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AN EMBEDDING OF CLASSICAL LOGIC IN S4

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§1. Introduction. There are well-known embeddings of intuitionistic logic into S4 and of classical logic into S5. In this paper we give a related embedding of (first order) classical logic directly into (first order) S4, with or without the Barcan formula. If one reads the necessity operator of S4 as 'provable', the translation may be roughly stated as: truth may be replaced by provable consistency. A proper statement will be found below. The proof is based ultimately on the notion of complete sequences used in Cohen's technique of forcing [1], and is given in terms of Kripke's model theory [3], [4].

In [6] McKinsey and Tarski defined a translation from nonmodal to modal propositional formulas, which has been extended to first order formulas by Prawitz [7]. See also Schütte [8]. The following is a variant of that translation.

For any nonmodal formulas, A, X, and Y:

If A is atomic, let $A° = \Box A$, and let

$$(X \wedge Y)° = \Box(X° \wedge Y°),$$
$$(X \vee Y)° = \Box(X° \vee Y°),$$
$$(X \supset Y)° = \Box(X° \supset Y°),$$
$$(\sim X)° = \Box \sim X°,$$
$$[(\forall x)X(x)]° = \Box(\forall x)[X(x)]°,$$
$$[(\exists x)X(x)]° = \Box(\exists x)[X(x)]°.$$

That is, to apply ° to a formula is to put $\Box$ before every subformula.

Letting I be first order intuitionistic logic, C be first order classical logic, S4 be first order S4 (see, e.g. Schütte [8]), and S5 be first order S5,

$$\vdash_I X \quad \text{iff} \quad \vdash_{S4} X°, \qquad \vdash_C X \quad \text{iff} \quad \vdash_{S5} X°.$$

REMARK. In the more customary version of this translation the corresponding cases above are replaced by

$$(X \wedge Y)° = X° \wedge Y°,$$
$$(X \vee Y)° = X° \vee Y°,$$
$$[(\exists x)X(x)]° = (\exists x)[X(x)]°.$$

That either version may be used follows easily by an induction on degree together with

$$\vdash_{S4} (\Box X \wedge \Box Y) \equiv \Box(\Box X \wedge \Box Y),$$
$$\vdash_{S4} (\Box X \vee \Box Y) \equiv \Box(\Box X \vee \Box Y),$$
$$\vdash_{S4} (\exists x)\Box X(x) \equiv \Box(\exists x)\Box X(x).$$

We define an analogous translation from nonmodal to modal formulas as follows:

If A is atomic, let $A^* = \Box\Diamond A$, and let

$$(X \wedge Y)^* = \Box\Diamond(X^* \wedge Y^*),$$
$$(X \vee Y)^* = \Box\Diamond(X^* \vee Y^*),$$
$$(X \supset Y)^* = \Box\Diamond(X^* \supset Y^*),$$
$$(\sim X)^* = \Box\Diamond \sim X^*,$$
$$[(\forall x)X(x)]^* = \Box\Diamond(\forall x)[X(x)]^*,$$
$$[(\exists x)X(x)]^* = \Box\Diamond(\exists x)[X(x)]^*.$$

That is, to apply $*$ to a formula is to put $\Box\Diamond$ before every subformula.

Let C and S4 be as above. Let T be the trivial modal logic in which $\Box X \equiv \Diamond X \equiv X$. We will show

THEOREM. *For any nonmodal formula X, and any modal logic L such that* S4 $\subseteq L \subseteq T$, $\vdash_C X$ *iff* $\vdash_L X^*$.

REMARKS. Unlike the above $\circ$ translation, only one case can be simplified here. The corresponding case above may be replaced by $(X \wedge Y)^* = (X^* \wedge Y^*)$. This follows from

$$\vdash_{S4} (\Box\Diamond X \wedge \Box\Diamond Y) \equiv \Box\Diamond(\Box\Diamond X \wedge \Box\Diamond Y).$$

A few words about the origin and significance of this translation may be in order. In [2] we showed that, suitably interpreted, the notion of a Kripke model for intuitionistic logic could replace the notion of forcing for obtaining the Cohen independence results in set theory. (A connection between the two notions was first remarked in [5].) The transition between intuitionistic logic and classical logic used there was the well-known theorem

$$\vdash_C X \quad \text{iff} \quad \vdash_I \sim\sim X$$

where X contains no universal quantifiers. Instead of intuitionistic logic models, S4 models could have been used. Thus, one can produce a Kripke S4 model in which if A is an axiom of ZF, A^* is valid, but $[\sim(V = L)]^*$ is also. Then the theorem stated above provides the classical independence of the axiom of constructibility. ($\Gamma \vDash \Box\Diamond X$ is analogous to 'X is weakly forced'.)

We give a purely semantic proof of the theorem although a proof theoretic one is possible (and not difficult). We feel that a semantic proof frees us of the peculiarities of a particular formalization, as well as being of interest for its own sake. For convenience, we will continue to use the symbol $\vdash$, but it may be read as asserting validity rather than provability.

§2. **Model theory preliminaries.** The notion of a Kripke S4 model is from [3], [4]; the definition is stated here to establish notation, which is based on [2]. We use formulas with parameters. We will use $x, y, \quad \cdot$ for variables, and $a, b, \cdots$ for parameters. 'Formula' means closed formula, i.e. with no free variables.

We use the convention that if $\mathscr{P}$ is a map ranging over sets of parameters, $\hat{\mathscr{P}}(x)$ is the set of all formulas with parameters from $\mathscr{P}(x)$.

By an S4 *model* we mean an ordered quadruple $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ where $\mathscr{G}$ is a nonempty set, $\mathscr{R}$ is a reflexive, transitive relation on $\mathscr{G}$, $\mathscr{P}$ is a map from elements of $\mathscr{G}$ to nonempty sets of parameters satisfying, for any $\Gamma, \Gamma^* \in \mathscr{G}$,

$$\Gamma\mathscr{R}\Gamma^* \Rightarrow \mathscr{P}(\Gamma) \subseteq \mathscr{P}(\Gamma^*),$$

and $\vDash$ is a relation between elements of $\mathscr{G}$ and formulas, having the following properties (we use $\nvDash$ for not-$\vDash$): for any $\Gamma \in \mathscr{G}$:

If $X \notin \hat{\mathscr{P}}(\Gamma)$, $\Gamma \nvDash X$.

If $X, Y \in \hat{\mathscr{P}}(\Gamma)$, then

(1) $\Gamma \vDash (X \wedge Y)$ iff $\Gamma \vDash X$ and $\Gamma \vDash Y$,

(2) $\Gamma \vDash (X \vee Y)$ iff $\Gamma \vDash X$ or $\Gamma \vDash Y$,

(3) $\Gamma \vDash (X \supset Y)$ iff $\Gamma \nvDash X$ or $\Gamma \vDash Y$,

(4) $\Gamma \vDash \sim X$ iff $\Gamma \nvDash X$,

(5) $\Gamma \vDash (\forall x)X(x)$ iff $\Gamma \vDash X(a)$ for all $a \in \mathscr{P}(\Gamma)$,

(6) $\Gamma \vDash (\exists x)X(x)$ iff $\Gamma \vDash X(a)$ for some $a \in \mathscr{P}(\Gamma)$,

(7) $\Gamma \vDash \square X$ iff   for all $\Gamma^* \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Gamma^*$, $\Gamma^* \vDash X$,

(8) $\Gamma \vDash \Diamond X$ iff   for some $\Gamma^* \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Gamma^*$, $\Gamma^* \vDash X$.

A motivation for this model theory is the following. $\mathscr{G}$ is the collection of all possible states of affairs. To say $\Gamma \mathscr{R} \Gamma^*$ is to say if Γ is the situation now, Γ^* might be the situation later. $\mathscr{P}(\Gamma)$ is the collection of all objects existing in Γ. $\Gamma \vDash X$ means, in the situation Γ, X is true.

A formula X is *valid* in the S4 model. $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ if, for each $\Gamma \in \mathscr{G}$, such that $X \in \hat{\mathscr{P}}(\Gamma)$, $\Gamma \vDash X$. A proof may be found in [3], [4], or [8] (with a slightly different notion of model) that

$$\vdash_{S4} X \quad \text{iff} \quad X \text{ is valid in all S4 models.}$$

Call an S4 model a trivial model, or a T model, if $\mathscr{G}$ has only one element. It is easy to show

$$\vdash_T X \quad \text{iff} \quad X \text{ is valid in all trivial models.}$$

REMARK. We will use the convention that in the model $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$, for $\Gamma \in \mathscr{G}$, Γ^* will stand for an arbitrary element of $\mathscr{G}$ such that $\Gamma \mathscr{R} \Gamma^*$.

We use the term *truth set* modified from Smullyan [9] as follows. Let P be some nonempty set of parameters. We call a set $\mathscr{T}$ of nonmodal formulas a *truth set with respect to P* if, for any nonmodal formulas X and Y with all parameters from P,

(1) $(X \wedge Y) \in \mathscr{T}$ iff $X \in \mathscr{T}$ and $Y \in \mathscr{T}$,

(2) $(X \vee Y) \in \mathscr{T}$ iff $X \in \mathscr{T}$ or $Y \in \mathscr{T}$,

(3) $(X \supset Y) \in \mathscr{T}$ iff $X \notin \mathscr{T}$ or $Y \in \mathscr{T}$,

(4) $(\sim X) \in \mathscr{T}$ iff $X \notin \mathscr{T}$,

(5) $(\forall x)X(x) \in \mathscr{T}$ iff $X(a) \in \mathscr{T}$ for all $a \in P$,

(6) $(\exists x)X(x) \in \mathscr{T}$ iff $X(a) \in \mathscr{T}$ for some $a \in P$.

For any nonmodal formula X, $\vdash_C X$ if and only if X belongs to any set $\mathscr{T}$ which is a truth set with respect to some P where P includes at least all the parameters of X.

§3. **Proof of the embedding theorem.** We now proceed with a proof of the theorem of §1. The principal half of the proof is based on the proof in [2] of a connection between intuitionistic and classical logic, which in turn was based on the notion of complete sequences in Cohen [1]. It somewhat resembles the Henkin completeness proof for classical logic.

We begin with the simpler half of the proof.

THEOREM 1. *Let X be any modal formula and let $X^\#$ be like X except that all occurrences of $\square$ and $\diamondsuit$ have been deleted. Then if $\vdash_T X$, $\vdash_C X^\#$.*

PROOF. If $\nvdash_C X^\#$, by the result stated in §2, there is a set of parameters P, including those of $X^\#$, and a truth set $\mathscr{T}$ with respect to P, such that $X^\# \notin \mathscr{T}$. We construct a T model $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ as follows. Let $\mathscr{G} = \{\mathscr{T}\}$, let $\mathscr{T}\mathscr{R}\mathscr{T}$, let $\mathscr{P}(\mathscr{T}) = P$. If A is atomic, let $\mathscr{T} \vDash A$ if $A \in \mathscr{T}$. $\vDash$ may be extended uniquely to all formulas in such a way that $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ is a model. Clearly, for any $Z \in \hat{\mathscr{P}}(\mathscr{T})$, $\mathscr{T} \vDash \square Z$ iff $\mathscr{T} \vDash \diamondsuit Z$ iff $\mathscr{T} \vDash Z$. It then follows that $\mathscr{T} \vDash Z^\#$ iff $\mathscr{T} \vDash Z$. Furthermore, since $Z^\#$ is nonmodal, $\mathscr{T} \vDash Z^\#$ iff $Z^\# \in \mathscr{T}$. Hence $\mathscr{T} \nvDash X$, but $X \in \hat{\mathscr{P}}(\mathscr{T})$, so $\nvdash_T X$.

COROLLARY. *Let X be any nonmodal formula. If $\vdash_T X^*$, then $\vdash_C X$.*

Before proceeding with the converse, let us define X' to be like X^* but without the initial occurrence of $\square\diamondsuit$. Thus $X^* = \square\diamondsuit X'$.

LEMMA 1. *Let $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ be any S4 model. Let $\Gamma \in \mathscr{G}$ and $X \in \hat{\mathscr{P}}(\Gamma)$ where X is a nonmodal formula. Then*

(1) $\Gamma \vDash X^* \Rightarrow$ *for every Γ^*, $\Gamma^* \vDash X^*$,*

(2) $\Gamma \nvDash X^* \Rightarrow$ *for some Γ^*, $\Gamma^* \vDash (\sim X)^*$,*

(3) $\Gamma \vDash [(\exists x)X(x)]^* \Rightarrow$ *for some Γ^* and for some $a \in \mathscr{P}(\Gamma^*)$, $\Gamma^* \vDash [X(a)]^*$,*

(4) $\Gamma \nvDash [(\forall x)X(x)]^* \Rightarrow$ *for some Γ^* and for some $a \in \mathscr{P}(\Gamma^*)$, $\Gamma^* \vDash [\sim X(a)]^*$.*

PROOF. (1) Since X^* begins with an occurrence of $\square$, and $\Gamma \vDash \square Y$ implies $\Gamma^* \vDash \square Y$, this is immediate.

(2) If $\Gamma \nvDash X^*$, $\Gamma \nvDash \square\diamondsuit X'$. Then for some Γ^*, $\Gamma^* \vDash \square \sim X'$. But $\vdash_{S4} \square \sim X' \supset \square\diamondsuit \sim \square\diamondsuit X'$, so $\Gamma^* \vDash \square\diamondsuit \sim \square\diamondsuit X'$, i.e. $\Gamma^* \vDash (\sim X)^*$.

(3) If $\Gamma \vDash [(\exists x)X(x)]^*$, $\Gamma \vDash \square\diamondsuit(\exists x)[X(x)]^*$. So $\Gamma \vDash \diamondsuit(\exists x)[X(x)]^*$. Then for some Γ^*, $\Gamma^* \vDash (\exists x)[X(x)]^*$. Thus for some $a \in \mathscr{P}(\Gamma^*)$, $\Gamma^* \vDash [X(a)]^*$.

(4) If $\Gamma \nvDash [(\forall x)X(x)]^*$, $\Gamma \nvDash \square\diamondsuit(\forall x)[X(x)]^*$, so $\Gamma \vDash \diamondsuit\square(\exists x)\sim [X(x)]^*$. Then for some Γ^*, $\Gamma^* \vDash \square(\exists x)\sim [X(x)]^*$. $\Gamma^* \vDash (\exists x)\sim [X(x)]^*$. So for some $a \in \mathscr{P}(\Gamma^*)$, $\Gamma^* \vDash \sim [X(a)]^*$. $\Gamma^* \nvDash [X(a)]^*$. Now by part (2) we are done.

Let $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ be an S4 model and $\Gamma \in \mathscr{G}$. We describe the construction of a *complete sequence* in $\mathscr{G}$ beginning with Γ.

Let $X_1, X_2, X_3, \cdots$ be an enumeration of all nonmodal formulas.

Let $\Gamma_1 = \Gamma$. Suppose we have defined Γ_n. Consider X_n. We have several cases.

If $X_n \notin \hat{\mathscr{P}}(\Gamma_n^*)$ for any Γ_n^*, then let $\Gamma_{n+1} = \Gamma_n$.

If $X_n \in \hat{\mathscr{P}}(\Gamma_n^*)$ for some Γ_n^*, choose one such Γ_n^*, call it γ_n.

Case (1a). $\gamma_n \vDash X_n^*$ and X_n *is not of the form* $(\exists x)A(x)$. Then let $\Gamma_{n+1} = \gamma_n$.

Case (1b). $\gamma_n \vDash X_n^*$ and X_n *is of the form* $(\exists x)A(x)$. Then by the previous lemma, for some γ_n^* and some $a \in \mathscr{P}(\gamma_n^*)$, $\gamma_n^* \vDash [A(a)]^*$. Let $\Gamma_{n+1} = \gamma_n^*$.

Case (2a). $\gamma_n \nvDash X_n^*$ and X_n *is not of the form* $(\forall x)A(x)$. Then for some γ_n^*, $\gamma_n^* \vDash (\sim X_n)^*$. Let $\Gamma_{n+1} = \gamma_n^*$.

Case (2b). $\gamma_n \nvDash X_n^*$ and X_n *is of the form* $(\forall x)A(x)$. Then for some γ_n^* and for some $a \in \mathscr{P}(\gamma_n^*)$, $\gamma_n^* \vDash [\sim A(a)]^*$. Let $\Gamma_{n+1} = \gamma_n^*$.

REMARK. In the above construction, if $X_n \in \hat{\mathscr{P}}(\gamma_n)$ and $\gamma_n \nvDash X_n^*$, then $\Gamma_{n+1} \vDash (\sim X_n)^*$. If this falls under Case (2a), this is immediate. In Case (2b) it follows since

$$\vdash_{S4} \square\diamondsuit \sim \square\diamondsuit A'(a) \supset \square\diamondsuit \sim \square\diamondsuit(\forall x)\square\diamondsuit A'(x)$$

i.e.

$$\vdash_{S4} [\sim A(a)]^* \supset [\sim(\forall x)A(x)]^*.$$

Let $\mathscr{C} = \{\Gamma_1, \Gamma_2, \Gamma_3, \cdots\}$. We call $\mathscr{C}$ a *complete sequence beginning with* Γ.

Let $\overline{\mathscr{C}} = \{X \mid \text{for some } \Gamma_n \in \mathscr{C}, \Gamma_n \vDash X^*\}$. Call a formula X *relevant* to $\overline{\mathscr{C}}$ if all the parameters of X belong to $P = \bigcup_{\Gamma_n \in \mathscr{C}} \mathscr{P}(\Gamma_n)$.

We proceed to show $\overline{\mathscr{C}}$ is a classical truth set with respect to P.

LEMMA 2. *For any relevant nonmodal formula* X, *exactly one of* X *or* $\sim X$ *belongs to* $\overline{\mathscr{C}}$.

PROOF. (1) Suppose both X and $\sim X$ belonged to $\overline{\mathscr{C}}$. Then for some Γ_n and Γ_m in $\mathscr{C}$, $\Gamma_n \vDash X^*$ and $\Gamma_m \vDash (\sim X)^*$. By construction, either $\Gamma_n \mathscr{R} \Gamma_m$ or $\Gamma_m \mathscr{R} \Gamma_n$. Say $\Gamma_n \mathscr{R} \Gamma_m$. By Lemma 1, $\Gamma_m \vDash X^*$. But $\vdash_{S4} \Box \Diamond \sim \Box \Diamond X' \supset \sim \Box \Diamond X'$, i.e. $\vdash_{S4} (\sim X)^* \supset \sim X^*$. So, since $\Gamma_m \vDash (\sim X)^*$, $\Gamma_m \vDash \sim X^*$, $\Gamma_m \nvDash X^*$. A contradiction. Similarly if $\Gamma_m \mathscr{R} \Gamma_n$.

(2) Suppose $X \notin \overline{\mathscr{C}}$. For some n, $X = X_n$. Since $X_n \notin \overline{\mathscr{C}}$, $\gamma_n \nvDash X_n^*$. By the remark above, $\Gamma_{n+1} \vDash (\sim X_n)^*$ so $\sim X_n = \sim X \in \overline{\mathscr{C}}$.

LEMMA 3. *For any relevant nonmodal formulas* X *and* Y, *the following holds*:

(1) $(X \lor Y) \in \overline{\mathscr{C}}$ *iff* $X \in \overline{\mathscr{C}}$ *or* $Y \in \overline{\mathscr{C}}$,

(2) $(X \land Y) \in \overline{\mathscr{C}}$ *iff* $X \in \overline{\mathscr{C}}$ *and* $Y \in \overline{\mathscr{C}}$,

(3) $(X \supset Y) \in \overline{\mathscr{C}}$ *iff* $X \notin \overline{\mathscr{C}}$ *or* $Y \in \overline{\mathscr{C}}$.

PROOF. (1) $\vdash_{S4} (\Box \Diamond \sim \Box \Diamond X' \land \Box \Diamond \sim \Box \Diamond Y') \supset \Box \Diamond \sim \Box \Diamond (\Box \Diamond X' \lor \Box \Diamond Y')$,

i.e. $\vdash_{S4} [(\sim X)^* \land (\sim Y)^*] \supset [\sim (X \lor Y)]^*$.

Suppose $X \notin \overline{\mathscr{C}}$ and $Y \notin \overline{\mathscr{C}}$. By Lemma 2, $\sim X \in \overline{\mathscr{C}}$ and $\sim Y \in \overline{\mathscr{C}}$. Then for some $\Gamma_n, \Gamma_m \in \mathscr{C}$, $\Gamma_n \vDash (\sim X)^*$ and $\Gamma_m \vDash (\sim Y)^*$. Either $\Gamma_n \mathscr{R} \Gamma_m$ or $\Gamma_m \mathscr{R} \Gamma_n$. Say the latter. Then using Lemma 1, $\Gamma_n \vDash (\sim X)^* \land (\sim Y)^*$, thus $\Gamma_n \vDash [\sim (X \lor Y)]^*$. So $\sim (X \lor Y) \in \overline{\mathscr{C}}$ and by Lemma 2, $X \lor Y \notin \overline{\mathscr{C}}$.

The converse follows similarly using

$\vdash_{S4} (\Box \Diamond X' \lor \Box \Diamond Y') \supset \Box \Diamond (\Box \Diamond X' \lor \Box \Diamond Y')$,

i.e. $\vdash_{S4} (X^* \lor Y^*) \supset (X \lor Y)^*$.

(2) is done like (1), using

$\vdash_{S4} (\Box \Diamond X' \land \Box \Diamond Y') \equiv \Box \Diamond (\Box \Diamond X' \land \Box \Diamond Y')$,

i.e. $\vdash_{S4} (X^* \land Y^*) \equiv (X \land Y)^*$.

(3) follows using the following

$\vdash_{S4} (\Box \Diamond \sim \Box \Diamond X' \lor \Box \Diamond Y') \supset \Box \Diamond (\Box \Diamond X' \supset \Box \Diamond Y')$,

i.e. $\vdash_{S4} [(\sim X)^* \lor Y^*] \supset [X \supset Y]^*$

and $\vdash_{S4} (\Box \Diamond X' \land \Box \Diamond \sim \Box \Diamond Y') \supset \Box \Diamond \sim \Box \Diamond (\Box \Diamond X' \supset \Box \Diamond Y')$,

i.e. $\vdash_{S4} [X^* \land (\sim Y)^*] \supset [\sim (X \supset Y)]^*$.

LEMMA 4. *Let* $X(x)$ *be a relevant nonmodal formula. Then*

(1) $(\forall x) X(x) \in \overline{\mathscr{C}} \Leftrightarrow X(a) \in \overline{\mathscr{C}}$ *for all* $a \in P$.

(2) $(\exists x) X(x) \in \overline{\mathscr{C}} \Leftrightarrow X(a) \in \overline{\mathscr{C}}$ *for some* $a \in P$.

PROOF. (1) $\vdash_{S4} [\Box \Diamond (\forall x) \Box \Diamond X'(x)] \supset [\Box \Diamond X'(a)]$,

i.e. $\vdash_{S4} [(\forall x) X(x)]^* \supset [X(a)]^*$

from which half of (1) follows. Conversely, suppose $(\forall x) X(x) \notin \overline{\mathscr{C}}$. For some n, $(\forall x) X(x) = X_n$, so $\gamma_n \nvDash X_n^*$, $\gamma_n \nvDash [(\forall x) X(x)]^*$. By Case (2b) of the construction of $\mathscr{C}$,

$\Gamma_{n+1} \vDash [\sim X(a)]^*$ for some $a \in \mathscr{P}(\Gamma_{n+1})$. Thus $\sim X(a) \in \overline{\mathscr{C}}$, so by Lemma 2, $X(a) \notin \overline{\mathscr{C}}$.

(2) $\vdash_{S4} [\Box \Diamond X'(a)] \supset [\Box \Diamond (\exists x) \Box \Diamond X'(a)]$,

i.e. $\vdash_{S4} [X(a)]^* \supset [(\exists x)X(x)]^*$

so we have half of (2). Conversely, suppose $(\exists x)X(x) \in \overline{\mathscr{C}}$. For some n, $(\exists x)X(x) = X_n$. If $\gamma_n \nvDash X_n^*$, i.e. $\gamma_n \nvDash [(\exists x)X(x)]^*$, by Case (2a) of the construction of $\mathscr{C}$, $\Gamma_{n+1} \vDash [\sim (\exists x)X(x)]^*$, so $\sim (\exists x)X(x) \in \overline{\mathscr{C}}$, contradicting Lemma 2. Thus $\gamma_n \vDash [(\exists x)X(x)]^*$. Now by Case (1b) of the construction of $\mathscr{C}$, $\Gamma_{n+1} \vDash [X(a)]^*$ for some $a \in \mathscr{P}(\Gamma_{n+1})$. Thus $X(a) \in \overline{\mathscr{C}}$.

Thus we have shown that $\overline{\mathscr{C}}$ is a truth set with respect to P.

THEOREM 2. *Let X be any nonmodal formula. If $\vdash_C X$ then $\vdash_{S4} X^*$.*

PROOF. Suppose $\nvdash_{S4} X^*$. Then for some model $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$, for some $\Gamma \in \mathscr{G}$, $X^* \in \hat{\mathscr{P}}(\Gamma)$, but $\Gamma \nvDash X^*$. By Lemma 1, for some Γ^*, $\Gamma^* \vDash (\sim X)^*$. Construct a complete sequence $\mathscr{C}$ beginning with Γ^*. $\sim X \in \overline{\mathscr{C}}$ so $X \notin \overline{\mathscr{C}}$. But the above lemmas show that $\overline{\mathscr{C}}$ is a classical truth set with respect to P, and all the parameters of X are in P. Thus, as we remarked in §2, $\nvdash_C X$.

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