathcal{M}, \Delta \Vdash A$ for every $\Delta \in \mathcal{G}$ then $\mathcal{M}, \Gamma \Vdash t{:}_X A$ for some proof term t, where X is the set of domain constants appearing in A.

Theorem 10.2 *The canonical model is fully explanatory.*

Proof Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{A}, \mathcal{I}\rangle$ be a canonical model. Let $\Gamma \in \mathcal{G}$, and assume A is a closed $\mathcal{D}^*$-formula that lives in Γ. Also assume that $\mathcal{M}, \Gamma \nVdash t{:}_X A$ for every proof term t, where X is the set of witness variables appearing in A. We show that for some $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \nVdash A$.

By our assumption and the Truth Lemma, $t{:}_X A \notin \mathsf{form}(\Gamma)$ for every proof term t. Using this we first show $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A\}$ is consistent.

Well, suppose not, that is, suppose $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A\}$ is inconsistent. We derive a contradiction. Assuming the inconsistency, there are $\forall\vec{y_1}B_1, \ldots, \forall\vec{y_n}B_n \in (\mathsf{form}(\Gamma))^\sharp$ so that $\vdash \forall\vec{y_1}B_1 \to \forall\vec{y_2}B_2 \to \ldots \to \forall\vec{y_n}B_n \to A$, where association is to the right, and $\vec{y_i}$ are the free L-variables of B_i. For each i, since $\forall\vec{y_i}B_i \in (\mathsf{form}(\Gamma))^\sharp$ then $u_i{:}_{X_i}B_i \in \mathsf{form}(\Gamma)$ for some u_i, where X_i is the set of witness variables in B_i, and $u_i{:}_{X_i}B_i$ is a closed $\mathcal{D}^*$-formula. The L-variables in $\vec{y_i}$ cannot occur in X_i so by repeated use of Axiom **B5** there is a proof term t_i such that $u_i{:}_{X_i}B_i \to t_i{:}_{X_i}\forall\vec{y_i}B_i$ is provable. (In fact, t_i consists of iterated applications of **gen** operators, but we do not need the details.)

By the Internalization Theorem, 6.2, $w{:}_\emptyset(\forall\vec{y_1}B_1 \to \forall\vec{y_2}B_2 \to \ldots \to \forall\vec{y_n}B_n \to A)$ is provable, for some proof term w. Using axiom **B2** (and **A3** as well) each of the following is provable (association is to the left in the proof terms).

$$
\begin{array}{r}
w{:}_\emptyset(\forall\vec{y_1}B_1 \to \forall\vec{y_2}B_2 \to \ldots \to \forall\vec{y_n}B_n \to A)\\
t_1{:}_{X_1}\forall\vec{y_1}B_1 \to (w \cdot t_1){:}_{X_1}(\forall\vec{y_2}B_2 \to \ldots \to \forall\vec{y_n}B_n \to A)\\
t_1{:}_{X_1}\forall\vec{y_1}B_1 \to t_2{:}_{X_2}\forall\vec{y_2}B_2 \to (w \cdot t_1 \cdot t_2){:}_{X_1 \cup X_2}(\forall\vec{y_3}B_3 \to \ldots \to \forall\vec{y_n}B_n \to A)\\
\vdots\\
t_1{:}_{X_1}\forall\vec{y_1}B_1 \to t_2{:}_{X_2}\forall\vec{y_2}B_2 \to \ldots \to t_n{:}_{X_n}\forall\vec{y_n}B_n \to (w \cdot u_1 \cdot u_2 \cdot \ldots \cdot u_n){:}_{X_1 \cup X_2 \cup \ldots \cup X_n}A
\end{array}
$$

This, combined with the provability of each $u_i{:}_{X_i}B_i \to t_i{:}_{X_i}\forall\vec{y_i}B_i$, gives us provability of the following.

$$u_1{:}_{X_1}B_1 \to u_2{:}_{X_2}B_2 \to \ldots \to u_n{:}_{X_n}B_n \to (w \cdot u_1 \cdot u_2 \cdot \ldots \cdot u_n){:}_{X_1 \cup X_2 \cup \ldots \cup X_n}A$$

For each i, $u_i{:}_{X_i}B_i \in \mathsf{form}(\Gamma)$, so by maximal consistency, $(w \cdot u_1 \cdot u_2 \cdot \ldots \cdot u_n){:}_{X_1 \cup X_2 \cup \ldots \cup X_n} A \in \mathsf{form}(\Gamma)$. It follows using Axioms **A2** and **A3** that $(w \cdot u_1 \cdot u_2 \cdot \ldots \cdot u_n){:}_X A \in \mathsf{form}(\Gamma)$, where X is the set of witness variables appearing in A. But this contradicts the assumption that $t{:}_X A \notin \mathsf{form}(\Gamma)$ for every proof term t.

We have now shown that $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A\}$ is consistent. Let $\mathsf{var}(\Delta)$ extend $\mathsf{var}(\Gamma)$ with the addition of a countable set of members of $\mathbf{V}$, so that countably many members are still omitted. Extend the consistent set $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A\}$ to a set $\mathsf{form}(\Delta)$ that is maximally consistent and E-complete with respect to $L(\mathsf{var}(\Delta))$, with members of $\mathsf{var}(\Delta)$ as witnesses. Then $\Delta = \langle \mathsf{form}(\Delta), \mathsf{var}(\Delta) \rangle \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$. Since $\neg A \in \mathsf{form}(\Delta)$, by the Truth Lemma $\mathcal{M}, \Delta \not\Vdash A$, which completes the proof. ■

Corollary 10.3 *FOLP is complete with respect to the class of* fully explanatory *models.*

11 Mkrtychev Models

Possible world semantics for FOLP has both a semantic and a syntactic component. That is, it uses all the semantic material of first-order modal models, and it has an evidence function which depends on syntactic details of formulas. Possible world semantics is flexible and provides a plausible intuition, but in fact an entirely syntactic-based approach is possible. For propositional LP, Mkrtychev models were the original non-arithmetic semantics, and they are entirely syntactic in nature. They have turned out to be of considerable use, for example in determining justification logic complexity. Rather nicely, they carry over to FOLP in a direct way.

Definition 11.1 An *Mkrtychev FOLP model* is a structure, $\mathcal{M} = \langle\langle \mathcal{D}, \mathcal{I} \rangle, \mathcal{E} \rangle$ where $\langle \mathcal{D}, \mathcal{I} \rangle$ is a classical first-order model, and $\mathcal{E}$ is an evidence function. Conditions are as follows.

$\mathcal{D}$ is the *domain* of the model, a non-empty set.

$\mathcal{I}$ is an *interpretation*, assigning to each n-place relation symbol of language L some n-ary relation on $\mathcal{D}$.

$\mathcal{E}$ is an *evidence function*, mapping proof term t and formula A with individual constants from $\mathcal{D}$ to a boolean truth value, $\mathcal{E}(t, A)$. In the present context an evidence function must meet the following conditions.

- $\cdot$ **Condition** $\mathcal{E}(s, A \to B) \wedge \mathcal{E}(t, A) \to \mathcal{E}((s \cdot t), B)$.
- $+$ **Condition** $\mathcal{E}(s, A) \vee \mathcal{E}(t, A) \to \mathcal{E}((s + t), A)$.
- ! **Condition** $\mathcal{E}(t, A) \to \mathcal{E}(!t, t{:}_X A)$ where X is the set of all members of $\mathcal{D}$ that appear in A.
- **Instantiation Condition** If $a \in \mathcal{D}$ then $\mathcal{E}(t, A(x)) \to \mathcal{E}(t, A(a))$.
- $\mathbf{gen}_x$ **Condition** $\mathcal{E}(t, A) \to \mathcal{E}(\mathsf{gen}_x(t), \forall x A)$.

We write $\mathcal{M} \Vdash A$ to symbolize that *closed* formula A of language $L(\mathcal{D})$ is true in Mkrtychev model $\mathcal{M}$. The truth conditions are as follows.

Atomic For an n place relation symbol Q and $k_1, \ldots, k_n \in \mathcal{D}$, $\mathcal{M} \Vdash Q(k_1, \ldots, k_n) \iff \langle k_1, \ldots, k_n \rangle \in \mathcal{I}(R)$.

Propositional $\mathcal{M} \Vdash (A \to B) \iff \mathcal{M} \not\Vdash A$ or $\mathcal{M} \Vdash B$, and similarly for other connectives.

Quantifier $\mathcal{M} \Vdash \forall x A(x) \iff \mathcal{M} \Vdash A(a)$ for every $a \in \mathcal{D}$.

Justification Term Assume $t{:}_X A(\vec{x})$ is closed and $\vec{x}$ are all the free variables of A. (Note that since the formula is closed, members of X must be individual constants in $\mathcal{D}$.)

$\mathcal{M} \Vdash t{:}_X A(\vec{x}) \iff \mathcal{E}(t, A(\vec{x}))$ and $\mathcal{M} \Vdash A(\vec{a})$ for all $\vec{a}$ in $\mathcal{D}$.

Example 11.2 Example 4.2 showed the non-validity of the formula $t{:}_{\{x,y\}}Q(x,y) \to t{:}_{\{x\}}Q(x,y)$, where $Q(x,y)$ is atomic and the individual variables are the ones displayed. It did so with a two-world model. Here is a Mkrtychev model that also shows non-validity, incidentaly answering a question posed in the Remark following Example 4.3.

Let $\mathcal{M} = \langle\langle\mathcal{D},\mathcal{I}\rangle,\mathcal{E}\rangle$ be the Mkrtychev model specified as follows. $\mathcal{D} = \{a,b\}$. $\mathcal{I}(Q) = \{\langle a,b\rangle\}$. For every proof term t, $\mathcal{E}(t,X)$ is true if X is a formula with individual constants from $\mathcal{D}$ but no free individual variables, and false if there are individual variables in X.

First we note that $\mathcal{E}$ meets the conditions required of an evidence function in Mkrtychev models. Here are two sample cases. For the $\cdot$ Condition, if $\mathcal{D}((s{\cdot}t), B)$ is false B contains individual variables, but then so does $A \to B$ so $\mathcal{E}(s, A \to B)$ is false and the implication is true. For the ! Condition, $t{:}_X A$ contains no free individual variables if members of X are all individual constants from $\mathcal{D}$, so the consequent of the implication is true.

Now $\mathcal{M} \Vdash Q(\vec{a},b)$ because $\langle\vec{a},b\rangle \in \mathcal{I}(Q)$. Then $\mathcal{M} \Vdash t{:}_{\{\vec{a},b\}}Q(\vec{a},b)$ because $\mathcal{E}(t, Q(\vec{a},b))$ is also true. But $\mathcal{M} \not\Vdash t{:}_{\{\vec{a}\}}Q(\vec{a},y)$ because $\mathcal{E}(t, Q(\vec{a},y))$ is false.

Example 4.3 gives another example of a Mkrtychev model.

It is straightforward to check that Mkrtychev models are essentially one-world FOLP models, and so we have soundness with respect to them. It is also straightforward to check that each possible world in the canonical model is an Mkrtychev model, and completeness with respect to such models follows.

12 Conclusion

We have given a possible world semantics, with and without a fully explanatory condition, and an Mkrtychev semantics, for the first-order logic FOLP. Thus the same set of semantic tools available propositionally is also available when quantifiers are present. In the propositional case, LP turned out to be one of a family of *justification logics*, and the same is true when quantification is added. The propositional justification logics J, JT and J4, and some others, all have monotonic first-order versions, and all the results given here adapt to them in a straightforward way. The situation with quantified logics involving symmetry has not yet been investigated. Work is underway on first-order justification logics with constant domain semantics, but this is very much still in progress.

References

[1] S. N. Artemov. The logic of justification. *The Review of Symbolic Logic*, 1(4):477–513, Dec. 2008.

[2] S. N. Artemov and M. C. Fitting. Justification logic. In E. N. Zalta, editor, *Stanford Encyclopedia of Philosophy*, chapter Logic, Justification. Stanford University, http://plato.stanford.edu/, on-line edition, 2010.

[3] S. N. Artemov and T. Yavorskaya (Sidon). First-order logic of proofs. Technical Report TR–2011005, CUNY Ph.D. Program in Computer Science, May 2011.

[4] M. Fitting. The logic of proofs, semantically. *Annals of Pure and Applied Logic*, 132(1):1–25, Feb. 2005.

[5] M. C. Fitting. Possible world semantics for first order LP. Technical Report TR-2011010, CUNY Ph.D. Program in Computer Science, September 2011.

[6] A. Mkrtychev. Models for the logic of proofs. In S. Adian and A. Nerode, editors, *Logical Foundations of Computer Science, 4th International Symposium, LFCS'97, Yaroslavl, Russia, July 6–12, 1997, Proceedings*, volume 1234 of *Lecture Notes in Computer Science*, pages 266–275. Springer, 1997.