Chapter 1
Tableaus and Dual Tableaus

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Abstract
In a sense, tableaus and dual tableaus are the same thing, just as tableaus and sequent calculi are the same thing. There are mathematical ideas, and there are presentations of them. For applications, representing linear operators as matrices is wonderfully helpful, but for proving results about linear operators a more abstract approach is simpler and clearer. The form of mathematical structures matters psychologically for people, though perhaps it matters little to the god of mathematics who kept Paul Erdős's book of proofs. Tableaus work towards an obvious contradiction, dual tableaus work towards an obvious truth. Which is best? Who asks the question? That determines the answer. Here we examine the basics of tableaus and dual tableaus and their connections, looking only at the most fundamental of logics. That should be enough to make the general ideas plain.

> "I'll tell you all my ideas about Looking-glass House. First, there's the room you can see through the glass—that's just the same as our drawing room, only the things go the other way. I can see all of it when I get upon a chair—all but the bit behind the fireplace. ... Well then, the books are something like our books, only the words go the wrong way; I know that, because I've held up one of our books to the glass, and then they hold up one in the other room. ... But oh, Kitty! now we come to the passage. You can just see a little peep of the passage in Looking-glass House, if you leave the door of our drawing-room wide open: and it's very like our passage as far as you can see, only you know it may be quite different on beyond. Oh, Kitty! how nice it would be if we could only get through into Looking-glass House! I'm sure it's got, oh! such beautiful things in it!"
>
> Excerpts From: Lewis Carroll. *Through the Looking-Glass*.

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1.1 Introduction

The tableau proof format is highly malleable. Machinery can be added or subtracted to produce versions suitable for a range of logics. Indeed, even the spelling of the name varies: the standard plural is ”tableaux” but I have always preferred ”tableaus.” It’s a minor point, but I will follow my preferences here (notice to copy editor). What is common to all tableau systems is that proofs somehow involve trees, and proof steps move from formulas to subformulas, thus reducing formula complexity. At one time I would have said that tableau systems are always refutation systems, but dual tableaus are a prominent counter-example to this. It is the job of the present paper to explain something of tableaus and dual tableaus and their relationships to each other. It will be seen that while connections are very close, differences stem from varying intuitions of how one determines what it is that makes a formula true under all pertinent circumstances. But, you will see.

The sequent calculus, for both classical and intuitionistic logic, was introduced in (Gentzen 1935). While its primary function was meant to be theoretical, it was also employed for proof discovery by making use of it upside down. For instance, in (Wang 1960) it was made the basis of a very early automated theorem prover using this upside down idea. Beth, with independent semantical motivation, introduced a two column tree proof method in (Beth 1955, 1956, 1959), essentially making the upside down version of the sequent calculus into a thing in itself, under the name *semantic tableaus*. Beth’s machinery was rather awkward in practice, and was simplified to its modern version in (Smullyan 1968) which has been highly influential. Essentially the same mechanism was also presented in (Lis 1960), though this paper did not become generally known until much later.

Dual tableaus originated in (Rasiowa and Sikorski 1960), were used in (Binkley and Clark 1967), were extended to some modal logics in (Snyder 1971), and underwent further development largely due to Ewa Orlowska and her students. A recent and detailed presentation can be found in (Orłowska and Golińska-Pilarek 2011). Dual tableaus are, quite literally, dual to tableaus as they are customarily presented. Proofs can be translated back and forth between tableaus and dual tableaus. So one could develop in detail everything needed of a formal presentation for just one of tableaus or dual tableaus, and extract a treatment of the other system by translation. But psychology plays an important role in the creation/discovery of mathematics. Tableaus are refutation systems, while dual tableaus more directly search for forward proofs. This difference changes how one thinks about what one is doing, and hence what one is doing in fact. Our machinery shapes the things we make when we wield the tools.

Dual tableaus have found a major application in the treatment of logics formulated using relational algebras. This is a topic that will not be considered here. We concentrate on the tableau/dual tableau machinery itself, looking at classical and intuitionistic propositional logic. Other logics could easily be added to the mix, but once the basic ideas are understood additional examples should be developed by the reader. It’s how one truly understands. We do not pick either tableau or dual tableau as primary, but present both in parallel. Rather than using proof-theoretic methods, we rely exclusively on semantical machinery. This can give us equivalence with axiomatic systems, for instance, but proofs are not central here. Proof theoretical machinery is important and useful, but semantics is sometimes clearer and simpler, especially as a first exposure to material. See (Dawson and Goré 2017) for closely related semantic based work, formalizing the meta-theory of the dual tableau calculus for intuitionistic logic presented in Section 1.3.1 using HOL.

1.2 Classical Propositional Logic and the Basic Ideas

Formulas are built up from a countable list of propositional letters using the connectives $\wedge$, $\vee$, $\supset$, and $\neg$. Of course not all are needed, but this will change when we come to intuitionistic logic, so we might as well have them now. Also in proofs we use *signed* formulas: if X is a formula, $T\,X$ and $F\,X$ are signed formulas which, informally say that X is true, or false respectively. Signs are not necessary classically. Instead of $F\,X$ we could use $\neg X$, and $T\,X$ can simply be X itself. But again things change when we come to intuitionistic logic, and signs do no harm now.

We have just encountered a subtle but important point. When we said that $T\,X$ could be read informally as asserting that X is true, there was an ambiguity. Were we talking about true under some interpretation of propositional letters, or under every interpretation? Tableaus assume we are talking about *some* interpretation, which leads to satisfiability being basic. Dual tableaus assume we are talking about *every* interpretation, which makes tautology-hood basic. This is the essential conceptual distinction between tableaus and dual tableaus. A standard tableau proof system is a refutation system. To show X must be a tautology, begin by supposing it is not. Then $F\,X$ must be possible. Derive a contradiction. Hence X no matter what. On the other hand a dual tableau is a kind of search for a proof. We want $T\,X$ no matter what. What would we need for this, then what would we need for that, and so on, until we reach something we obviously have.

We used the word “interpretation” above, but that was informal. More properly, we call an assignment of truth values, true and false, to propositional letters a *boolean valuation*. A boolean valuation extends uniquely to a mapping from all propositional formulas to truth values in the usual way, and we make no distinction between a valuation as a mapping on propositional letters and on all formulas. We do extend boolean valuations to signed formulas: $T\,X$ is true under a boolean valuation if X is true, and $F\,X$ is true if X is false. We will talk about satisfiability of a set of signed formulas, meaning its members all map to true under some boolean valuation. This is a simple extension of satisfiability for sets of unsigned formulas.

We begin by presenting the dual tableau rules, and then a tableau counterpart. Finally we discuss soundness and completeness for the two proof systems.

1.2.1 Classical Propositional Dual Tableaus

We do not use the standard notation for dual tableaus, but rather a straightforward alternative that helps bring out similarity of ideas between tableaus and the dual version. Customarily a dual tableau is a tree of sets of formulas, but instead we present a dual tableau as a tree of signed formulas. In Section 1.2.5 we briefly discuss the more common presentation of classical dual tableaus, and we use it extensively when we come to intuitionistic logic.

Figure 1.1 shows a schematic form of the dual tableau rules for classical propositional logic. Here is the intuition. Suppose we want to verify that $X \wedge Y$ is true under every boolean valuation. Then we must produce verifications for both X and Y. Hence the rule: from $T X \wedge Y$ divide into two cases, one with $T X$ and one with $T Y$. Similarly to refute (verify the negation of) $X \wedge Y$ it is enough to refute either X or Y. Hence the rule: from $F X \wedge Y$ we remain in a single case, but we list both $F X$ and $F Y$, either of which is sufficient to work with. The other rules have similar motivations.

$$\frac{T \neg X}{F X} \quad \frac{F \neg X}{T X}$$

$$\frac{F X \wedge Y}{\begin{array}{c} F X \\ F Y \end{array}} \quad \frac{T X \vee Y}{\begin{array}{c} T X \\ T Y \end{array}} \quad \frac{T X \supset Y}{\begin{array}{c} F X \\ T Y \end{array}}$$

$$\frac{T X \wedge Y}{T X \mid T Y} \quad \frac{F X \vee Y}{F X \mid F Y} \quad \frac{F X \supset Y}{T X \mid F Y}$$

Fig. 1.1 Classical Propositional Dual Tableau Rules

A dual tableau proof of formula X begins with $T X$; we want to find what is needed to verify X under every boolean valuation. Dual tableau proofs have a tree structure, and so we start with a tree consisting of only a root node labeled $T X$. Trees are "grown" by thinking of the various cases in Figure 1.1 as *branch extension rules*. These are of two types. One type is *non-branching*: if a certain signed formula occurs on a branch, the branch can be extended with some new signed formulas. The other type of rule is *branching*, in which the end of a branch forks and a new signed formula is added to the end of each fork. It is customary to display both tableaus and dual tableaus as branching downward. Taking our discussion of rule motivation above into account, we see that branching should be thought of conjunctively—each branch is a task, and the tasks associated with every branch must be accomplished. Each branch individually should be thought of disjunctively; success with any item on a branch is sufficient for that branch.

Suppose $T X \vee Y$ occurs on a branch, that is, we would like a verification of $X \vee Y$. We add $T X$ and $T Y$ to the branch—a verification of either X or Y would suffice. Intuitively it would be redundant to do this a second time, and similarly for the other rules. Here is some simple machinery to handle this issue.

Definition 1 (Inactive and Active). On a dual tableau branch:

1. $F\,X \wedge Y$ is inactive if both $F\,X$ and $F\,Y$ are present;
2. $T\,X \vee Y$ is inactive if both $T\,X$ and $T\,Y$ are present;
3. $T\,X \supset Y$ is inactive if both $F\,X$ and $T\,Y$ are present;
4. $T\,X \wedge Y$ is inactive if one of $T\,X$ or $T\,Y$ are present;
5. $F\,X \vee Y$ is inactive if one of $F\,X$ or $F\,Y$ are present;
6. $F\,X \supset Y$ is inactive if one of $T\,X$ or $F\,Y$ are present;
7. $T\,\neg X$ is inactive if $F\,X$ is present;
8. $F\,\neg X$ is inactive if $T\,X$ is present.

If a signed formula is not inactive on a dual tableau branch, it is *active* on that branch. We say a dual tableau meets a *single use restriction* if rules are only applied to active signed formulas on a branch.

Dual tableaus are sound and complete with or without a single use restriction, but a single use restriction is better for proof search. Indeed, it easily gives us decidability, since we will either conclude our proof search successfully or run out of things to do. The order of branch extension rule applications is *non-deterministic*. All applicable rules can be applied in any order, though since we pay attention to single-use restrictions, after a rule is applied to a signed formula occurrence *on a branch*, the signed formula is not subject to further rule applications *on that branch*.

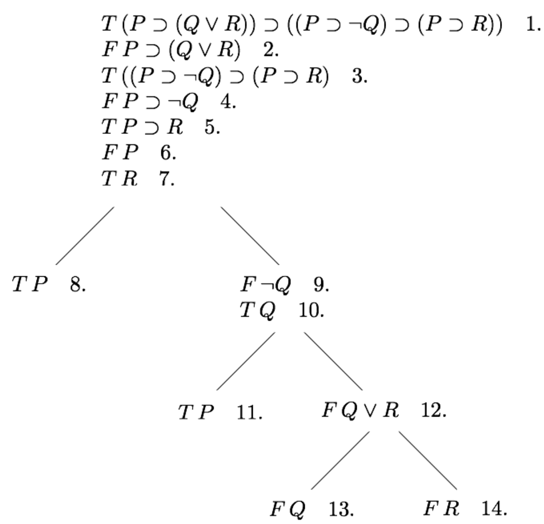

Fig. 1.2 Classical Propositional Dual Tableau

Figure 1.2 shows a dual tableau construction, for the signed propositional formula $T\,(P \supset (Q \vee R)) \supset ((P \supset \neg Q) \supset (P \supset R))$. Think of 1 as the goal—we want a verification of $(P \supset (Q \vee R)) \supset ((P \supset \neg Q) \supset (P \supset R))$. Semantically we need that under any boolean valuation, either $P \supset (Q \vee R)$ should be false or $(P \supset \neg Q) \supset (P \supset R)$ should be true and hence goal 1 is replaced with (the set of) goals 2 and 3. In a similar way 3 is replaced with 4 and 5, and 5 is replaced with 6 and 7. At this point we have 2, 4, 6, and 7 as active goals. We must show every boolean valuation satisfies one of them. Notice the use of non-determinism by the way. We have not applied rules in the "obvious" order, to 1, then to 2, then to 3, and so on. Instead we have singled out those rule applications that did not induce branching, simply to avoid repeating work on each branch that could be done just once, before branching occurs. If we succeed, success is all that matters. In fact, most provable formulas will have many dual tableau proofs, generally of various sizes.

Now to continue, we apply a rule to 4. To refute $P \supset \neg Q$, we must verify P and refute $\neg Q$. Hence goal 4 gets replaced with two goals, 8 and 9. Notice that we can stop work on the left-most branch. It contains both $F\,P$ and $T\,P$ and under any boolean valuation we must have one of these. We continue work on the right branch.

Trivially 9 is replaced with 10. Then 2 is replaced with 11 and 12. As before, we can stop work on the branch ending with 11 because it contains both $F\,P$ and $T\,P$.

Finally 12 is replaced with 13 and 14. The branch ending with 13 is a "success" because of 10 and 13, and so is the branch ending with 14 because of 7 and 14. The initial problem has been reduced to trivial verification.

Definition 2. A dual tableau branch is *closed* (or *axiomatic*) if it contains both $T\,X$ and $F\,X$ for some X. A dual tableau is *closed* if every branch is closed. And a classical propositional dual tableau proof for a formula X is a closed tableau with $T\,X$ at the root, constructed using the rules in Figure 1.1. A dual tableau is *atomically closed* if every branch is closed because it contains a propositional letter with both T and F signs.

We will show soundness in Section 1.2.3 with no restrictions, so adding single use and atomic closure requirements also gives us sound systems. We will show completeness in Section 1.2.4 with both single use and atomic closure restrictions so removing them also gives us complete proof systems.

1.2.2 Classical Propositional Tableaus

As we noted in Section 1.1, a tableau proof system is actually a *refutation* system. To prove a formula X one shows the assumption that X could be false under *some* boolean valuation leads to a contradiction. Then, a tableau to show that a propositional formula X is a tautology begins with $F\,X$, and this is the root of a tableau proof tree.

Next, the tree is expanded using *tableau branch extension rules*. As with dual tableaus there are two types, non-branching and branching. Figure 1.3 gives the classical propositional tableau branch extension rules. It should be noted that the rules are the same as in Figure 1.1, but with the signs reversed.

$$\frac{T\,\neg X}{F\,X} \quad \frac{F\,\neg X}{T\,X}$$

$$\frac{T\,X \wedge Y}{\begin{array}{c}T\,X\\ T\,Y\end{array}} \quad \frac{F\,X \vee Y}{\begin{array}{c}F\,X\\ F\,Y\end{array}} \quad \frac{F\,X \supset Y}{\begin{array}{c}T\,X\\ F\,Y\end{array}}$$

$$\frac{F\,X \wedge Y}{F\,X \mid F\,Y} \quad \frac{T\,X \vee Y}{T\,X \mid T\,Y} \quad \frac{T\,X \supset Y}{F\,X \mid T\,Y}$$

Fig. 1.3 Classical Propositional Tableau Rules

Just as with dual tableaus, the order of rule application is non-deterministic, and we have a single use principle. But the informal motivation becomes the mirror image of that for dual tableaus. Now branching is thought of disjunctively, while signed formulas on the same branch are thought of conjunctively. Instead of being true under all boolean valuations, we want truth under some boolean valuation—satisfiability. For instance, the tableau rule for $T\,X \wedge Y$ informally tells us that if some boolean valuation assigns truth values to a set of formulas in a way that makes $X \wedge Y$ true, it will make both X and Y true. The corresponding rule with F tells us that if a boolean valuation makes $X \wedge Y$ false, it will make one of X or Y false (or possibly both, of course).

Figure 1.4 shows a tableau construction, for the signed propositional formula $F\,(P \supset (Q \vee R)) \supset ((P \supset \neg Q) \supset (P \supset R))$. Unlike with dual tableaus, don't think of 1 as the goal, but as something we want to show impossible. If no boolean valuation can make $(P \supset (Q \vee R)) \supset ((P \supset \neg Q) \supset (P \supset R))$ false, it must be a tautology.

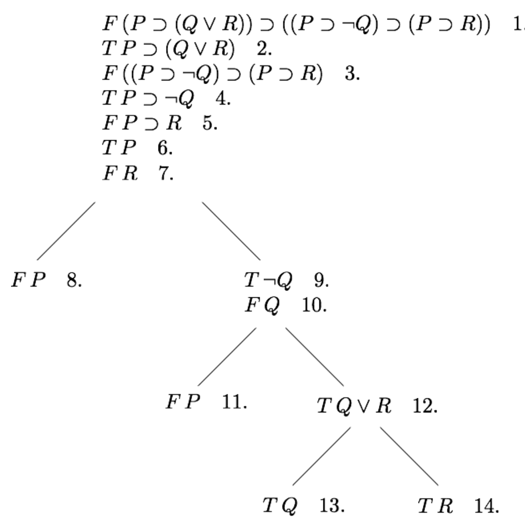

Fig. 1.4 Classical Propositional Tableau

If some boolean valuation falsifies $(P \supset (Q \vee R)) \supset ((P \supset \neg Q) \supset (P \supset R))$, it must make $P \supset (Q \vee R)$ true and $(P \supset \neg Q) \supset (P \supset R)$ false so 1 is replaced with 2 and 3 (recall, branches are now understood conjunctively). In a similar way 3 is replaced with 4 and 5, and 5 is replaced with 6 and 7. We now are left with 2, 4, 6, and 7 active—the set of these four must be satisfiable provided $(P \supset (Q \vee R)) \supset ((P \supset \neg Q) \supset (P \supset R))$ can be falsified.

We apply a rule to 4. If $P \supset \neg Q$ is true, either P is false or $\neg Q$ is true. So if we replace 4 with 8 on the left branch and with 9 on the right branch, the signed formulas on one of these branches must be a satisfiable set. The left branch cannot be satisfiable because it contains both 6 and 8.

Continuing with the right branch, 9 is replaced with 10 and 2 is replaced with 11 and 12. The branch ending with 11 cannot be satisfied because it contains both 6 and 11. Next 12 is replaced with 13 and 14. The branch ending with 13 is not satisfiable because of 10 and 13, and neither is the branch ending with 14 because of 7 and 14.

If $(P \supset (Q \vee R)) \supset ((P \supset \neg Q) \supset (P \supset R))$ could be falsified, some tableau branch would be satisfiable. None are. The formula cannot be falsified, and hence must be a tautology.

Here is all this, made official.

Definition 3. A tableau branch is *(atomically) closed* if it contains both $T\,X$ and $F\,X$ for some (atomic) X. A tableau is *(atomically) closed* if every branch is (atomically) closed. A classical propositional tableau proof for a formula X is a closed tableau with $F\,X$ at the root, constructed using the rules in Figure 1.3.

Informal readings of the dual tableau and tableau construction process have motivations that are something like mirror images. Then it should come as no surprise that the dual tableau proof in Figure 1.2 and the tableau proof in Figure 1.4 are identical except that T's and F's have been exchanged!

1.2.3 Soundness

Given the mirror image motivations relating tableaus and dual tableaus, one should not be astonished to find that a soundness argument for one system (tableaus) is root downward, while for the other (dual tableaus) it is leaf upward. We begin with dual tableaus.

Associate a formula with each dual tableau as follows. First for signed formulas: with $T\,X$ we associate X, and with $F\,X$ we associate $\neg X$. Next, for dual tableau branches: to each branch we associate the disjunction of the formulas associated with the signed formulas on the branch. And finally, for dual tableaus themselves: to each dual tableau we associate the conjunction of the formulas associated with the branches. This simply formalizes the informal reading of dual tableaus that we have used all along. There is some ambiguity here, however. Disjunction and conjunction are binary operations, while we have talked about disjunctions and conjunctions of an arbitrary number of formulas. But both operations are commutative and associative semantically, so we can simply ignore these finer points.

Example 1. Here is a dual tableau. It is not closed, and not all applicable rules have been applied.

$$
\begin{array}{ccc}
 & T\,(X \wedge Y) \supset (Y \wedge X) \quad 1. & \\
 & F\,X \wedge Y \quad 2. & \\
 & T\,Y \wedge X \quad 3. & \\
 & \swarrow \qquad \searrow & \\
T\,Y \quad 4. & & T\,X \quad 5.
\end{array}
$$

The formula associated with this dual tableau is the following, ignoring details of parenthesizing disjunctions.

$$
\begin{array}{l}
[((X \wedge Y) \supset (Y \wedge X)) \vee \neg(X \wedge Y) \vee (Y \wedge X) \vee Y] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge \\
\qquad\qquad\qquad\qquad\qquad [((X \wedge Y) \supset (Y \wedge X)) \vee \neg(X \wedge Y) \vee (Y \wedge X) \vee X]
\end{array}
$$

Now there is a kind of reverse induction step. We leave it to you to show that if dual tableau $\mathscr{T}_2$ results from the application of a single branch extension rule to dual tableau $\mathscr{T}_1$, then if the formula associated with $\mathscr{T}_2$ is a tautology, so is the formula associated with $\mathscr{T}_1$. An inspection of the rules in Figure 1.1 and a little thought should convince you of this.

Since $X \vee \neg X$ is a tautology, the formula associated with a closed dual tableau branch must be a tautology, and hence the same is true of the formula associated with a closed dual tableau.

If X has a a dual tableau proof, there must be a sequence of tableaus $\mathscr{T}_1, \mathscr{T}_2, \ldots,$ $\mathscr{T}_n$, where $\mathscr{T}_1$ consists of just a root node labeled $T\,X$, each $\mathscr{T}_{k+1}$ is the result of applying a single dual tableau branch extension rule to its predecessor $\mathscr{T}_k$, and with $\mathscr{T}_n$ closed.. The formula associated with $\mathscr{T}_n$ is a tautology, hence so is the formula associaed with $\mathscr{T}_{n-1}$, and so on backward, until we conclude that the formula associated with $\mathscr{T}_1$ is a tautology. But this is just X itself. Thus a formula having a dual tableau proof is a tautology—soundness.

Since the soundness proof for dual tableaus showed tautology-hood was preserved going up the branches of a dual tableau, it should be expected that the proof for tableaus proceeds by showing satisfiability is preserved going down the branches of a tableau.

Recall that for tableaus, branching is a kind of disjunction, while single branches act conjunctively. Then, call a tableau branch *satisfiable* if the set of signed formulas on it is satisfiable, and call a tableau satisfiable if one of its branches is satisfiable. We leave it to the reader to show: if a tableau rule, from Figure 1.3, is applied to a satisfiable tableau, the result is another satisfiable tableau.

Now, suppose X has a tableau proof, but is not a tautology—we derive a contradiction. Since X is not a tautology, some boolean valuation falsifies X, and hence $\{F\,X\}$ is a satisfiable set. This means the tableau proof begins with a satisfiable tableau. Then every subsequent tableau must also be satisfiable. Since X has a tableau proof, a closed tableau can be constructed, starting with $F\,X$, and this must be satisfiable, which is obviously impossible since each branch contains a direct contradiction.

1.2.4 Completeness

A standard way of proving completeness for both tableaus and dual tableaus is to show that from a systematic but failed attempt to construct a proof one can extract enough information to create a counter-model. This not only gives completeness, but provides a decision procedure (in the propositional case). We follow an alternative route here since it is quicker, easier to describe, and generalizes well to non-classical logics. It follows a pattern familiar from axiom system completeness proofs, where one works with maximally consistent sets of formulas. There are some variations on the usual theme, however. First, while an appropriate version of consistency can be introduced for tableaus, not surprisingly for dual tableaus we need a dual version. We call this being *non-tautologous*. And second, when working with axiomatics one shows what is often called a *truth lemma*—belonging to a maximally consistent set is equivalent to being true under some boolean valuation. When working with tableaus or dual tableaus we can only show half of this equivalence, though this is still enough for the purpose.

As is our general pattern here, we begin with dual tableaus. Up to now a dual tableau construction began with a single signed formula, $T\,X$, where X is the formula we are trying to prove. We now allow a dual tableau to start with a finite set S of signed formulas. The members of S are used as labels for a root node, it's only child, the only child of that, and so on. The order does not matter. We refer to a dual tableau beginning in this way as a dual tableau *for S*. Using this terminology, a dual tableau proof of X is a closed dual tableau for $\{T\,X\}$. Note that if we start a dual tableau with a set of signed formulas, some of these may be inactive at the start (Definition 1) because their consequences may already be in the set.

Definition 4. We call a set S of signed formulas, possibly infinite, *tautologous* if there is a closed dual tableau for some finite subset of S (Definition 2). Assume that a single-use restriction is imposed, and branch closure must be atomic. We call S *non-tautologous* if it is not tautologous.

If a set is tautologous, trivially so is every extension, so dually if a set is non-tautologous, so is every non-empty subset. The familiar Lindenbaum construction shows that every axiomatically consistent set of formulas extends to a maximal such set. This carries over to dual tableaus quite directly.

Theorem 1 (after Lindenbaum). *If S is a non-tautologous set of signed formulas, S extends to a maximal non-tautologous set. That is, there is an extension of S that is non-tautologous with no proper non-tautologous extension.*

Proof. Standard Sketch.
Suppose S is non-tautologous. Enumerate all signed formulas: Z_1, Z_2, ... Define a sequence of sets as follows. $S_0 = S$. Then set $S_{n+1} = S_n \cup \{Z_n\}$ if that is non-tautologous, and otherwise $S_{n+1} = S_n$. $S_\infty = S_0 \cup S_1 \cup S_2 \cup \ldots$. It is straightforward to show S_∞ is maximally non-tautologous.

Now assume M is a maximally non-tautologous set of signed formulas. Let's look at the dual tableau rules for $\wedge$ from Figure 1.1 and see what they tell us about M. We claim the following.

1. If $F\,X \wedge Y \in M$ then both $F\,X \in M$ and $F\,Y \in M$.
2. If $T\,X \wedge Y \in M$ then (at least) one of $T\,X \in M$ or $T\,Y \in M$.

We show item 1; item 2 is handled similarly. Suppose $F\,X \wedge Y \in M$ but $F\,X \notin M$ or $F\,Y \notin M$. We derive a contradiction, namely that M is tautologous.

If $F\,X \notin M$, since M is maximal then $M \cup \{F\,X\}$ is tautologous, and hence so is $M \cup \{F\,X, F\,Y\}$. Similarly if $F\,Y \notin M$. So by our assumptions, $M \cup \{F\,X, F\,Y\}$ is tautologous, and hence there is a finite subset M_0 of M such that there is a closed dual tableau $\mathscr{T}$ for $M_0 \cup \{F\,X, F\,Y\}$. It may be that $F\,X \wedge Y \in M_0$, but if not we can add it since $M_0 \cup \{F\,X \wedge Y\}$ will still be a subset of M, and we will still have a closed tableau for $M_0 \cup \{F\,X \wedge Y\} \cup \{F\,X, F\,Y\}$. From now on, we assume $F\,X \wedge Y \in M_0$. Note that with $F\,X \wedge Y \in M_0$, in constructing a closed tableau for $M_0 \cup \{F\,X, F\,Y\}$, $F\,X \wedge Y$ is inactive at the start.

Now we construct a dual tableau just for M_0 itself, as follows. Begin with the members of M_0. Note that $F\,X$ and $F\,Y$ are not in M_0 since they are not in M, and hence $F\,X \wedge Y$ is active at this point. As the first rule application in the new dual tableau, use $F\,X \wedge Y$ to add $F\,X$ and $F\,Y$ to the branch. At this point, $F\,X \wedge Y$ becomes inactive, so it is no longer subject to any rule application. And since we require atomic closure, $F\,X \wedge Y$ cannot contribute to branch closure. In effect, it can play no further role in our dual tableau construction. It is as if we have M_0 without $F\,X \wedge Y$, but with $F\,X$ and $F\,Y$ to work with. Now continue the construction by doing exactly what was done in the dual tableau $\mathscr{T}$ for $M_0 \cup \{F\,X, F\,Y\}$. Of course this produces a closed dual tableau, thus showing that M_0, and hence M is tautologous, our contradiction.

There are similar conditions for $\vee$, $\neg$, and $\supset$ whose statement and proof we leave to the reader.

Now completeness is easy. First, an informal argument. Suppose Z does not have a dual tableau proof—there is no closed tableau for $T\,Z$. Then $\{T\,Z\}$ is a non-tautologous set and so can be extended to a maximally non-tautologous set M by Theorem 1. In effect, M tells us what would be needed for Z to be true, so if we do the opposite of what M says, we will have a way in which Z would be false, and we can do this because M does not close off all possibilities—it is non-tautologous after all.

To turn this into a formal argument we begin with the following.

Theorem 2 (Truth Lemma). *Let M be a maximally non-tautologous set. Let v be the boolean valuation such that, for each propositional letter P, $v(P)$ is true just when $F\,P \in M$. Then for each formula X:*

- *If $T\,X \in M$ then X is false under v.*
- *If $F\,X \in M$ then X is true under v.*

Proof. The verification of this involves lots of cases. Here are two of them. We leave the rest to the reader.

First a base case for the induction. Suppose P is atomic and $T\,P \in M$. Then $F\,P \notin M$ since M is non-tautologous. By definition, v assigns false to P.

Next, one of the induction steps. Suppose $F\,X \wedge Y \in M$ and the result is known for formulas of lower degree. As we showed earlier, since M is *maximally* non-tautologous, $F\,X \in M$ and $F\,Y \in M$. By the induction hypothesis, X and Y are both true under v, hence so is $X \wedge Y$.

We now have completeness as follows. Suppose that formula Z has no dual tableau proof. Then $\{T\,Z\}$ is non-tautologous; extend to a maximal non-tautologous set M by Theorem 1. Create a boolean valuation v by setting each propositional letter P to be true under v exactly when $F\,P$ is in M. Then we appeal to Theorem 2. $T\,Z \in M$ so Z is false under v, and hence Z is not a tautology. Equivalently, any tautology must have a dual tableau proof.

We have shown completeness for dual tableaus. A completeness argument for tableaus along these lines is well-known, and we just sketch it. Call a set S of signed formulas *consistent* if no tableau for any finite subset of S closes. Every consistent set can be extended to a maximally consistent set, along the lines of Theorem 1. Just as maximally non-tautologous sets respect the dual tableau rules, maximally consistent sets respect the tableau rules. For instance, if M is maximally consistent, and $T\,X \wedge Y \in M$, then both $T\,X \in M$ and $T\,Y \in M$. Any maximally consistent set M can be used to create a boolean valuation by doing what M says at the atomic level, rather than doing the opposite as we did with maximally non-tautologous sets. Such a boolean valuation will satisfy the entire of M. From this, completeness follows immediately. We leave the details to you (of course).

Notice that the completeness arguments for dual tableaus and for tableaus are essentially the same thing, but with one being the mirror image of the other. The difference is conceptual. With tableaus we begin with $F\,X$ and we search for a way X could be false. A failed proof gives us such a way. With dual tableaus, beginning with $T\,X$, we are searching for what we need to guarantee the truth of X, and a failed proof convinces us there is no such guarantee.

1.2.5 What Dual Tableaus "Really" Look Like

Tableaus are customarily presented as trees with (signed) formulas as node labels. In order to emphasize relationships between the two systems, we have presented dual tableaus the same way, but this is not how dual tableaus are usually shown. In our version a dual tableau is a kind of dynamic object. At various stages in its construction some signed formulas are active, some are inactive, and this changes as branches grow. In the customary presentation of dual tableaus, *sets* of (signed) formulas appear as node labels. These sets contain just the formulas that are active at the corresponding stage of a dual tableau construction as we have shown it. Thus a dual tableau in its usual formulation is a static object, summing up the dynamic history of a dual tableau construction in our sense.

Figure 1.5 shows a dual tableau in its customary form. Sets appear as node labels, but enclosing curly brackets are commonly omitted, as we have done here. The proofs in Figures 1.2 and 1.5 are the same except for the change in display style. A few moments comparison should make the connections clear.

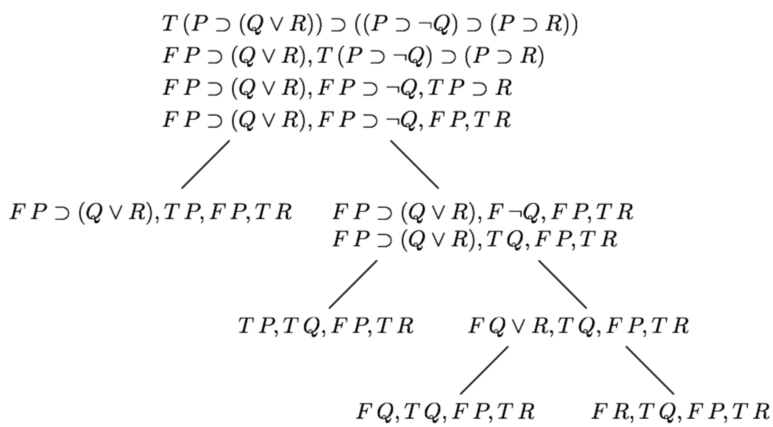

Fig. 1.5 A Dual Tableau As It Usually Appears

We should note that there is a similar presentation for tableaus, using sets of formulas instead of single formulas. These are the *block tableaus* of (Smullyan 1968, Chapter XI §1), where a connection with the work in (Hintikka 1955) is also pointed out. Ultimately, the connection between dual tableaus, tableaus, and sequent calculi is both close and complex. At least a portion of the history can be found in (Anellis 1990).

1.3 Intuitionistic Logic

Intuitionistic tableau systems are well-known. Dual tableaus for propositional intuitionistic logic are in the literature, (Orłowska and Golińska-Pilarek 2011, Chapter 8) for instance, but this dual tableau formulation makes essential use of a relational formulation of logic. Here we strip that away, presenting a simple, basic dual tableau intuitionistic proof system. We try to provide plausible motivation based on the Brouwer, Heyting, Kolmogorov (BHK) interpretation of intutionistic logic, and perhaps this will be enough to convince the reader that the system succeeds, even before seeing soundness and completeness proofs. We have discussed at some length the way in which classical dual tableaus are dual to tableaus. We feel the point has been made, and now we omit the details of a tableau version entirely, leaving a formulation to the reader.

1.3.1 Intuitionistic Dual Tableaus

When working with classical logic, Boolean truth is central. For an intuitionist, *true* is replaced with *proved*. But proved using what machinery? And by who, David Hilbert or the person on the street? Here a certain amount of idealization is appropriate. Assume the prover is a competent constructively oriented mathematician who does not make mistakes, working in some standard area of mathematics. And it is not just what has been proved that matters, but what could be proved if the mathematician worked long enough and cleverly enough. Loosely, there is what has been done, and there is what could be done in the future, or at least in a possible future, since the mathematician might choose to pursue research in any one of a number of directions.

With classical logic we used signed formulas, following the informal idea that $T\,X$ says formula X is true and $F\,X$ says it is false. For intuitionistic logic we can informally understand $T\,X$ to say that X is proved, but $F\,X$ needs more discussion. It could be understood to say that X is *refuted*, or that X is *not proved*. Classically either a formula X is true or it is false so we always have one of $T\,X$ or $F\,X$, and this gave us a simple syntactical reason for closing dual tableau branches. We would like a similar dichotomy intuitionistically—branches close when they contain $T\,X$ and $F\,X$. Understanding $F\,X$ informally as saying that X is refuted will not work. There are many mathematical assertions that are neither provable nor refutable. But understanding $F\,X$ informally as saying that X has not been proved will fill our needs. For each mathematical assertion, at any stage of mathematical work, either it has been proved or it hasn't. We always have one of $T\,X$ or $F\,X$ with F understood as unproved. Refuted more properly corresponds to $\neg X$ being proved, expressed now by $T\,\neg X$. If X is refuted, X is unproved (assuming our mathematics is consistent), so $T\,\neg X$ informally entails $F\,X$, but not conversely. This is a basic difference between intuitionistic and classical machinery—classically $\neg X$ can be used in place of $F\,X$ so that signs can be dropped. They cannot be dropped for intuitionistic logic.

Initially we represented classical dual tableaus as trees of signed formulas. We noted in Section 1.2.5 that this is not how they appear in the literature, and for intuitionistic logic it is more convenient to do things as we did in Section 1.2.5, because in intuitionistic dual tableaus signed formulas both come *and go*. We show intuitionistic dual tableaus as *sets* of signed formulas, a *context*. Think of a context as representing a stage in the researches of our idealized mathematician. It changes as work goes on because hitherto unproved things may become proved. Using sets records context statically, instead of dynamically, the way our original presentation of classical dual tableaus did.

Some notational conventions. If S is a set of signed formulas and Z is a single signed formula, instead of writing $S \cup \{Z\}$ we will simply write S, Z to indicate the result of adding Z to S. As we did in Section 1.2.5, we will omit enclosing curly brackets when displaying sets in intuitionistic dual tableaus.

An intuitionistic dual tableau for X begins with the set consisting of $T\,X$. Unlike in the classical case this does not represent the goal of making X true no matter what, but instead of analyizing what it would take for our idealized mathematican to find a proof for X. At this starting point formal dual tableau appearances are the same as classical, though intuitions differ. We then proceed to expand the initial dual tableau tree, so we must formulate appropriate rules. In fact, there is more than one way that rules for intuitionistic logic can be created. It is common to trace things back to the work of Gentzen. Here we rely on the informal BHK understanding of intuitionistic principles, and these lead us to a tableau or dual tableau version that traces to (Beth 1959), and first appeared as a signed tableau system in (Fitting 1969). At this point the differing versions of intuitionistic dual tableau and tableau formulations are a matter of taste, but for proof-theoretical work the differences can be crucial.

Dual tableau rules represent backward searches. For example, the classical rule for $F\wedge$ in Figure 1.1 tells us that for $X \wedge Y$ to be false it is sufficient that either X or Y be false. (Indeed, it is necessary and sufficient.) Thus the problem of showing falsehood for a classical conjunction can be replaced by the problem of showing falsehood for one of the conjuncts. Intuitionistically, a proof of $X \wedge Y$ consists of a proof of both X and Y. Then $F\,X \wedge Y$, informally that the conjunction is unproved, can be replaced with $F\,X, F\,Y$ read disjunctively, one of X or Y is unproved. Similarly for $T\,X \wedge Y$, where branching is understood conjunctively. This motivates the intuitionistic dual tableau rules for $F\wedge$ and $T\wedge$ shown in Figure 1.6. They look like the classical rules, but do not have the same motivation or intuitive reading.

Following the BHK interpretation, an intuitionistic proof of a disjunction is a proof of one of the disjuncts. This is quite different than in classical logic, where one trivially has a proof of $X \vee \neg X$, but might not have a proof of either X or of $\neg X$. Reading T as intuitionistically proved, we still have that $T\,X \vee Y$ should behave like $T\,X$, $T\,Y$, and this motivates the $T\vee$ rule in Figure 1.6. The rule $F\vee$ is similarly understood.

If we find a proof of X, represented by $T\,X$, we cannot have a proof of $\neg X$ too, thus $F\,\neg X$. Informally, the disjunctive set $S \cup \{F\,\neg X, T\,X\}$ entails $S \cup \{F\,\neg X\}$, and trivially conversely too. Of course the simpler $S \cup \{T\,X\}$ also entails $S \cup \{F\,\neg X\}$, but in general it represents a stronger situation—the two sets are not equivalent. The $F\neg$ rule shown in Figure 1.6, then, simply amounts to replacing a situation by an equivalent. Similar considerations motivate the $F \supset$ rule. So far there are no formal rule differences between classical and intuitionistic.

Now we come to the two key cases, $T \supset$ and $T\neg$. We examine $T \supset$ in detail—the negation case is similar and is not discussed. The BHK understanding of implication is that we have a proof of $X \supset Y$ provided we have an algorithm that can convert any proof of X into a proof of Y. We build on this to informally answer the question: what would be sufficient to ensure we must be in the situation represented by $S, T X \supset Y$? Our strategy for understanding is to reason backward. If $S, T X \supset Y$ were not the case, what would follow? Ruling that consequence out would be sufficient to guarantee we must have $S, T X \supset Y$. First, some useful notation.

Definition 5. Let S be a set of signed formulas.

1. $S^T = \{T X \mid T X \in S\}$
2. $S^F = \{F X \mid F X \in S\}$
3. $S^\circ = \{X \mid T X$ or $F X$ is in $S\}$

Now, suppose $S, T X \supset Y$ were not the case, or equivalently, we do not have $S^T, S^F, T X \supset Y$. Since $S^T, S^F, T X \supset Y$ is understood disjunctively, if it were *not* the case then intuitively: we would not have proofs for any member of $(S^T)^\circ$, we would have proofs for all the members of $(S^F)^\circ$, and we would not have a proof of $X \supset Y$. Using the BHK understanding of implication, since we don't have a proof of $X \supset Y$, we lack a way of converting any discovered proof of X into one for Y. Then we cannot rule out the possibility of a future stage of our mathematical research at which we have found a proof of X but lack a proof of Y. Since we have proofs of the members of $(S^F)^\circ$ now, those proofs remain with us, and so members of $(S^F)^\circ$ will still be provable formulas at any future stage of our mathematical research. But, while at the present we do not have proofs for the members of $(S^T)^\circ$, we must allow for the possibility that future research will find proofs for some of them, and thus there is nothing definite to be said about the status of members of $(S^T)^\circ$ in the future. To summarize, if we do not have $S^T, S^F, T X \supset Y$ now, we must allow for a possible future in which we have a proof of X, but we do not have a proof of Y, all members of $(S^F)^\circ$ have proofs, and nothing is certain about the status of the members of $(S^T)^\circ$. Briefly, there is a possible future in which we can be certain we *do not* have $S^F, F X, T Y$.

Turning this around, we have argued informally that if at some later stage of our mathematical research we have $S^F, F X, T Y$, this suffices to ensure that we presently have $S, T X \supset Y$. This is the informal content of the rule for $T \supset$ in Figure 1.6. Notice that in this rule, context shift is critical. Intuitively we have moved from one stage of mathematical research to a possible future stage. In this shift some information is lost—members of S^T vanish. Very informally, information about what we do *not* know might not be preserved.

A dual tableau branch is *closed* if it has a node $S, T P, F P$. Formally this looks like classical closure, but the informal meaning is different. It represents a situation in which P is either proved or not proved, and this always is the case. As we did classically, we will require *atomic* closure—P must be atomic.

A version of our classical single use restrictions can still be imposed, along with the accompanying notions of active and inactive. Definition 1 carries over directly, except that talk of dual tableau branches is replaced with talk of sets of signed formulas. For instance, item 1 from that definition becomes: $F X \wedge Y$ is inactive in set S if both $F X$ and $F Y$ are present in S. We assume the reader can adjust the other conditions as well.

$$\frac{S, F\,X \wedge Y}{S, F\,X \wedge Y, F\,X, F\,Y} \quad \frac{S, T\,X \vee Y}{S, T\,X \vee Y, T\,X, T\,Y}$$

$$\frac{S, F\,X \vee Y}{S, F\,X \vee Y, F\,X \mid S, F\,X \vee Y, F\,Y} \quad \frac{S, T\,X \wedge Y}{S, T\,X \wedge Y, T\,X \mid S, T\,X \wedge Y, T\,Y}$$

$$\frac{S, F\,X \supset Y}{S, F\,X \supset Y, T\,X \mid S, F\,X \supset Y, F\,Y} \quad \frac{S, F\,\neg X}{S, F\,\neg X, T\,X}$$

$$\frac{S, T\,X \supset Y}{S^F, F\,X, T\,Y} \quad \frac{S, T\,\neg X}{S^F, F\,X}$$

Fig. 1.6 Intuitionistic Propositional Dual Tableau Rules

Figure 1.7 displays an example of a proof following the intuitionistic dual tableau rules, with single-use applications throughout. The example is abbreviated—In order to keep clutter down, *we do not show signed formula occurrences that are inactive*. Numbers have been added to aid discussion. Reasons are as follows: 2 is from 1 by $T \supset$, 3 is from 2 by $F \wedge$, 4 is from 3 by $F \neg$, 5 is from 4 by $T \neg$. Notice that in this last step, $T\,B$ has dropped out. Continuing, 6 and 7 are from 5 by $F \vee$, 8 is from 6 by $F \neg$, and 9 is from 7 also by $F \neg$. Both branches are closed.

$$\begin{array}{c}
T\,(A \wedge \neg B) \supset \neg(\neg A \vee B) \quad 1. \\
F\,A \wedge \neg B,\ T\,\neg(\neg A \vee B) \quad 2. \\
F\,A, F\,\neg B,\ T\,\neg(\neg A \vee B) \quad 3. \\
F\,A,\ F\,\neg B,\ T\,B,\ T\,\neg(\neg A \vee B) \quad 4. \\
F\,A,\ F\,\neg B,\ F\,\neg A \vee B \quad 5. \\
\begin{array}{cc}
\swarrow & \searrow \\
F\,A,\ F\,\neg B,\ F\,\neg A \quad 6. & F\,A,\ F\,\neg B,\ F\,B \quad 7. \\
F\,A,\ F\,\neg B,\ F\,\neg A,\ T\,A \quad 8. & F\,A,\ F\,\neg B,\ T\,B,\ F\,B \quad 9.
\end{array}
\end{array}$$

Fig. 1.7 An Intuitionistic Dual Tableau Example

1.3.2 Soundness

We show soundness relative to standard possible world intuitionistic models, without single-use assumptions. It follows that we also have soundness with single-use assumptions present. Here is the well-known definition of the semantics.

Definition 6 (Intuitionistic Model). $\mathscr{M} = \langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$ is a Kripke propositional intuitionistic model provided:

1. $\mathscr{G}$ is a non-empty set (of states).
2. $\mathscr{R}$ is a reflexive, transitive relation on $\mathscr{G}$.
3. $\Vdash$ is a relation between possible worlds and propositional letters meeting the condition: if $\Gamma \Vdash P$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \Vdash P$.

The truth-at-a-state relation $\Vdash$ in a model $\mathscr{M} = \langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$ extends to all formulas using the following conditions. For each $\Gamma \in \mathscr{G}$:

4. $\Gamma \Vdash X \wedge Y$ if and only if $\Gamma \Vdash X$ and $\Gamma \Vdash Y$,
5. $\Gamma \Vdash X \vee Y$ if and only if $\Gamma \Vdash X$ or $\Gamma \Vdash Y$,
6. $\Gamma \Vdash X \supset Y$ if and only if for every $\Delta \in \mathscr{G}$ with $\Gamma \mathscr{R} \Delta$, $\Delta \nVdash X$ or $\Delta \Vdash Y$,
7. $\Gamma \Vdash \neg X$ if and only if for every $\Delta \in \mathscr{G}$ with $\Gamma \mathscr{R} \Delta$, $\Delta \nVdash X$.

A formula X is valid in a model if it evaluates to true at all states, that is, $\Gamma \Vdash X$ for all $\Gamma \in \mathscr{G}$. X is valid if it is valid in every model.

Our goal is to show that if X has a proof using intuitionistic dual tableaus, then X is valid as just defined. We adapt the soundness argument for classical dual tableaus from Section 1.2.3. There we associated a formula with each classical dual tableau. We can do a similar thing now, but remember that while classically $F\,X$ and $\neg X$ can be identified, this is decidedly not so intuitionistically. Suppose we have an intuitionistic dual tableau branch with the set $\{F\,A, F\,B, T\,C, T\,D\}$ as a node label. Such a set is understood disjunctively, and can be read now as telling us that either C is provable or D is provable or A is not provable or B is not provable. We can reformulate this as: if both A and B are provable, then one of C or D is provable. Given the BHK understanding of intuitionistic implication (and disjunction), this corresponds to $(A \wedge B) \supset (C \vee D)$, and this is what we will use as our formula counterpart. There are still a few open translation cases, but they have standard treatments. We translate $\{F\,A, F\,B\}$ as $(A \wedge B) \supset \bot$ and $\{T\,C, T\,D\}$ as $\top \supset (C \vee D)$, where $\bot$ is absurdity and holds at no possible world of a Kripke intuitionistic model, while $\top$ holds at every world. Then with the usual understanding that the conjunction of the empty set is $\top$ and the disjunction of the empty set is $\bot$, the formula counterpart of a set S of signed formulas is simply $\bigwedge(S^F)^\circ \supset \bigvee(S^T)^\circ$.

Now the central item in showing soundness is to show that, for each dual tableau rule, if the formula counterpart(s) of the set(s) below the line are intuionistically valid, this is also the case for the formula counterpart of the set above the line. There are a number of cases with the ones for $\wedge$ and $\vee$ quite straightforward. We discuss the two implication cases in more detail—the negation cases are similar. One of the implication cases has peculiarities that distinguish it from its classical counterpart, and the reasons are centered in soundness issues.

First consider the $F \supset$ case. What must be shown is that intuitionistic validity of $[\bigwedge(S^F)^\circ \wedge (X \supset Y)] \supset [\bigvee(S^T)^\circ \vee X]$ and of $[\bigwedge(S^F)^\circ \wedge (X \supset Y) \wedge Y] \supset \bigwedge(S^T)^\circ]$ entail that of $[\bigwedge(S^F)^\circ \wedge (X \supset Y)] \supset \bigvee(S^T)^\circ]$. This is the case, though verification takes some work and we omit it.

Next consider first the $T \supset$ case. It must be shown that validity of the formula counterpart of $S^F, F\,X, T\,Y$ entails validity of the counterpart of $S, T\,X \supset Y$, that is, intuitionistic validity of $[\bigwedge(S^F)^\circ \wedge X] \supset Y$ entails that of $\bigwedge(S^F)^\circ \supset [\bigvee(S^T)^\circ \vee (X \supset$

$Y)]$. This is so, and has a rather simple proof that we omit. On the other hand, the classical form of the rule would have an occurrence of S below the line, not S^F. For this one would need to show that intuitionistic validity of $[\bigwedge(S^F)^\circ \wedge X] \supset [\bigvee(S^T)^\circ \vee Y]$ entails that of $\bigwedge(S^F)^\circ \supset [\bigvee(S^T)^\circ \vee (X \supset Y)]$, and this is not the case. The restriction to F-signed formulas in the rule is essential.

The final node on a closed branch of a dual intuitionistic tableau will be of the form $S, T\,P, F\,P$. The formula counterpart of this is $[\bigwedge(S^F)^\circ \wedge P] \supset [\bigvee(S^T)^\circ \vee P]$, and this is obviously intuitionistically valid. By the results sketched above, having an intuitionistically valid formula counterpart is a property that is propagated up branches. It follows that the formula counterpart of the top dual tableau node is intuitionistically valid. A dual tableau proof of X begins with the set containing only $T\,X$, and this has a formula counterpart $\top \supset X$, which must be valid, and hence also X is valid. Summarizing, if X has an intuitionistic dual tableau proof, X is intuitionistically valid.

As an illustrative example, suppose we take the dual tableau shown in Figure 1.7, turn it over, and replace each displayed set by its formula counterpart. Properly speaking, inactive signed formulas should also be taken into account—they were omitted in the dual tableau display. We still omit them since it makes no essential difference, but aids in reading. The result is shown in Figure 1.8. Now work from top to bottom in Figure 1.8. It is easy to check that 8 and 9 are valid. It is also easy to check that 6 is valid, using the fact that 8 is, that 7 is valid because 9 is, that 5 is valid because both 6 and 7 are, and so on. Finally, 1 is valid, which trivially implies that $(A \wedge \neg B) \supset \neg(\neg A \vee B)$ also is.

$$
\begin{array}{cc}
(A \wedge \neg B \wedge \neg A) \supset A \quad 8. & (A \wedge \neg B \wedge B) \supset B \quad 9. \\
(A \wedge \neg B \wedge \neg A) \supset \bot \quad 6. & (A \wedge \neg B \wedge B) \supset \bot \quad 7.
\end{array}
$$

$$
\begin{array}{c}
(A \wedge \neg B \wedge (\neg A \vee B)) \supset \bot \quad 5. \\
(A \wedge \neg B) \supset (B \vee \neg(\neg A \vee B)) \quad 4. \\
(A \wedge \neg B) \supset \neg(\neg A \vee B) \quad 3. \\
(A \wedge \neg B) \supset \neg(\neg A \vee B) \quad 2. \\
\top \supset ((A \wedge \neg B) \supset \neg(\neg A \vee B)) \quad 1.
\end{array}
$$

Fig. 1.8 An Intuitionistic Forward Proof Outline

We note that with a little more work we can replace the use of validity in the example above by provability in any standard intuitionistic axiom system. More generally, an axiomatic intuitionistic proof can be extracted from any intuitionistic dual tableau proof, The motivation for dual tableaus, that they amount to a search procedure for a proof, is thus justified in the intuitionistic case.

1.3.3 Completeness

We show completeness of the Intuitionistic Propositional Dual Tableau Rules from Figure 1.6, with a single-use assumptions imposed. It follows that we also have completeness without them. Some of the machinery of the classical completeness proof from Section 1.2.4 can be carried over, some of the machinery is new. Definition 4 identified something we called *tautologous*. Of course this would be bad terminology now, so we replace it by *I-tautologous*, where the *I* stands for 'intuitionistic'.

Definition 7. We call a set S of signed formulas, possibly infinite, *I-tautologous* if there is a closed intuitionistic dual tableau for some finite subset of S, where branch closure must be atomic. We call S *non-I-tautologous* if it is not I-tautologous.

Much of what was said in Section 1.2.4 about non-tautologous sets for classical dual tableaus carries over to non-I-tautologous sets for intuitionistic dual tableaus, with essentially no changes in proofs. The primary item is that Theorem 1, Lindenbaum's Lemma, continues to apply, so a non-I-tautologous set extends to a maximal one. We also showed the following, where M is a maximally non-tautologous set.

1. If $F\,X \wedge Y \in M$ then both $F\,X \in M$ and $F\,Y \in M$.
2. If $T\,X \wedge Y \in M$ then one of $T\,X \in M$ or $T\,Y \in M$.

These continue to hold if M is a maximally non-I-tautologous set, and with no essential change in argument. Classically it was noted that similar results held involving other connectives. For the intuitionistic dual tableau system this is so for $\vee$, but for $\neg$ and $\supset$ two of the cases are missing. Briefly, those cases where the intuitionistic rules have the same form as the classical rules give us the same conditions on maximality both classically and intuitionistically. Thus we have items 1–6 for M being a maximal non-I-tautologous set.

3. If $F\,X \vee Y \in M$ then one of $F\,X \in M$ or $F\,Y \in M$.
4. If $T\,X \vee Y \in M$ then both $T\,X \in M$ and $T\,Y \in M$.
5. If $F\,\neg X \in M$ then $T\,X \in M$.
6. If $F\,X \supset Y \in M$ then one of $T\,X \in M$ or $F\,Y \in M$.

Now we construct an intuitionistic *canonical model* $\mathscr{M} = \langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$, as follows. $\mathscr{G}$ is the collection of all maximally non-I-tautologous sets. For $\Gamma, \Delta \in \mathscr{G}$, set $\Gamma \mathscr{R} \Delta$ if $\Gamma^F \subseteq \Delta$. And for each atomic formula P, set $\Gamma \Vdash P$ if $F\,P \in \Gamma$. This determines $\mathscr{M}$, and a version of the Truth Lemma, Theorem 2, can be shown.

Theorem 3 (Intuitionistic Truth Lemma). *In canonical model $\mathscr{M} = \langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$, for each $\Gamma \in \mathscr{G}$ and for each formula X:*

- *If $T\,X \in \Gamma$ then $\Gamma \not\Vdash X$.*
- *If $F\,X \in \Gamma$ then $\Gamma \Vdash X$.*

Most of the cases are direct analogs of classical ones. We only consider the implication cases in detail. Let $\mathscr{M} = \langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$ be the canonical model and $\Gamma \in \mathscr{G}$.

Assume the Theorem holds for formulas simpler than $X \supset Y$. We have two cases to examine. Let Γ be an arbitrary member of $\mathscr{G}$.

Assume $F\,X \supset Y \in \Gamma$.

Let Δ be an arbitrary member of $\mathscr{G}$ with $\Gamma \mathscr{R} \Delta$. Then $\Gamma^F \subseteq \Delta$, so $F\,X \supset Y \in \Delta$. By item 6 above, we have one of $T\,X \in \Delta$ or $F\,Y \in \Delta$. By the induction hypothesis, we have one of $\Delta \not\Vdash X$ or $\Delta \Vdash Y$. Since Δ was arbitrary, $\Gamma \Vdash X \supset Y$.

Now suppose $T\,X \supset Y \in \Gamma$.

We first show that $\Gamma^F, F\,X, T\,Y$ is non-I-tautologous. Well, suppose not. Then there is a finite subset of it having a closed intuitionistic dual tableau, and without loss of generality we can assume it contains $F\,X$ and $T\,Y$, so this subset has the form $M_0, F\,X, T\,Y$ where $M_0 \subseteq \Gamma^F$. But then there is also a closed intuitionistic dual tableau for $M_0, T\,X \supset Y$, because we can start with $M_0, T\,X \supset Y$, apply the $T \supset$ rule getting $M_0, T\,X \supset Y, F\,X, T\,Y$ (leaving $T\,X \supset Y$ inactive), and then continue with steps copied from the closed dual tableau for $M_0, F\,X, T\,Y$. Since $M_0 \subseteq \Gamma$ and $T\,X \supset Y \in \Gamma$, it follows that Γ is I-tautologous, which is false since $\Gamma \in \mathscr{G}$.

We have established that $\Gamma^F, F\,X, T\,Y$ is non-I-tautologous. Then it extends to a maximal such set, Δ. Then $\Delta \in \mathscr{G}$, and $\Gamma \mathscr{R} \Delta$ since $\Gamma^F \subseteq \Delta$. Since $F\,X, T\,Y \in \Delta$, by the induction hypothesis $\Delta \Vdash X$ and $\Delta \not\Vdash Y$. It follows that $\Gamma \not\Vdash X \supset Y$.

Now completeness follows in the familiar way. If X has no intuitionistic dual tableau proof, there is no closed dual tableau for $T\,X$ so the set $\{T\,X\}$ is non-I-tautologous. This set extends to a maximal such set, Γ, which will be a possible world in the canonical model, and at it X will fail.

1.3.4 Intuitionistic Tableaus

An intuitionistic tableau proof system is now easy to formulate. Begin with the dual tableau rules from Section 1.3.1 but reverse the roles of the signs T and F. The resulting tableau system has been discussed in the literature, back as far as (Fitting 1969), again in (Fitting 1983), and in a somewhat different form in (Waaler and Wallen 1999). We leave it to the reader to carry out the details. It is a good way of ensuring understanding.

1.3.5 Logical Consequence

We have only talked about provability, classical and intuitionistic. The machinery makes it simple to bring consequence, or deduction from premises, into the picture, and here we briefly sketch how, concentrating on intuitionistic dual tableaus.

Suppose M is a set of formulas (not signed formulas) and X is a single formula. We proceed informally for now; think of $M \vdash_I X$ as meaning we can construct a proof of X provided we are supplied with proofs of the members of M. Using dual tableaus, we should be able to produce a closed tableau for TX, somehow bringing members of M into the tableau. The *Premise Rule* that does this is given in Figure 1.9.

$$\frac{S}{S, FY}$$
where Y is any member of M

Fig. 1.9 Intuitionistic Propositional Dual Tableau Premise Rule

Motivating this dual tableau rule Premise Rule is really quite simple. We follow the ideas of Section 1.3.1, using the BHK ideas informally. Recall that for intuitionistic dual tableaus TX represents that X must be proved (or in the present setting, that it must be derived from a set M), and FX represents that X should not have been derived. We start a tableau with TX with the idea that we want to establish we have sufficient conditions for X to have a derivation from M. In a derivation from a set M, the informal idea is that we will be supplied with proofs of members of M, from the outside so to speak. Then if $Y \in M$, Y has a proof, so FY informally is simply false. Since sets are understood disjunctively in dual tableaus, S and S, FY represent equivalent problems, so the Premise Rule simply replaces one task by another that is equivalent to it.

Figure 1.10 shows a small example of a dual tableau using the rule of Figure 1.9. It shows that $\neg\neg X \vdash_I \neg\neg(X \vee Y)$. In it, 2 is from 1 by $T\neg$, 3 is by the Premise Rule, 4 is from 3 by $F\neg$, 5 is from 4 by $T\neg$, 6 is from 5 by $F\neg$, 7 is from 6 by $T\vee$. The dual tableau is now closed.

$$\begin{array}{c}
T\neg\neg(X \vee Y) \quad 1. \\
F\neg(X \vee Y) \quad 2. \\
F\neg(X \vee Y), F\neg\neg X \quad 3. \\
F\neg(X \vee Y), F\neg\neg X, T\neg X \quad 4. \\
F\neg(X \vee Y), F\neg\neg X, FX \quad 5. \\
F\neg(X \vee Y), F\neg\neg X, FX, TX \vee Y \quad 6. \\
F\neg(X \vee Y), F\neg\neg X, FX, TX \vee Y, TX, TY \quad 7.
\end{array}$$

Fig. 1.10 Dual Tableau Deduction Example

All this so far has been informal and intuitive. Formally, $M \vdash_I X$ is defined to mean that in any intuitionistic model $\mathscr{M} = \langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$ and for any $\Gamma \in \mathscr{G}$, if $\Gamma \Vdash M$ then $\Gamma \Vdash X$, where $\Gamma \Vdash M$ means $\Gamma \Vdash Y$ for every $Y \in M$. Soundness and completeness results can be proved by an easy adaptation of the work in Sections 1.3.2 and 1.3.3; we leave this to the reader. Not surprisingly, the Premise Rule adapts to intuitionistic tableaus by switching signs: one can add TY to the end of any intuitionistic tableau branch for any premise Y. And all the machinery carries over directly to classical logic as well.

1.4 Conclusion

We have examined tableaus and dual tableaus for classical propositional logic in detail. For intuitionistic propositional logic we looked at dual tableaus, but largely skipped over tableaus. We could do this because the general pattern should have become clear. Tableau systems and dual tableau systems are, well, dual. There are many kinds of tableau systems in the literature for modal logics. Some are similar to the one we dualized for intuitionistic logic—formulas disappear from branches as well as appear. These are called *destructive* systems. Other modal tableau systems bring in extra machinery such as prefixes. All these have dual tableau counterparts, whose formulation should not be difficult. Similarly the labeled sequent calculus has dual tableau relatives. Indeed, one of the referees of the present paper suggested there should be some similarity between the labeled dual tableaus of (Orłowska and Golińska-Pilarek 2011) and the labeled sequent calculi found in (Negri 2005). A pertinent reference for comparing a variety of styles of labeled sequent calculi is (Indrzejczak 2010). But fundamentally, essentially all the variety of tableau mechanisms can be adapted to dual tableau formulations. Quantification too presents no difficulties. In (Orłowska and Golińska-Pilarek 2011) dual tableaus are employed for a relational formulation of many logics, and they play a central role. We have stopped very much short of an exhaustive examination, presenting basic ideas only.

It should be clear that tableaus and dual tableaus, and sequent calculi too, can all be seen as strongly equivalent. Systems of one kind can be reworked to become systems of another. The fundamental point is not the logic, but the psychologic. Tableaus are refutation systems—if we did not have what we wanted, it would eventually lead to a clear contradiction. Dual tableaus are searches for positive results—we want this, what would get it for us, what would get us that, and eventually we reach the obvious. Just as some logicians are more comfortable with sequents and others with tableaus, the same is the case with tableaus and dual tableaus. The psychology of mathematical proof discovery and proof presentation is important, and yet remains somehow a very individual thing.

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