AN ϵ-CALCULUS SYSTEM FOR FIRST-ORDER S4[1]

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§1. Introduction

We give a formulation of the first-order modal logic S4 patterned after the classical ϵ-calculus of Hilbert (see [3]) and prove (non-constructively) that it is a conservative extension of the usual first-order S4. Other modal logics may be similarly treated.

A natural first attempt at such a formulation would be to add S4 axioms and rules to a classical ϵ-calculus base. This fails, and the reason is not hard to find. If $X(x)$ is a formula with one free variable, x, classically ϵxX is intended to be the name of a constant such that, if $(\exists x)X(x)$ is true, $X(\epsilon xX)$ is true. But in a Kripke S4 model [2], $(\exists x)X(x)$ may be true in two possible worlds but yet there may be no single constant, c, such that $X(c)$ is true in both worlds. Thus ϵxX can not be thought of as the name of a constant in an ϵ-calculus S4. Instead we treat ϵxX as a function defined on the collection of possible worlds and such that, if $(\exists x)X(x)$ is true in some possible world, the value of ϵxX at that world is a constant, c, such that $X(c)$ is true there.

Unfortunately, there is no syntactic machinery in ordinary first-order S4 to handle ϵ-terms. They are neither constants nor variables,

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but rather 'world-dependent' terms, naming different constants in different worlds. Consequently we work with an extension of S4 due to Stalnaker and Thomason [5,6], which they created partly to treat definite descriptions, which are also 'world-dependent' terms. We add an abstraction operator, λ, to the language, so that $(\lambda x\ X)(t)$ is a formula if X is a formula with only x free and t is a 'world-dependent' term. We say $(\lambda x\ X)(t)$ is true in a given possible world if X(c) is true there, where c is the value of t at that world. Thus we can make the necessary distinction between $(\lambda x \Diamond X)(t)$ and $\Diamond(\lambda x\ X)(t)$. A fuller discussion of this point may be found in [5,6].

We give an ϵ-calculus S4 system and a suitable model theory. We mention various results which may be derived in it, and show it is a conservative extension of more customary formulations of first-order S4. The system we give is adequate to prove all theorems of first-order S4 (without parameters), but is not complete in terms of its own model theory. In a later paper we will extend the system, by adding more structural axioms, to produce a complete system.

§2. An ϵ-calculus S4 ($\epsilon S4^{\circ}$)

We take as primitive symbols $\wedge$, $\sim$, $\exists$, $\Diamond$, ϵ, λ,) and (, and use $\vee$, $\supset$, $\equiv$, $\forall$ and $\Box$ as abbreviations as usual. We are informal about parentheses. We assume a countable collection of n-place predicate letters for each natural number n, and countably many variables. (We choose not to have parameters in our basic system, though this is of no real significance.)

Following [3], when we use the words <u>formula</u> or <u>term</u> we mean there are no free variables present. In the more general situation we use <u>quasi-formula</u> or <u>quasi-term</u>. A proper definition of these concepts is straightforward, and contains the following clauses. If X is a quasi-formula, x is a variable and t is a quasi-term, $(\lambda x\ X)(t)$ is a quasi-formula. The free variables of $(\lambda x\ X)(t)$ are those of X except for x, together with those of t. Similarly, $\epsilon x X$ is a quasi-term; its free variables are those of X other than x. We will use x, y, z, ... to stand for variables, and t for a quasi-term. We may use subscripts.

We use $(\lambda x_1 \ldots x_n\ X)(t_1, \ldots, t_n)$ as an abbreviation for $(\lambda x_1(\lambda x_2 \ldots (\lambda x_n\ X)(t_n) \ldots)(t_2))(t_1)$. We denote by $X(x/t)$ the result of substituting t for free x in X. We often use $\underset{\sim}{x}$ and $\underset{\sim}{t}$ for sequences of variables and quasi-terms respectively, and write $(\lambda \underset{\sim}{x}\ X)(\underset{\sim}{t})$ for $(\lambda x_1 \ldots x_n\ X)(t_1, \ldots, t_n)$.

Let X be a quasi-formula whose variables are among $x_1, \ldots, x_n$. Let $t_1, \ldots, t_n$ be quasi-terms. If $(\lambda x_1 \ldots x_n\ X)(t_1, \ldots, t_n)$ is a _formula_ we call it a _λ-closure_ of X. If X has no free variables we consider it to be a λ-closure of itself.

We use the phrase _t is free for x in X_ in the standard way to mean that, on replacing free x by t in X, no free variable of t becomes bound by a quantifier, abstract or ϵ symbol of X.

The axioms and rules of $\epsilon S4^{\circ}$ are as follows.

RULES:

R1 $\dfrac{X \quad X \supset Y}{Y}$ where X and Y are _formulas_

R2 $\dfrac{X}{\Box X}$ where X is a _formula_

AXIOM SCHEMAS: Let X and Y be quasi-formulas. We take as axioms _all λ-closures of_ the following quasi-formulas.

First, structural axioms.

A1 if y is not free in X, but y is free for x in X, $(\lambda x\ X)(t) \equiv [\lambda y\ X(x/y)](t)$

A2 if x is not free in X, $(\lambda x\ X)(t) \equiv X$

A3 if $x_1 \neq x_2$, x_1 is not free in t_2, x_2 is not free in t_1, $(\lambda x_1 x_2\ X)(t_1, t_2) \equiv (\lambda x_2 x_1\ X)(t_2, t_1)$

A4 $[\lambda \underset{\sim}{x}\ (X \wedge Y)](\underset{\sim}{t}) \equiv [(\lambda \underset{\sim}{x}\ X)(\underset{\sim}{t}) \wedge (\lambda \underset{\sim}{x}\ Y)(\underset{\sim}{t})]$

A5 $(\lambda \underset{\sim}{x}\ {\sim}X)(\underset{\sim}{t}) \equiv {\sim}(\lambda \underset{\sim}{x}\ X)(\underset{\sim}{t})$

A6 if y is not free in any quasi-term of $\underset{\sim}{t}$ and y is not in the sequence $\underset{\sim}{x}$, $[\lambda \underset{\sim}{x}\ (\exists y)X](\underset{\sim}{t}) \equiv (\exists y)[(\lambda \underset{\sim}{x}\ X)(\underset{\sim}{t})]$

Next, propositional axioms

A7 X, where X is a tautology

A8 $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$

A9 $\Box X \supset X$

A10 $\Box X \supset \Box\Box X$

Finally, quantification

A11 $(\lambda x\ X)(t) \supset (\lambda x\ X)(\epsilon x X)$

A12 $(\lambda x\ \Diamond X)(t) \supset \Diamond(\lambda x\ X)(\epsilon x X)$

A13 $(\exists x)X \equiv (\lambda x\ X)(\epsilon x X)$

This completes the system $\epsilon S4^{\circ}$.

§3. $\epsilon S4^{\circ}$ model theory

We give a Kripke type model theory for $\epsilon S4^{\circ}$. It is based on that for first-order S4, as found in [1,2,4], extended along the lines of [5,6].

The system $\epsilon S4^{\circ}$ above has no constant symbols or parameters. For this section only, let us add them, and treat them as terms. We use a, b, c, ... to represent them.

By FS4 we mean ordinary first-order S4, as found in [1] or [4] say. Its language is that part of $\epsilon S4^{\circ}$ with parameters, not containing abstracts or ϵ-symbols. We begin with a model theory for FS4.

By an FS4 model we mean a quadruple, $\langle \mathcal{G}, \mathcal{R}, \models, \mathcal{P} \rangle$ where: $\mathcal{G}$ is a non-empty set; $\mathcal{R}$ is a transitive, reflexive relation on $\mathcal{G}$; $\mathcal{P}$ is a function on $\mathcal{G}$ ranging over non-empty sets of parameters; and $\models$ is a relation between elements of $\mathcal{G}$ and formulas of FS4. These are to satisfy the following, where $\Gamma \in \mathcal{G}$.

1) if $\Delta \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$ then $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Delta)$.

2) if $\Gamma \models X$, all parameters of X are in $\mathcal{P}(\Gamma)$.

3) if all constants of X and Y are in $\mathcal{P}(\Gamma)$ then

$\Gamma \models (X \wedge Y)$ if and only if $\Gamma \models X$ and $\Gamma \models Y$

$\Gamma \models \sim X$ if and only if not-$\Gamma \models X$.

4) if X is a quasi-formula with at most one free variable, x, and all parameters in $\mathcal{P}(\Gamma)$, then

$\Gamma \models (\exists x)X$ if and only if $\Gamma \models X(x/c)$ for some $c \in \mathcal{P}(\Gamma)$.

5) if all parameters of X are in $\mathcal{P}(\Gamma)$, then

$\Gamma \models \Diamond X$ if and only if for some $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta \models X$.

An FS4 formula X is called valid in the FS4 model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ if $\Gamma \models X$ for every $\Gamma \in \mathcal{G}$ such that all constants of X belong to $\mathcal{P}(\Gamma)$. Proofs may be found in [1,2,4] that the set of theorems of FS4 coincides with the set of formulas valid in all FS4 models.

By an $\epsilon S4^o$ model we mean a quintuple, $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}, \mathcal{F}>$ where: $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}>$ is an FS4 model (save that now $\models$ is a relation between elements of $\mathcal{G}$ and formulas of $\epsilon S4^o$) and $\mathcal{F}$ is a collection of functions defined on subsets of $\mathcal{G}$. These are to satisfy:

6) if ϵxX is a term, there is an element, $f_{\epsilon xX}$ in $\mathcal{F}$ such that: $f_{\epsilon xX}$ is a function with domain the set of Γ in $\mathcal{G}$ such that $\mathcal{P}(\Gamma)$ contains all parameters of X; if $\Gamma \in$ domain $f_{\epsilon xX}$ then $f_{\epsilon xX}(\Gamma) \in \Gamma$; if $\Gamma \models (\exists x)X$ then $\Gamma \models X(x/f_{\epsilon xX}(\Gamma))$.

[For simplicity in stating the next two items, if c is a parameter, let f_c be the function with domain the set of Γ in $\mathcal{G}$ such that $c \in \mathcal{P}(\Gamma)$, and whose value is given by $f_c(\Gamma) = c$.]

7) if $(\lambda x\ X)(t)$ is a formula,

$$\Gamma \models (\lambda x\ X)(t) \text{ if and only if } \Gamma \models X(x/f_t(\Gamma)).$$

8) if P is an n-place predicate letter and $t_1, \ldots, t_n$ are terms,

$$\Gamma \models P(t_1,\ldots,t_n) \text{ if and only if } \Gamma \models P(f_{t_1}(\Gamma),\ldots,f_{t_n}(\Gamma)).$$

An $\epsilon S4^o$ formula X is called valid in the $\epsilon S4^o$ model $<\mathcal{G}, \mathcal{R}, \models, \mathcal{P}, \mathcal{F}>$ if $\Gamma \models X$ for every $\Gamma \in \mathcal{G}$ such that all parameters of X are in $\mathcal{P}(\Gamma)$. We leave the reader to verify (by induction on the length of the proof)

THEOREM 3.1. All theorems of $\epsilon S4^o$ are valid in all $\epsilon S4^o$ models.

Moreover, any FS4 model $< \mathcal{G}, \mathcal{R}, \models, \mathcal{P} >$ can be extended to an $\epsilon S4^{o}$ model $< \mathcal{G}, \mathcal{R}, \models, \mathcal{P}, \mathcal{F} >$ by extending $\models$ and defining $\mathcal{F}$ by induction on the degree of formulas. Thus we have, using the above and the completeness of FS4,

THEOREM 3.2. Let X be a formula of FS4 with no parameters. If X is a theorem of $\epsilon S4^{o}$, X is a theorem of FS4.

§4. Development of $\epsilon S4^{o}$

In this section we merely sketch how $\epsilon S4^{o}$ can be developed as a practical calculus, and show that it extends the parameter-free part of FS4. We no longer allow parameters in $\epsilon S4^{o}$ formulas.

We use the notation $\vdash X$ to mean <u>all λ-closures of X are provable</u>. Our axiom schemas are of this form; our rules have analogous generalizations. Thus one may show: if X and Y are quasi-formulas,

1) $$\frac{\vdash X \qquad \vdash X \supset Y}{\vdash Y}$$

2) $$\frac{\vdash X}{\vdash \Box X}.$$

Next one may show a replacement theorem in the following form:

3) Let A, B, X and Y be quasi-formulas. Let Y be the result of replacing, in X, the quasi-formula A at some or all of its occurrences (except within quasi-terms) by B. Then

$$\frac{\vdash A \equiv B}{\vdash X \equiv Y}.$$

This is somewhat different than the usual form, but that follows using

4) (closure theorem) Let us denote by $\forall X$ any universal closure of the quasi-formula X. Then $\vdash X$ if and only if $\forall X$ is provable.

Finally we show that $\epsilon S4^o$ is an extension of the parameter-free part of FS4.

Let $X_1, X_2, \ldots, X_n$ be a proof of X_n in some FS4 axiom system, say that of [1] or [4]. Let $a_1, a_2, \ldots, a_k$ be all the parameters occurring in this proof, and let $x_1, x_2, \ldots, x_k$ be variables not used in any formula of the proof. For each $i = 1, 2, \ldots, n$, let $X_i^* = X_i(\underset{\sim}{a}/\underset{\sim}{x})$. We claim $\vdash X_i^*$. If X_i is an axiom of the FS4 system, this is straightforward. Modus ponens becomes 1) above, the rule of necessitation, 2), and the property corresponding to the rule of universal generalization is easily shown. Thus $\vdash X_n^*$. Now, if X_n has no parameters, $X_n^* = X_n$. Thus we have

THEOREM 4.5. If X has no parameters and is a theorem of FS4, then X is a theorem of $\epsilon S4^o$.

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