De Re, De Dicto, and Binding Modalities*

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Abstract

In classical logic the move from propositional to quantificational is profound but essentially takes one route, following a direction we are all familiar with. In modal logic, such a move shoots off in many directions at once. One can quantify over things or over intensions. Quantifier domains can be the same from possible world to possible world, shrink or grow as one moves from a possible world to an accessible one, or follow no pattern whatsoever. A long time ago, Kripke showed us how shrinking or growing domains related to validity of the Barcan and the converse Barcan formulas, bringing some semantic order into the situation. But when it comes to proof theory things get somewhat strange. Nested sequents for shrinking or growing domains, or for constant domains or completely varying domains, are relatively straightforward. But axiomatically some oddities are quickly apparent. A simple combination of propositional modal axioms and rules with standard quantificational axioms and rules proves the converse Barcan formula, making it impossible to investigate its absence. Kripke showed how one could avoid this, at the cost of using a less common axiomatization of the quantifiers. But things can be complicated and even here an error crept into Kripke's work that wasn't pointed out until 20 years later, by Fine.

Justification logic was created by Artemov, with a system called LP which is related to propositional $\mathsf{S4}$. This was extended to a quantified version by Artemov and Yavorskaya, for which a possible world semantics was supplied by Fitting. Subsequently Artemov and Yavorskaya introduced what they called *binding modalities*, by transferring ideas back from quantified LP to $\mathsf{S4}$. In this paper we continue the investigation of binding modalities, but for K, and show that they provide a natural intuition for Kripke's non-standard axiomatization, and relate directly to the distinction between *de re* and *de dicto*. Unkike in Kripke's treatment, the heavy lifting is done through a generalization of the modal operator, instead of a restriction on quantifier axiomatizations.

1 Introduction

In his famous 1963 paper, [13], Saul Kripke analyzed quantificational modal logic. His paper introduced the now familiar connection between the Barcan formula and monotonicity of domains, and the converse Barcan formula and anti-monotonicity of domains. While the analysis was primarily semantic, it was important to say something about connections with axiomatic approaches

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to quantified modal logics, which were already well-established in the literature. But here a problem arose. Prior had shown in [15] that the Barcan formula was provable in first-order S5, and the converse Barcan formula was provable in first-order versions of modal logics more generally, meaning that Kripke's semantics apparently characterized some logical systems not corresponding to axiomatizations. Kripke traced the source of the problem to the inoccuous looking universal instantiation axiom, $\forall x\varphi(x) \supset \varphi(y)$, and its interplay with the usual rule of necessitation in modal logics. He proposed that using a less common way of axiomatizing classical quantificational logic would solve the problem—work with a system in which only closed formulas can appear in proofs, and hence only closed formulas are provable. As he noted, such a system had already appeared in Quine's well-known [16]. With a correction added later by Kit Fine, [9], Kripke's proposal did work successfully for modal varying domain logics, as well as for first-order S5.

Remarkably, Kripke's solution was re-discovered independently many years later, albeit in disguise. This time the source was justification logic, which is a kind of explicit counterpart of modal logic. It has had a long and substantial development—see [1, 3] for details but, except for this introduction, they will not actually be needed in this paper. The idea behind justification logics is that the simple assertion that some formula is necessary, $\Box\varphi$, is replaced by an *explicit* version, $t{:}\varphi$, informally asserting that φ is necessary for reason t, or that t is a justification for the necessity of φ. Most modal logics are now known to have justification logic counterparts, and there is a precise characterization of what it means to be a counterpart. The use of a justification logic counterpart for S4, known as LP for *logic of proofs*, allowed Artemov to complete Gödel's program of finding an arithmetical semantics for propositional intuitionistic logic, [2].

All justification logics were propositional until 2011, [4], when Sergei Artemov and Tatiana Yavorskaya created a quantified version of LP, allowing them to extend the work on arithmetic semantics for intuitionistic logic. A possible world semantics for their system was introduced in [10]. A central problem that Artemov and Yavorskaya solved was how to formulate a quantified system so that what is called *internalization* would be provable, where internalization is a justification logic analog of the necessitation rule for modal logics. The key lay in discovering how to handle internalization for applications of the universal generalization rule. Following a proof theoretical motivation, they separated two different roles that free variables play in first-order proofs. On the one hand they serve as place holders, so that a proof can be seen as a kind of proof template from which one can manufacture proof after proof by substituting different terms for instances of a free variable throughout the template. The other role played by a free variable is that of a formal symbol subject to binding using the universal generalization rule. The two roles are, in fact, incompatible. Artemov and Yavorskaya introduced syntactic machinery to distinguish the two roles.

It was clear that the justification logic machinery Artemov and Yavorskaya introduced could be modified to apply to modal logic itself, and this was done in [6], where the machinery acquired the name *binding modalities*. They provided a binding modality version of quantified S4, including monotonicity. They worked with S4 because this logic is the modal counterpart of the first-order LP justification logic they had created earlier. As it happens, S4 has a very well behaved binding modalities version, so the connection with Kripke's ideas was missed. Here we will see that binding modalities for K needs ideas that exactly mirror those of Kripke. But for both Kripke and Artemov/Yavorskaya the approach taken was was basically a proof theoretic one. We will see that when approached semantically, binding modalities have a very natural meaning in their own right. As was, in fact, noted in [6] binding modalities provides an intuitively plausible way of capturing aspects of the familiar *de re*/*de dicto* distinction. In this paper the collection of ideas and machinery just discussed will be examined, with *de re* and *de dicto* as the pegs on which everything hangs. We work specifically with varying domain K. The choice of K itself plays no special role other than being the weakest normal modal logic and hence at the base of all the others. We discuss briefly how other domain conditions can be brought in. We also provide an axiomatization and prove soundness and completeness for varying domain K.

2 Binding Modalities—*De Re/De Dicto*

The familiar *de re/de dicto* distinction has been a long standing bone of contention in intentional logic. It is a problem with a four valued solution space: accept the distinction, reject it, or reduce one notion to the other, which could be done in either of two directions. We present one way of embracing the distinction, hopefully with minimum disruption but with some nice technical benefits. We begin with a discussion of one of the problems orbiting around *de re/de dicto*.

We use Kripke modal models somewhat informally for now, with formal details later on. The models are first-order, and we will think of them as having varying domains. No special assumptions are made about accessibility, so our models are appropriate for K. We write $\mathcal{M}, \Gamma \Vdash \varphi$ to mean formula φ is true at possible world Γ of modal model $\mathcal{M}$. Our problem comes from the presence of free variables in φ; how should we understand $\mathcal{M}, \Gamma \Vdash \varphi(x)$? Of course the usual solution is to assume we have a valuation in mind, so free variables get assigned domain members, and the problem is avoided. But still, the question makes sense as asked. For instance, one commonly understands the assertion that $\varphi(x)$ is true in a *classical* model as meaning it is true with respect to every assignment of values to x from the domain of the model. Very loosely, this corresponds to the rule of universal generalization. Modally we could do a similar thing, provided there are no modal operators present. But how should we understand something like $\mathcal{M}, \Gamma \Vdash \Box\varphi(x)$? (For simplicity in the discussion let us suppose that in $\Box\varphi(x)$ the initial $\Box$ is the only modal operator present.) The formula is ambiguous, with two plausible meanings.

Meaning 1: $\mathcal{M}, \Gamma \Vdash \Box\varphi(x)$ is to be taken as true provided that for every assignment of a value, say a, from the quantificational domain associated with Γ, and for every accessible possible world Δ, $\mathcal{M}, \Delta \Vdash \varphi(x)$ is true when x is assigned the value a. Briefly, $\mathcal{M}, \Delta \Vdash \varphi(a)$ is true for all accessible Δ and all a in the domain of Γ. This amounts to a *de re* usage. We choose an individual, a, in Γ, and ask how that individual behaves in Δ.

Meaning 2: $\mathcal{M}, \Gamma \Vdash \Box\varphi(x)$ is to be taken as true provided $\mathcal{M}, \Delta \Vdash \varphi(x)$ is true for every accessible world Δ (just as we understand things in the propositional modal case). Since no further modal operators are present, this simply means $\mathcal{M}, \Delta \Vdash \varphi(a)$ is true for every assignment of a value a to x, where a is taken from the domain of Δ. This is a *de dicto* reading since we have reduced things to the truth of a formula at Δ, with no individuals identified across worlds.

The two meanings are not identical, since the domains associated with Γ and with the various accessible worlds Δ need not be the same.

Binding Modalities were introduced in [5, 6]. Here is the idea, very informally. Suppose φ is a formula and X *is a finite set of variables*. Then $\Box_X\varphi$ counts as a formula. To say it is valid is to say it holds universally when free variables in φ that appear in X are understood in a *de re* sense and free variables in φ that are not in X are understood in a *de dicto* sense. Now it is time for the formal details.

3 Binding Modalities—Syntax

We assume a mostly standard first-order modal language. We have relation symbols of all arities, typically written P, Q, ... We do not have function symbols or constants. We have a countable set of individual variables $\mathbf{V} = \{x_1, x_2, x_3, \ldots\}$, and we will generally write x, y, $\ldots$ for arbitrary members of $\mathbf{V}$. An atomic formula is a relation symbol followed by the appropriate number of individual variables separated by commas, or is $\bot$, representing falsehood. Formulas are built up from atomic formulas using the logical connectives, $\wedge$, $\vee$, $\supset$, and the universal quantifier (we take negation and the existential quantifier to be defined). We write quantified formulas as $\forall x\varphi$, where x is an individual variable and φ is a formula. The central new rule of formation is the following.

Modal Formation Rule: For every finite subset X of $\mathbf{V}$ and for every formula φ, the modality $\Box_X\varphi$ is a formula.

An objection might be raised that, since finite sets are involved, $\Box_X\varphi$ is not really syntax. This is actually a minor point. Every finite set of variables can be coded syntactically, say by a list of its members separated by commas. We would also need axioms to say that order change and repetition of members produces equivalent results. This would not be hard to do, but would seriously obscure the central points. We will use finite sets as honorary syntactic objects, in the same spirit that they are commonly used when working with Gentzen sequents.

Notation Convention: For a finite set of free variables X and a single free variable y, we write $\Box_{Xy}\varphi$ as shorthand for $\Box_{X\cup\{y\}}\varphi$, and *it is assumed that y does not occur in X.*

For free and bound variable occurrences we use the standard definition, augmented with the following.

Free 1: All variable occurrences in the finite set X are free occurrences in $\Box_X\varphi$.

Free 2: All free variable occurrences in the formula φ involving variables that are in the set X are free occurrences in $\Box_X\varphi$.

For example, suppose $\varphi(x, z)$ has free occurrences of x and z. Let ψ be the formula $\Box_{\{x,y\}}\varphi(x, z)$. In ψ the occurrences of x and y in the subscript of $\Box$ are free. Also in ψ all free occurrences of x in $\varphi(x, z)$ are free, and no occurrences of z are free. Note that the *set* of variables occurring free in a formula $\Box_X\varphi$ is simply X.

We also need the notion of a variable y being *free for* occurrences of x in a formula φ, which informally means that replacing free x occurrences by occurrences of y won't lead to any accidental variable binding. The definition has its usual clauses, see [14] for instance. In addition we have the following.

Free For: An individual variable y is free for x in $\Box_X\varphi$ provided y is free for x in φ and, either y does not occur free in φ or $y \in X$.

We will write $\varphi(x/y)$ to denote the result of substituting variable y for free occurrences of x in $\varphi(x)$, where it is understood that y is free for x in φ. We often write $\vec{x}, \vec{y}, \vec{z}, \ldots$ for finite sequences of individual variables, where sequence length can be inferred from context, and we write $\varphi(\vec{x}/\vec{y})$ to denote the result of substituting terms of $\vec{y}$ for free occurrences of the corresponding terms of $\vec{x}$ in φ, all with appropriate free-for conditions understood. If no confusion will result, we may write $\varphi(\vec{x}/\vec{y})$ as $\varphi(\vec{y})$ provided the formula $\varphi(\vec{x})$ is understood from context.

4 Binding Modalities—Semantics

First-order modal models are generally familiar, and are what we use even though binding modalities are being considered. We give the basic varying domain semantics definitions, partly to establish notation. In Section 5 we connect conditions involving binding modalities to conditions on the domain function, with monotonicity and anti-monotonicity playing their familiar role.

Definition 4.1 (Skeleton) A skeleton is a structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ where $\mathcal{G}$ is a non-empty set (of possible worlds or states), $\mathcal{R}$ is a binary relation on $\mathcal{G}$ (of accessibility), and $\mathcal{D}$ is a function that maps members of $\mathcal{G}$ to non-empty sets (domains). The *domain of the skeleton* is defined to be $\cup_{w \in \mathcal{G}} \mathcal{D}(w)$.

Definition 4.2 (Model) A (first-order modal) model is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ where $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ is a skeleton (on which the model is *based*) and $\mathcal{I}$ is an *interpretation function*, assigning to each n-ary relation symbol P and to each possible world $\Gamma \in \mathcal{G}$ an n-ary relation $\mathcal{I}(P, \Gamma)$ on the domain of the skeleton. In this context we also refer to the domain of the skeleton as *the domain of the model* and denote it by $\mathcal{D}_{\mathcal{M}}$.

Note that the interpretation function is not restricted, at a world, only to members of the domain of that world. For instance, for a one-place relation symbol P we could have $a \in \mathcal{I}(P, \Gamma)$ while $a \notin \mathcal{D}(\Gamma)$, though we must have $a \in \mathcal{D}_{\mathcal{M}}$, and so $a \in \mathcal{D}(\Delta)$ for some $\Delta \in \mathcal{G}$. Conditions can be placed on accessibility relations but we do not do so now, which means that at the propositional level our modal logic is K. Also conditions can be placed on domain functions but we do not do this either (yet).

It is standard to use *valuations* in working with first-order models. Valuations map free variables to members of a domain, and we commonly talk about the truth value of a formula φ with respect to a valuation v. In this paper we follow an alternate convention, allowing members of the domain of a model to appear directly in formulas as if they were constants; in fact we call them *model constants*. We call formulas containing these model constants *model formulas*. Very strictly speaking, a model formula is not a formula, but any assertion about it can easily be rephrased using proper formulas and valuations. In a model formula, members of the domain of the model can only appear in what would be free variable positions. A formula, as usually understood, is simply a model formula with no members of the domain of a model appearing as constants. Here is all this said more formally.

Definition 4.3 (Model Formula) Let $\mathcal{M}$ be a model. A *model formula of* $\mathcal{M}$, or more simply an $\mathcal{M}$ *formula*, is the result of substituting members of $\mathcal{D}_{\mathcal{M}}$ for some or all of the free variable occurrences in a formula. We refer to members of $\mathcal{D}_{\mathcal{M}}$ as *model constants* in their appearance in a modal formula. If the context makes it clear which model is involved, we may refer to a model formula of $\mathcal{M}$ simply as a model formula.

We extend the notation for substitution, from Section 3. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be a model. We write $\vec{a}$, $\vec{b}$, $\vec{c}$, ... for finite sequences of members of the domain $\mathcal{D}_{\mathcal{M}}$ of the model. The notation $\varphi(\vec{x}/\vec{a})$ denotes the model formula in which members of $\vec{a}$ replace free occurrences of corresponding members of $\vec{x}$ in φ. Since model constants are not variables, we can think of a model constant as being automatically free for any variable in a formula.

It is convenient to write $\vec{a} \in \mathcal{D}_{\mathcal{M}}$ instead of $\vec{a} \in \mathcal{D}_{\mathcal{M}}^n$. Likewise we write $\{\vec{x}\}$ for the set whose members are the terms of $\vec{x}$, and similarly for $\{\vec{a}\}$. If $\varphi(\vec{a})$ contains no free individual variables, we say it is a *closed model formula*. A closed model formula with no members of the model domain appearing as constants is simply a closed formula in the usual sense.

Definition 4.4 (Truth in a Model) Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be a model, and let φ be an $\mathcal{M}$ formula. We write $\mathcal{M}, \Gamma \Vdash \varphi$ to indicate that φ is true at possible world Γ of model $\mathcal{M}$. The conditions defining this are as follows.

(0) Let $\vec{x}$ be all the free variables of $\varphi(\vec{x})$; $\mathcal{M}, \Gamma \Vdash \varphi(\vec{x})$ if and only if $\mathcal{M}, \Gamma \Vdash \varphi(\vec{a})$ for every $\vec{a} \in \mathcal{D}(\Gamma)$. (This reduces truth for formulas with free variables to truth for closed model formulas. These, in turn, are covered by the following.)

(1) Let $P\vec{a}$ be an atomic closed model formula, where $\vec{a} \in \mathcal{D}_{\mathcal{M}}$. $\mathcal{M}, \Gamma \Vdash P\vec{a}$ if and only if $\vec{a} \in \mathcal{I}(P, \Gamma)$

(2) $\mathcal{M}, \Gamma \nVdash \bot$.

(3) Let $\varphi \supset \psi$ be a closed model formula; $\mathcal{M}, \Gamma \Vdash \varphi \supset \psi$ if and only if $\mathcal{M}, \Gamma \nVdash \varphi$ or $\mathcal{M}, \Gamma \Vdash \psi$ (and similarly for the other propositional connectives).

(4) Let $\forall x \varphi(x)$ be a closed model formula; $\mathcal{M}, \Gamma \Vdash \forall x \varphi(x)$ if and only if $\mathcal{M}, \Gamma \Vdash \varphi(a)$ for each $a \in \mathcal{D}(\Gamma)$.

(5) Let $\Box_{\{\vec{a}\}}\varphi(\vec{x}, \vec{a})$ be a closed model formula; $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}}\varphi(\vec{x}, \vec{a})$ if and only if $\mathcal{M}, \Delta \Vdash \varphi(\vec{x}, \vec{a})$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$. (Note that this reduces things back to case 0.)

Definition 4.5 (Validity) For a model $\mathcal{M}$ and an $\mathcal{M}$ formula φ, we say φ is *valid in* $\mathcal{M}$ if it is true at every possible world Γ of $\mathcal{M}$. φ is simply *valid* if it is valid in every model $\mathcal{M}$ such that φ is an $\mathcal{M}$ formula.

Example 4.6 Suppose $\varphi(\vec{a}, \vec{x})$ is an $\mathcal{M}$ formula, where $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$. Assume $\vec{a}$ are all the model constants, and $\vec{x}$ are all the free variables in the formula. Invoking case (0) from Definition 4.4, to say $\varphi(\vec{a}, \vec{x})$ is valid in $\mathcal{M}$ is to say that, at each world $\Gamma \in \mathcal{G}$, and for each $\vec{b} \in \mathcal{D}(\Gamma)$, $\mathcal{M}, \Gamma \Vdash \varphi(\vec{a}, \vec{b})$.

Example 4.7 Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be a model where $\mathcal{G} = \{\Gamma, \Delta\}$, under $\mathcal{R}$ each possible world is accessible from the other, $\mathcal{D}(\Gamma) = \{a\}$, $\mathcal{D}(\Delta) = \{b\}$, $\mathcal{I}(P, \Gamma) = \{a\}$, and $\mathcal{I}(P, \Delta) = \{b\}$ (where $a \neq b$). Schematically we have what is shown in Figure 1.

In this model, $\mathcal{M}, \Gamma \Vdash Px$ because every instance from the domain of Γ is true at Γ, namely Pa. Similarly at Δ, so Px is valid in this model. Further, $\mathcal{M}, \Gamma \Vdash \Box_{\emptyset} Px$ because Px is true at all worlds accessible from Γ, that is, at Δ. On the other hand, if we had $\mathcal{M}, \Gamma \Vdash \Box_{\{x\}} Px$, by Definition 4.4 case (0) we should have every Γ instance of Px true at Δ, and hence $\mathcal{M}, \Delta \Vdash Pa$, and we do not have this. Thus $\mathcal{M}, \Gamma \nVdash \Box_{\{x\}} Px$, and similarly $\mathcal{M}, \Delta \nVdash \Box_{\{x\}} Px$.

$$\Gamma \boxed{a} \Vdash Pa \quad \longleftrightarrow \quad \Delta \boxed{b} \Vdash Pb$$

Figure 1: A Simple Model

5 *De Re/De Dicto* Conversions

As we have said, the idea is that binding modalities distinguish between individual variables playing a *de dicto* role and those playing a *de re* role. More specifically, in $\Box_{\{\vec{x}\}}\varphi(\vec{x}, \vec{y})$, the variables of $\vec{x}$ are *de re* while those of $\vec{y}$ are *de dicto*. Various people have, from time to time, made a case that *de dicto* usages could be converted into *de re*, or conversely. In this section we look at what direct conversions amount to, in terms of binding modalities.

We get a trivial case out of the way first. In considering issues of *de re* and *de dicto* conversion, we have vacuous occurrences for free. The proof is quite simple, but will serve to set up notation and machinery for subsequent use.

Theorem 5.1 *$\Box_{Xy}\varphi \equiv \Box_X\varphi$ is valid provided y is a free variable that does not occur free in φ.*

Proof Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ and assume $\Box_{Xy}\varphi \equiv \Box_X\varphi$ is an $\mathcal{M}$ formula. To keep notation a bit simpler we assume X contains no model constants. Let $\Gamma \in \mathcal{G}$; we show $\mathcal{M}, \Gamma \Vdash \Box_{Xy}\varphi \equiv \Box_X\varphi$. Let $X = \vec{x}$, and let $\vec{z}$ be the free variables in φ other than those in $\vec{x}$, so $\varphi = \varphi(\vec{x}, \vec{z})$ (since y is not free in φ).

Suppose $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{x}\}y}\varphi(\vec{x}, \vec{z})$. Then for each $\vec{a}, b \in \mathcal{D}(\Gamma)$, $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}b}\varphi(\vec{a}, \vec{z})$, and so for each $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{z})$. Since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}}\varphi(\vec{a}, \vec{z})$, and since $\vec{a}$ was arbitrary, $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{x}\}}\varphi(\vec{x}, \vec{z})$. The converse implication is similar. ■

Now we move on to the serious cases, where vacuous occurrences are not guaranteed..

Definition 5.2 (*De Re/De Dicto* Conversion) This is first at the level of models and then at the level of skeletons.

Model Level: Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be a model. We say $\mathcal{M}$ admits *de re* to *de dicto* conversion provided, for every $\mathcal{M}$ formula φ, for every finite set X, for every variable y (note, not a model constant) and for every $\Gamma \in \mathcal{G}$, if $\mathcal{M}, \Gamma \Vdash \Box_{Xy}\varphi$ then $\mathcal{M}, \Gamma \Vdash \Box_X\varphi$. Under similar circumstances $\mathcal{M}$ admits *de dicto* to *de re* conversion if $\mathcal{M}, \Gamma \Vdash \Box_X\varphi$ implies $\mathcal{M}, \Gamma \Vdash \Box_{Xy}\varphi$.

Skeleton Level: Let $\mathcal{S} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be a skeleton. We say $\mathcal{S}$ admits *de re* to *de dicto* conversion if every model based on $\mathcal{S}$ admits *de re* to *de dicto* conversion. Likewise $\mathcal{S}$ admits *de dicto* to *de re* conversion if every model based on $\mathcal{S}$ admits *de dicto* to *de re* conversion.

In propositional modal logic it is standard to associate frame conditions with the validity of particular modal schemes in all models based on frames meeting those conditions. In the first order setting we have the additional machinery of quantifier domains and, as Kripke showed us, these too can be made use of in similar ways, with skeletons from Definition 4.1, replacing frames.

Definition 5.3 (Monotonicity and Anti-Monotonicity) Let $\mathcal{S} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be a skeleton. $\mathcal{S}$ is *monotonic* if, for every $\Gamma, \Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$. $\mathcal{S}$ is *anti-monotonic* if, for every $\Gamma, \Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$ implies $\mathcal{D}(\Delta) \subseteq \mathcal{D}(\Gamma)$. A model is monotonic or anti-monotonic if the skeleton on which it is based is monotonic or anti-monotonic.

Theorem 5.4 *Let $\mathcal{S}$ be a skeleton.*

De Dicto *to* De Re*: $\mathcal{S}$ admits* de dicto *to* de re *conversion if and only if $\mathcal{S}$ is monotonic,*

De Re *to* De Dicto*: $\mathcal{S}$ admits* de re *to* de dicto *conversion if and only if $\mathcal{S}$ is anti-monotonic.*

Proof For the following, assume $\mathcal{S} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ is a skeleton.

De Dicto to *De Re* $\Leftarrow$ monotonic: Suppose $\mathcal{S}$ is monotonic and $\mathcal{M}$ is built on $\mathcal{S}$. Let $\Gamma \in \mathcal{G}$, and suppose $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}}\varphi(\vec{a}, \vec{x}, y)$; we show $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}y}\varphi(\vec{a}, \vec{x}, y)$. To show this, let b be an arbitrary member of $\mathcal{D}(\Gamma)$; we show $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}b}\varphi(\vec{a}, \vec{x}, b)$. And to show this, let $\Delta \in \mathcal{G}$ be arbitrary with $\Gamma\mathcal{R}\Delta$; we show $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{x}, b)$.

Since $b \in \mathcal{D}(\Gamma)$ we have $b \in \mathcal{D}(\Delta)$ by monotonicity. And since $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}}\varphi(\vec{a}, \vec{x}, y)$ then $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{c}, d)$ for every $\vec{c}, d \in \mathcal{D}(\Delta)$. In particular, $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{c}, b)$ for every $\vec{c} \in \mathcal{D}(\Delta)$, and hence $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{x}, b)$.

De Dicto to *De Re* $\Rightarrow$ monotonic: Suppose $\mathcal{S}$ is not monotonic; we show it does not admit *de dicto* to *de re* conversion. Since $\mathcal{S}$ is not monotonic, there are $\Gamma, \Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ but $\mathcal{D}(\Gamma) \not\subseteq \mathcal{D}(\Delta)$; say $d \in \mathcal{D}(\Gamma)$ but $d \notin \mathcal{D}(\Delta)$. Let P be a unary predicate symbol, and build a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ on $\mathcal{S}$ by setting the interpretation $\mathcal{I}$ so that P is interpreted at every world as being true of exactly the members of the domain of that world, and the interpretation of other predicate symbols doesn't matter. We have $\mathcal{M}, \Gamma \Vdash \Box_{\emptyset} P(y)$ because, for every Ω with $\Gamma\mathcal{R}\Omega$, we have $\mathcal{M}, \Omega \Vdash P(y)$. But we do not have $\mathcal{M}, \Gamma \Vdash \Box_{\{y\}} P(y)$ because $d \in \mathcal{D}(\Gamma)$ but we do not have $\mathcal{M}, \Gamma \Vdash \Box_{\{d\}} P(d)$, and we do not have this because $\Gamma\mathcal{R}\Delta$ but we do not have $\mathcal{M}, \Delta \Vdash P(d)$ since $d \notin \mathcal{D}(\Delta)$.

De Re to *De Dicto* $\Leftarrow$ anti-monotonic: This is very similar to the monotonic case. Suppose $\mathcal{S}$ is anti-monotonic and $\mathcal{M}$ is built on $\mathcal{S}$. Let $\Gamma \in \mathcal{G}$, and suppose $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}y} \varphi(\vec{a}, \vec{x}, y)$; we show $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}} \varphi(\vec{a}, \vec{x}, y)$. To show this, let $\Gamma\mathcal{R}\Delta$, where Δ is arbitrary; we must show that $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{x}, y)$. And to show this, let $d \in \mathcal{D}(\Delta)$ be arbitrary; we must show $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{x}, d)$. Since we have anti-monotonicity, $d \in \mathcal{D}(\Gamma)$, and since we have $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}y} \varphi(\vec{a}, \vec{x}, y)$, we have $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}d} \varphi(\vec{a}, \vec{x}, d)$, and $\mathcal{M}, \Delta \Vdash \varphi(\vec{a}, \vec{x}, d)$ follows.

De Re to *De Dicto* $\Rightarrow$ anti-monotonic: We show the contrapositive, with an argument rather similar to the monotonic case. Suppose $\mathcal{S}$ is not anti-monotonic, and $\Gamma, \Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$, but $\mathcal{D}(\Delta) \not\subseteq \mathcal{D}(\Gamma)$, say $d \in \mathcal{D}(\Delta)$ but $d \notin \mathcal{D}(\Gamma)$. Let P be a unary predicate symbol, and build a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ on $\mathcal{S}$ by setting the interpretation $\mathcal{I}$ so that P is interpreted at every world as being true of exactly the members of the members of $\mathcal{D}(\Gamma)$.

We have $\mathcal{M}, \Gamma \Vdash \Box_{\{y\}} P(y)$ because we have $\mathcal{M}, \Gamma \Vdash \Box_{\{e\}} P(e)$ for every $e \in \mathcal{D}(\Gamma)$, and we have this because for every Ω we have $\mathcal{M}, \Omega \Vdash P(e)$ for all $e \in \mathcal{D}(\Gamma)$. But if we had $\mathcal{M}, \Gamma \Vdash \Box_{\emptyset} P(y)$, since $\Gamma\mathcal{R}\Delta$ we must have $\mathcal{M}, \Delta \Vdash P(y)$, and hence we must have $\mathcal{M}, \Delta \Vdash P(d)$ since $d \in \mathcal{D}(\Delta)$, but we do not since $d \notin \mathcal{D}(\Gamma)$ and so d is not in the interpretation of P at Δ.

■

6 Binding Modalities—Varying Domain Axiomatics

A modal logic incorporating binding modalities was first axiomatized in [5, 6]. It derived from work in [4], and [1, Chapter 10], on the quantified version of the justification logic LP, which itself is related to propositional $\mathsf{S4}$. Consequently the binding modalities logic that Artemov and Yavorskaya presented was a quantified version of $\mathsf{S4}$. In their approach what we called *De Dicto* to *De Re* conversion was built into the axiomatization, and this considerably simplified things. Here we want to be completely general, so we give axioms and rules for a first-order version of modal K with binding modalities, but with no assumptions of *De Re*/*De Dicto* conversion. The intended varying domain semantics is as described in Section 4. Since binding modalities are not standard, there is no established naming convention here. We will call this logic FOBMK, for "first-order, binding modalities, K". Because of the use of binding modalities, and the distinctions it allows, the underlying axiomatization of classical first-order logic can be of the more conventional sort, allowing free variables.

The rule of necessitation is a problem, as we might expect. If we were to apply it in its usual form to $\forall x\varphi(x) \supset \varphi(y)$, to get $\Box(\forall x\varphi(x) \supset \varphi(y))$, we have a free variable, y, in the scope of the necessity sign, and should $\varphi(y)$ be understood in the *de re* or *de dicto* sense? Of course this is exactly the issue that binding modalities were designed to address. It is well-known that for propositional modal logics, the necessitation rule can be replaced by the assumption that for each axiom we also have that axiom with any number of prefixed necessity operators. We will do something similar here. This allows us considerable control because we do not have to treat each axiom the same way, and in fact we do not.

The axioms and rules are given in Figure 2. The two rules called *axiom necessitation* could really have been included in the axiom list, but they replace a conventional rule of necessitation, so we called them rules to emphasize that.

FOBMK–A-1 All tautologies

FOBMK–A-2 $\forall x\varphi(x) \supset \varphi(y)$ provided y is free for x in φ

FOBMK–A-3 $\forall x(\varphi \supset \psi) \supset (\varphi \supset \forall x\psi)$ provided x does not occur free in φ

FOBMK–A-4 $\Box_{Xy}\varphi \equiv \Box_X\varphi$ provided y does not occur free in φ

FOBMK–A-5 $\Box_X\varphi \supset \Box_X\forall y\varphi$ provided $y \notin X$

FOBMK–A-6 $\Box_X(\varphi \supset \psi) \supset (\Box_X\varphi \supset \Box_X\psi)$

FOBMK–R-1 (*modus ponens*) $\vdash \varphi,\ \vdash \varphi \supset \psi \ \Rightarrow \vdash \psi$

FOBMK–R-2 (*universal generalization*) $\vdash \varphi \ \Rightarrow \vdash \forall x\varphi$

FOBMK–R-3 (*axiom necessitation 1*) For every axiom φ except for instances of FOBMK–A-2, $\vdash \Box_{X_1}\Box_{X_2}\ldots\Box_{X_n}\varphi$ for $n > 0$ and arbitrary finite sets $X_1, X_2, \ldots, X_n$

FOBMK–R-4 (*axiom necessitation 2*) For an instance of FOBMK–A-2, $\forall x\varphi(x) \supset \varphi(y)$ where y is free for x in φ, $\vdash \Box_{X_1}\Box_{X_2}\ldots\Box_{X_n}(\forall x\varphi(x) \supset \varphi(y))$ where $n > 0$, $X_1, X_2, \ldots, X_n$ are arbitrary finite sets, and $y \notin X_n$

Figure 2: FOBMK Axioms and Rules

The usual proof of the distributivity of necessitation over conjunction goes through for FOBMK with no changes. Thus we have $\vdash \Box_X(\varphi \wedge \psi) \equiv (\Box_X\varphi \wedge \Box_X\psi)$ for every finite set X of individual variables.

Since we have two versions of axiom necessitation we do not get a derived necessitation rule in full generality. Still, the use of binding modalities allows us to get a partial version. This can be strengthened when we take monotonicity and anti-monotonicity into account, as we will see in Section 7.

Theorem 6.1 (FOBMK Necessitation) *Let X be a finite set of individual variables. Call an* FOBMK *axiomatic proof X* acceptable *provided that whenever $\forall x\varphi(x) \supset \varphi(y)$ occurs in the proof, where y is free for x in φ, then $y \notin X$. In* FOBMK*, if $\vdash \varphi$ and there is an X acceptable proof, then $\vdash \Box_X\varphi$.*

Proof The argument is by induction on proof length. Assume the result is known for proofs whose length is less than i and we now have a proof of length i. Let $\varphi_1, \varphi_2, \ldots, \varphi_i$ be an X acceptable FOBMK proof. We show $\Box_X\varphi_i$ is provable. There are several cases.

If φ_i is an axiom, we have $\Box_X\varphi_i$ using either FOBMK–R-3 or FOBMK–R-4. Similarly if φ_i was itself added to the proof using FOBMK–R-3 or FOBMK–R-4 themselves.

For FOBMK–R-1, suppose φ_i follows from earlier lines of the proof φ_j and $\varphi_j \supset \varphi_i$ using *modus ponens*. By the induction hypothesis we have $\Box_X \varphi_j$ and $\Box_X(\varphi_j \supset \varphi_i)$. $\Box_X \varphi_i$ follows using axiom FOBMK–A-6.

For FOBMK–R-2, suppose φ_i is $\forall y \varphi_j$ where $j < i$, and we have a proof of $\Box_X \varphi_j$. If $y \notin X$, we construct a proof of $\Box_X \forall y \varphi_j$ using FOBMK–A-5. Now suppose $y \in X$. Let $\widehat{X}$ be X with y removed. Since the proof is also $\widehat{X}$ acceptable, by the induction hypothesis we have a proof of $\Box_{\widehat{X}} \varphi_j$, so by FOBMK–A-5 we have a proof of $\Box_{\widehat{X}} \forall y \varphi_j$. Since y is not free in $\forall y \varphi_j$, we get $\Box_X \forall y \varphi_j$ using FOBMK–A-4. ■

Corollary 6.2 *If* $\vdash \varphi$ *then* $\vdash \Box_\emptyset \varphi$.

Corollary 6.3 (Quantifier Absorbtion) $\vdash \Box_X \psi \equiv \Box_X \forall y_1 \forall y_2 \cdots \forall y_n \psi$ *provided* $y_1, y_2, \ldots, y_n \notin X$.

Proof In the following, assume $y_1, y_2, \ldots, y_n \notin X$.

$\Box_X \psi \supset \Box_X \forall y_n \psi$ is axiom FOBMK–A-5, and hence provable. Another instance of FOBMK–A-5 is $\Box_X \forall y_n \psi \supset \Box_X \forall y_{n-1} \forall y_n \psi$, hence $\Box_X \psi \supset \Box_X \forall y_{n-1} \forall y_n \psi$ is provable. And so on.

For the implication in the other direction:

(1) $\forall y_n \psi \supset \psi$ FOBMK–A-2

(2) $\Box_X(\forall y_n \psi \supset \psi)$ FOBMK–R-4 since $y_n \notin X$

(3) $\Box_X(\forall y_n \psi \supset \psi) \supset (\Box_X \forall y_n \psi \supset \Box_X \psi)$ FOBMK–A-6

(4) $\Box_X \forall y_n \psi \supset \Box_X \psi$ *modus ponens*

(5) $\forall y_{n-1} \forall y_n \psi \supset \forall y_n \psi$ FOBMK–A-2

(6) $\Box_X(\forall y_{n-1} \forall y_n \psi \supset \forall y_n \psi)$ FOBMK–R-4 since $y_{n-1} \notin X$

(7) $\Box_X \forall y_{n-1} \forall y_n \psi \supset \Box_X \forall y_n \psi$, using FOBMK–A-6, 6, and *modus ponens*

(8) $\Box_X \forall y_{n-1} \forall y_n \psi \supset \Box_X \psi$ from 4 and 7 by classical logic

And so on. ■

7 Monotonicity and Anti-Monotonicity

We briefly discuss how one can add axioms to FOBMK to capture the *De Re/De Dicto* conversions discussed in Section 5, and thus bring in monotonicity and anti-monotonicity conditions. This section is a small interlude, since we are primarily interested in axiomatizing varying domain semantics as such in this paper.

The monotonicity condition on models is that a move to an accessible world may increase the quantification domain; if $\Gamma \mathcal{R} \Delta$ in a model, then $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$. Kripke showed in [13] that this semantic condition corresponds to the axiomatic converse Barcan scheme, $\Box \forall y \varphi \supset \forall y \Box \varphi$. The form this takes with binding modalities can be seen in Theorem 5.4. In the Artemov-Yavorskaya paper that introduced binding modalities, [6], motivation came from thinking of $\Box$ as representing provability, following Gödel in [12]. With provability in mind, the converse Barcan scheme is quite natural: if we have proved that φ holds of everything, then for each particular item we can prove that φ holds of it. Artemov and Yavorskaya assumed a stronger version of our FOBMK–A-4, breaking it into two implications. First they had $\Box_{Xy}\varphi \supset \Box_X\varphi$ provided y does not occur free in φ, which is half of our axiom. And second they had $\Box_X\varphi \supset \Box_{Xy}\varphi$, with no conditions. In their understanding, based on the familiar machinery of axiomatic provability, one could freely announce that a variable was being 'pulled out' to serve as a parameter that could be substituted for in a proof, and thus was no longer subject to quantification. While reasonable in their context, it turns out that this is exactly what corresponds to the converse Barcan formula.

Before starting formalities, we make some comments on the form that the converse Barcan scheme takes when binding modalities are present. Similar comments apply to the Barcan scheme too. In its usual form the converse Barcan scheme is $\Box\forall y\varphi \supset \forall y\Box\varphi$. In this paper the $\Box$ operator is replaced with a binding version, and we would like the binding to be as exact as possible. In the antecedent of the converse Barcan scheme, $\Box\forall y\varphi$, the variable y is quantified, and hence bound. Then in a binding modalities version, $\Box_X\forall y\varphi$, it does not matter whether y occurs in X since we have axiom FOBMK–A-4. The narrowest version would assume $y \notin X$. The consequent, $\forall y\Box\varphi$, becomes $\forall y\Box_X\varphi$ when binding modalities are brought in, and if $y \notin X$ then y would not be free in $\Box_X\varphi$, so the quantifier in $\forall y\Box_X\varphi$ would be vacuous. Then here we want to have $y \in X$. So, finally, our binding modalities version of converse Barcan is $\Box_X\forall y\varphi \supset \forall y\Box_{Xy}\varphi$ with the usual condition that $y \notin X$.

Theorem 7.1 (Converse Barcan Equivalents) *In axiomatic* FOBMK *the following are equivalent, following the convention that $y \notin X$:*

(1) $\Box_X\varphi \supset \Box_{Xy}\varphi$

(2) $\forall y(\Box_X\varphi \supset \Box_{Xy}\varphi)$

(3) $\Box_X\varphi \supset \forall y\Box_{Xy}\varphi$

(4) $\Box_X\forall y\varphi \supset \forall y\Box_{Xy}\varphi$

Proof items (1) and (2) are equivalent using universal generalization FOBMK–R-2 and universal instantiation FOBMK–A-2. Items (2) and (3) are equivalent because y is not free in $\Box_X\varphi$. Items (3) and (4) are equivalent using Quantifier Absorbtion, Theorem 6.3. ■

We could choose any of the four versions from Theorem 7.1 to add as new axiom schema. We follow Artemov and Yavorskaya, using $\Box_X\varphi \supset \Box_{Xy}\varphi$, which seems simplest. We also need a version of axiom necessitation to go with our new axiom scheme; we add it to the FOBMK–R-3 version. That is, we allow $\vdash \Box_{X_1}\Box_{X_2}\ldots\Box_{X_n}(\Box_X\varphi \supset \Box_{Xy}\varphi)$ for all finite sets X_i. One nice consequence is that the restriction imposed in FOBMK–R-4 vanishes. The result is a formulation of what we call FOBMKmon similar to the one for binding modalities S4 given in [4, 6]. The following makes all this official.

Definition 7.2 Axiom system FOBMKmon results from FOBMK by adding the axiom scheme $\Box_X\varphi \supset \Box_{Xy}\varphi$ for $y \notin X$, and extending rule FOBMK–R-3 to include this new axiom scheme thus:

$$\vdash \Box_{X_1}\Box_{X_2}\ldots\Box_{X_n}(\Box_X\varphi \supset \Box_{Xy}\varphi) \text{ for any finite sets } X_1, X_2, \ldots, X_n, \text{ where } y \notin X.$$

Theorem 7.3 *In* FOBMKmon *we have full necessitation: if $\vdash \varphi$ then $\vdash \Box_X\varphi$ for every finite set X.*

Proof Adding the new case to rule FOBMK–R-3 without modifying rule FOBMK–R-4 does not affect the proof of Theorem 6.1 or its Corollary 6.2. So we have that $\vdash \varphi$ implies $\vdash \Box_\emptyset\varphi$. Now $\vdash \Box_X\varphi$ follows by repeated use of instances of the new axiom scheme. ■

Corollary 7.4 *The restriction that $y \notin X_n$ in FOBMK–R-4 can be dropped in* FOBMKmon. *That is, FOBMK–R-4 can be folded into FOBMK–R-3.*

Proof $\forall x\varphi \supset \varphi(x/y)$ is provable since it is an axiom. Then by repeated use of the theorem above, $\Box_{X_1}\Box_{X_2}\ldots\Box_{X_n}\varphi$ is provable for any finite $X_1, \ldots, X_n$ without restriction. ■

As Kripke also showed in [13], the Barcan formula corresponds to the semantic condition of anti-monotonicity. Binding modalities provides us with fewer options here than in the anti-monotonic case.

Theorem 7.5 (Versions of the Barcan formula) *In axiomatic* FOBMK *the following are equivalent, where $y \notin X$:*

(1) $\forall y\Box_{Xy}\varphi \supset \Box_X\varphi$

(2) $\forall y\Box_{Xy}\varphi \supset \Box_X\forall y\varphi$

Proof Use Quantifier Absorbtion, Theorem 6.3. ■

Definition 7.6 Axiom system FOBMKantimon results from FOBMK by adding the axiom scheme $\forall y\Box_{Xy}\varphi \supset \Box_X\varphi$ for $y \notin X$, and extending rule FOBMK–R-3 to include this new axiom scheme thus:

$$\vdash \Box_{X_1}\Box_{X_2}\ldots\Box_{X_n}(\forall y\Box_{Xy}\varphi \supset \Box_X\varphi) \text{ for any finite sets } X_1, X_2, \ldots, X_n, \text{ where } y \notin X.$$

Finally, for constant domains we simply combine our earlier systems.

Definition 7.7 Axiom system FOBMKcons is the combination of FOBMKmon and FOBMKantimon. Equivalently, add to FOBMK the axiom scheme $\forall y\Box_{Xy}\varphi \equiv \Box_X\varphi$.

8 Soundness

Now we return to a consideration of FOBMK alone, and show our axiomatization matches our semantics. In this section we show soundness in the usual way: each axiom is valid, and validity is preserved by the rules. But truth in models, Definition 4.4, is directly about model formulas, while axioms are in a language that does not allow members of a model domain to appear. Further, axioms can contain free variable occurrences, while we have given a central role to *closed* model formulas—in Definition 4.5 we characterized validity of a model formula with free variables so that it is equivalent to the validity of its universal closure. (Recall that at a particular possible world the universal quantifier ranges over only the members of the domain *of that world.*) It is useful now to introduce something stronger, essentially allowing universal closure with respect to the domain of a model, and not just with respect to the domain of a world.

Definition 8.1 (Strong Validity) Let $\varphi(\vec{x})$ be a formula not containing members of a domain, but having all its free variables in $\vec{x}$, and let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ be a model. $\varphi(\vec{x})$ is *strongly valid* in $\mathcal{M}$ provided, for each $\Gamma \in \mathcal{G}$ and each $\vec{a} \in \mathcal{D}_S$, $\mathcal{M}, \Gamma \Vdash \varphi(\vec{a})$.

Note that since $\mathcal{D}(\Gamma) \subseteq \mathcal{D}_S$, strong validity in a model implies validity in that model.

Theorem 8.2 *Each of the axioms FOBMK–A-1, FOBMK–A-3, FOBMK–A-4, FOBMK–A-5, and FOBMK–A-6 are strongy valid. Also* $\forall y(\forall x\varphi \supset \varphi(x/y))$ *is strongly valid where* y *is free for* x *in* φ*, and hence FOBMK–A-2 is valid.*

Proof We check FOBMK–A-4, FOBMK–A-5, and FOBMK–A-2, leaving FOBMK–A-1, FOBMK–A-3, and FOBMK–A-6 to the reader. For the rest of this proof $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ is an arbitrary model and Γ is an arbitrary world in $\mathcal{G}$.

For FOBMK–A-4, to keep notation simple we work with a representative special case. Let us assume φ is an $\mathcal{M}$ formula that has only x and w as free variables, so we write it as $\varphi(x, w)$, and assume $X = \{x, z\}$. We show strong validity of $\Box_{\{x,y,z\}}\varphi(x, w) \equiv \Box_{\{x,z\}}\varphi(x, w)$ in $\mathcal{M}$.

Let a, b, c be arbitrary members of $\mathcal{D}_S$ and apply the substitution $(x/a, y/b, z/c)$ getting the closed model formula $\Box_{\{a,b,c\}}\varphi(a, w) \equiv \Box_{\{a,c\}}\varphi(a, w)$. We show this is true at Γ. To do this we must show $\mathcal{M}, \Gamma \Vdash \Box_{\{a,b,c\}}\varphi(a, w)$ if and only if $\mathcal{M}, \Gamma \Vdash \Box_{\{a,c\}}\varphi(a, w)$. For both, this reduces to the same condition, that for any $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash \varphi(a, w)$, so we are done.

For FOBMK–A-5, again to keep notation simple assume we have an $\mathcal{M}$ formula $\varphi = \varphi(x, y, w)$ where all free variables are displayed, and $X = \{x, z\}$. We show strong validity of $\Box_{\{x,z\}}\varphi(x, y, w) \supset \Box_{\{x,z\}}\forall y\varphi(x, y, w)$.

Let a, c be arbitrary members of $\mathcal{D}_S$ and apply the substitution $(x/a, z/c)$, getting closed model formula $\Box_{\{a,c\}}\varphi(a, y, w) \supset \Box_{\{a,c\}}\forall y\varphi(a, y, w)$. We show this is true at Γ. Assume $\mathcal{M}, \Gamma \Vdash \Box_{\{a,c\}}\varphi(a, y, w)$. Let Δ be an arbitrary member of $\mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$. Using Definition 4.4 part 5, $\mathcal{M}, \Delta \Vdash \varphi(a, y, w)$. Then by Definition 4.4 part 0, for every $b, d \in \mathcal{D}(\Delta)$, $\mathcal{M}, \Delta \Vdash \varphi(a, b, d)$. Rephrasing this, for each $d \in \mathcal{D}(\Delta)$, $\mathcal{M}, \Delta \Vdash \varphi(a, b, d)$ for every $b \in \mathcal{D}(\Delta)$, so by Definition 4.4 part 4, $\mathcal{M}, \Delta \Vdash \forall y\varphi(a, y, d)$ for each $d \in \mathcal{D}(\Delta)$. Since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash \Box_{\{a,c\}}\forall y\varphi(a, y, w)$ by Definition 4.4 part 5 again.

Now assume y is free for x in $\mathcal{M}$ formula $\varphi(x)$. We show strong validity of $\forall y(\forall x\varphi(x) \supset \varphi(y))$ in $\mathcal{M}$. Let φ' be the result of substituting members of $\mathcal{D}_S$ for individual variables of φ other than x and y. We want to show that $\forall y(\forall x\varphi'(x) \supset \varphi'(y))$ is true at Γ. But this is immediate using Definition 4.4 part 4.

It now follows that, for each $a \in \mathcal{D}(\Gamma)$, $\mathcal{M}, \Gamma \Vdash \forall x\varphi'(x) \supset \varphi'(a)$, and so we have validity of FOBMK–A-2. ■

Note that, while we proved validity of $\forall x\varphi(x) \supset \varphi(y)$, it is easy to construct examples to show it is not *strongly* valid.

Corollary 8.3 *Let* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ *be a model,* X *be a finite set of individual variables, and* ψ *be a model formula. If* ψ *is strongly valid in* $\mathcal{M}$ *so is* $\Box_X\psi$*.*

Proof Assume $\psi(\vec{x}, \vec{y})$ is a strongly valid model formula in $\mathcal{M}$, where all the free individual variables are shown. To show $\Box_{\{\vec{x}\}}\psi(\vec{x}, \vec{y})$ is strongly valid in $\mathcal{M}$ it must be shown that $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}}\psi(\vec{a}, \vec{y})$ for every $\Gamma \in \mathcal{D}$ and every $\vec{a} \in \mathcal{D}_S$. Let Δ be an arbitrary member of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$; we must show $\mathcal{M}, \Delta \Vdash \psi(\vec{a}, \vec{b})$ for every $\vec{b} \in \mathcal{D}_S$. But this is the case because $\psi(\vec{x}, \vec{y})$ is strongly valid in $\mathcal{M}$. ■

Theorem 8.4 *The rules FOBMK–R-1 and FOBMK–R-2 preserve weak (and strong) validity, and rules FOBMK–R-3, and FOBMK–R-4 add strong validity formulas.*

Proof We leave FOBMK–R-1 and FOBMK–R-2 to the reader. For FOBMK–R-3, all axioms except for FOBMK–A-2, are strongly valid by Theorem 8.2, and we can repeatedly apply Corollary 8.3.

For FOBMK–R-4 suppose φ is $\varphi(x, \vec{z}, \vec{w})$, where the $\vec{z}$ and $\vec{w}$ are disjoint and do not contain x. Also suppose y is free for x in this formula, and $X = \{\vec{z}\}$. We first show $\Box_X(\forall x\varphi \supset \varphi(x/y))$ is strongly valid, or using current notation, $\Box_{\{\vec{z}\}}(\forall x\varphi(x, \vec{z}, \vec{w}) \supset \varphi(y, \vec{z}, \vec{w}))$ is strongly valid. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ be a model, $\Gamma \in \mathcal{G}$, and $\vec{a} \in \mathcal{D}_S$. We must show $\mathcal{M}, \Gamma \Vdash \Box_{\{\vec{a}\}}(\forall x\varphi(x, \vec{a}, \vec{w}) \supset \varphi(y, \vec{a}, \vec{w}))$. This means that for each $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ we must show $\mathcal{M}, \Delta \Vdash \forall x\varphi(x, \vec{a}, \vec{w}) \supset \varphi(y, \vec{a}, \vec{w})$. To do this we must show $\mathcal{M}, \Delta \Vdash \forall x\varphi(x, \vec{a}, \vec{b}) \supset \varphi(c, \vec{a}, \vec{b})$ for every $\vec{b} \in \mathcal{D}(\Delta)$ and for every $c \in \mathcal{D}(\Delta)$. This is equivalent to showing $\mathcal{M}, \Delta \Vdash \forall y(\forall x\varphi(x, \vec{a}, \vec{b}) \supset \varphi(y, \vec{a}, \vec{b}))$ for every $\vec{b} \in \mathcal{D}(\Delta)$, and we have this because $\forall y(\forall x\varphi(x, \vec{z}, \vec{w}) \supset \varphi(y, \vec{z}, \vec{w}))$ is strongly valid by Theorem 8.2. Now, as in the previous case, we can repeatedly apply Corollary 8.3. ■

9 Completeness

Completeness proofs for axiomatizations of varying domain modal logics are rather rare, compared to the monotonic version. The proof we give is similar to the very general construction in [7]. Of course we have binding modalities and that paper does not, but that does allow a kind of free logic, with inner and outer domains, both of which vary, and we do not have this here. A trace of the dual domains can be seen, however, in our use of what we call *specification pairs*, in which one member plays the role of an inner domain and the other plays that of the outer version. But here it this simply a technical device, forced on us by circumstances. The proof in [7] can be used as a guide to extend the present argument to the monotonic and anti-monotonic cases, but this would be too much detail for here.

Completeness is proved Henkin style so the language is enlarged with witnesses for existential statements. These are sometimes taken to be formal constants, but we prefer to use variables that are restricted so they never occur bound (obviously they behave like constants in this respect). This keeps the variety of machinery down and simplifies things in some respects.

Definition 9.1 (Variable Extension) The language, as defined in Section 3, has a set $\mathbf{V} = \{x_1, x_2, x_3, \ldots\}$ of individual variables. We now add a countable set of new variables $\mathbf{W} = \{w_1, w_2, w_3, \ldots\}$ *that are not allowed to occur bound in formulas.* For this section we call members of $\mathbf{V} \cup \mathbf{W}$ *individual variables*, members of $\mathbf{V}$ (our original variables) *basic variables*, and the newly added members of $\mathbf{W}$ *witness variables.* For a formula φ in our extended language, by $\mathsf{v}(\varphi)$ we mean the set of *witness* variables occurring in φ. The axiom system of Section 6 is enlarged to allow formulas containing witness variables, but in free positions only. By *basic language and basic system* we mean the language and the axiomatic system as originally given, without the extension by witness variables. We call a formula with witness variables *closed* if no *basic* variable occurrences are free. (Thus in a closed formula, the free variables are all witness variables present and the bound variables are all basic variables present.)

Allowing witness variables that can only occur in free positions brings with it some peculiarities that should be noted. Axiom FOBMK–A-2 allows for $\forall x\varphi(x) \supset \varphi(w)$ where w is a witness variable (and x is basic). But Rule FOBMK–R-4 is inapplicable. In particular we cannot even have the $n = 1$ case which would allow us to conclude $\Box_X(\forall x\varphi(x) \supset \varphi(w))$, because w cannot occur in X by the conditions on FOBMK–R-4, but if it does not, its occurrence as part of $\varphi(w)$ in $\Box_X(\forall x\varphi(x) \supset \varphi(w))$ would be a bound occurrence, precisely because it does not occur in X. Also if w is a witness variable, a new instance of axiom FOBMK–A-4 is $\Box_{Xw}\varphi \equiv \Box_X\varphi$ provided w does not occur in φ.

The proviso, as actually stated with FOBMK–A-4, is that w does not occur *free* in φ, but since it cannot occur bound, this is equivalent to requiring that it does not occur at all.

We will use witness variables to supply domains of quantification at possible worlds of a canonical model. Different worlds can have different domains, so we will work with many subsets of $\mathbf{W}$ as quantificational domains. We will require that such subsets omit countably many witness variables, so 'new' witness variables are always available. Since the countable set $\mathbf{W}$ can always be split up into countably many disjoint countable subsets, this is not a problem and we omit the details.

Our models, as defined in Section 4, allow us to evaluate at a possible world formulas that contain members of the model domain that are not in the domain of that possible world itself. In constructing our Henkin-style model we need to be careful with this. The overall domain of the model we construct will be the entire set of witness variables, but possible worlds will be sets of formulas *that are required to omit countably many witness variables*, so we should not expect a truth lemma to match up semantic behavior at a possible world with membership in that world for every formula. We need some restriction on the set of formulas for which we can expect such a match-up to work out. For these purposes, for each possible world in the model that we will construct we specify what are, informally speaking, inner and outer sets of witness variables, where the inner set is the one we quantify over, and the outer set is the one for which truth lemma behavior is well behaved. Here are the formalities.

Definition 9.2 (Specification Pair) Let $\mathbf{W}_1 \subseteq \mathbf{W}_2$ be sets of witness variables. We call $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ a *specification pair*, and use it to restrict formulas and proofs as follows.

A $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ *formula* is any formula allowing basic variables free and bound, and witness variables in free positions but with all witness variables coming from $\mathbf{W}_2$.

A $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ *axiomatic proof* is any proof using the system of Section 6, allowing witness variables but in which only $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ formulas are allowed, and in addition if an instance of axiom FOBMK–A-2 occurs, say $\forall x\varphi \supset \varphi(x/w)$ where w is a witness variable, then w must be in $\mathbf{W}_1$. A proof proves its last line. We write $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2 \rangle} \varphi$ if φ is a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ formula having a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ proof.

Since witness variables cannot occur bound, a substitution (x/y) in which y is a witness variable is always a free substitution, which is pertinent to axiom FOBMK–A-2 and rule FOBMK–R-4. This simplifies some things. But on the other hand we cannot apply the generalization rule FOBMK–R-2 if we have a proof of $\varphi(w)$, where w is a witness variable. Fortunately we will have something almost as good in Corollary 9.4.

Theorem 9.3 *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ proof, let $w \in \mathbf{W}_2$, and let x be a basic variable that does not appear in the proof. Then $\varphi_1(w/x), \varphi_2(w/x), \ldots, \varphi_n(w/x)$ is also a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ proof.*

Proof This is an easy argument on proof length, which we leave to the reader. ■

Corollary 9.4 (Generalization) *Suppose $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2 \rangle} \varphi(w)$ where w is a witness variable in $\mathbf{W}_2$. Then $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2 \rangle} \forall x\varphi(x)$ for some basic variable x not appearing in $\varphi(w)$.*

Proof Assume we have a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ proof of $\varphi(w)$. Let x be any basic variable that does not occur in the proof (and hence not in $\varphi(w)$), and replace occurrences of w throughout the proof with occurrences of x. By Theorem 9.3 this will still be a proof, but of $\varphi(x)$. Now rule FOBMK–R-2 can be applied and $\forall x\varphi(x)$ is thus provable. ■

The next few items are rather technical, but play an important role in our completeness proof.

Corollary 9.5 *Let φ be a $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ formula. Recall from Definition 9.1 that $\mathsf{v}(\varphi)$ is the set of witness variables of φ. If $\mathsf{v}(\varphi) \cap \mathbf{W}_1 = \emptyset$ and $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \varphi$ then $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi)}\varphi$.*

Proof Assume $\mathsf{v}(\varphi) \cap \mathbf{W}_1 = \emptyset$, and consider a specific $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ proof of φ. We begin by eliminating some possible trouble spots from the proof. Suppose an instance of Axiom FOBMK–A-2, $\forall x\psi \supset \psi(x/w)$, appears in the proof, where w is a witness variable. Using Theorem 9.3, this witness variable can be replaced by a new basic variable throughout the proof and the result will still be a $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ proof. Further, since the original proof was a $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ proof, w must come from $\mathbf{W}_1$ and since $\mathsf{v}(\varphi) \cap \mathbf{W}_1 = \emptyset$, w cannot occur in φ, and so φ is unchanged by the replacement. In this way we can replace, one by one, all witness variables that occur in the original proof in instances of Axiom FOBMK–A-2 by basic variables without changing what is being proved, while keeping it a $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ proof. Let us assume this has been done.

We now have a $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ proof $\varphi_1, \varphi_2, \ldots, \varphi_n = \varphi$ in which no instantiation introduced by FOBMK–A-2 involves a witness variable, only a basic variable. We show that, for each i, $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_i)}\varphi_i$, and hence $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi)}\varphi$.

The proof is inductive. Suppose it is known that $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_j)}\varphi_j$ for all $j < i$; we show $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_i)}\varphi_i$.

Case FOBMK–A-2: φ_i is $\forall x\psi \supset \psi(x/y)$. The variable y cannot be a witness variable since these have been replaced, so it must be a basic variable. As a basic variable y cannot be in $\mathsf{v}(\varphi_i)$, so we have $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_i)}(\forall x\varphi \supset \varphi(x/y))$ using rule FOBMK–R-4.

Case all other axioms: φ_i is a an axiom other than FOBMK–A-2. Use Rule FOBMK–R-3.

Case FOBMK–R-1: Suppose $j, k < i$, $\varphi_k = \varphi_j \supset \varphi_i$, and we have $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_k)}\varphi_k$ and $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_k \supset \varphi_i)}(\varphi_k \supset \varphi_i)$. Using FOBMK–A-4, $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_k \supset \varphi_i)}\varphi_k$. Using FOBMK–A-6 we can get $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_k \supset \varphi_i)}\varphi_k \supset \Box_{\mathsf{v}(\varphi_k \supset \varphi_i)}\varphi_i$. By *modus ponens*, $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_k \supset \varphi_i)}\varphi_i$. Then by FOBMK–A-4 again, $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \Box_{\mathsf{v}(\varphi_i)}\varphi_i$.

Case FOBMK–R-2 Using the induction hypothesis, FOBMK–A-5, and the fact that only basic variables can be quantified while members of $\mathsf{v}(\varphi_i)$ are witness variables.

Case FOBMK–R-3 By FOBMK–R-3 itself.

Case FOBMK–R-4 Using FOBMK–R-4 itself.

■

Definition 9.6 Let S be a set of $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ formulas and φ be a single such formula. We write $S \vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \varphi$ if $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} (\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$ for some $\psi_1, \ldots, \psi_n \in S$. We say S is $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ *inconsistent* if $S \vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \bot$, and S is $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ *consistent* if it is not $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ inconsistent.

Theorem 9.7 *Let $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ and $\langle \mathbf{W}_3, \mathbf{W}_4\rangle$ be specification pairs where $\mathbf{W}_2 \cap \mathbf{W}_3 = \emptyset$ and $\mathbf{W}_2 \subseteq \mathbf{W}_4$, Also let S be a set of $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ formulas and φ be a single such formula. If $S \vdash_{\langle \mathbf{W}_3, \mathbf{W}_4\rangle} \varphi$ then $S \vdash_{\langle \mathbf{W}_1, \mathbf{W}_2\rangle} \varphi$.*

Proof The members of S and φ are all $\langle \mathbf{W}_1, \mathbf{W}_2\rangle$ formulas. Since $\mathbf{W}_2 \subseteq \mathbf{W}_4$ these are also $\langle \mathbf{W}_3, \mathbf{W}_4\rangle$ formulas. Assume $S \vdash_{\langle \mathbf{W}_3, \mathbf{W}_4\rangle} \varphi$. Then there is a $\langle \mathbf{W}_3, \mathbf{W}_4\rangle$ proof $\varphi_1, \varphi_2, \ldots, \varphi_n$, where $\varphi_n = (\psi_1 \wedge \ldots \wedge \psi_k) \supset \varphi$ for some $\psi_1, \ldots, \psi_k \in S$. Let $w_1, w_2, \ldots, w_m$ be all the witness variables in $\mathbf{W}_4 - \mathbf{W}_2$ that occur in this proof, let $x_1, \ldots, x_m$ be distinct basic variables that do not occur in the proof, and let σ be the substitution $(w_1/x_1, \ldots, w_m/x_m)$. By repeated application of Theorem 9.3, $\varphi_1\sigma, \ldots, \varphi_n\sigma$ is a $\langle \mathbf{W}_3, \mathbf{W}_4 \rangle$ proof. We show it is also a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ proof.

Throughout $\varphi_1, \ldots, \varphi_n$, σ replaces all witness variables from $\mathbf{W}_4$ except those that are also in $\mathbf{W}_2$, so the only witness variables in $\varphi_1\sigma, \ldots, \varphi_n\sigma$ are in $\mathbf{W}_2$. Since $\varphi_1, \ldots, \varphi_n$ was a $\langle \mathbf{W}_3, \mathbf{W}_4 \rangle$ proof, if some φ_i is an instance of FOBMK–A-2 and introduced a witness variable, that witness variable must have come from $\mathbf{W}_3$. Since $\mathbf{W}_3 \subseteq \mathbf{W}_4$, that witness variable must be in $\mathbf{W}_4$. Since $\mathbf{W}_2 \cap \mathbf{W}_3 = \emptyset$, that witness variable cannot be in $\mathbf{W}_2$. Then σ replaced it with a basic variable. So, all witness variables in $\varphi_1, \ldots, \varphi_n$ introduced by FOBMK–A-2 have been replaced by basic variables in $\varphi_1\sigma, \ldots, \varphi_n\sigma$, and thus vacuously all are in $\mathbf{W}_1$. It follows that $\varphi_1\sigma, \ldots, \varphi_n\sigma$ is a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ proof.

Finally φ_n is $(\psi_1 \wedge \ldots \wedge \psi_k) \supset \varphi$ which is a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ formula, thus containing no members of $\mathbf{W}_4$ that are not in $\mathbf{W}_2$. Then $\varphi_n\sigma = (\psi_1 \wedge \ldots \wedge \psi_k) \supset \varphi$, and $\varphi_1\sigma, \ldots, \varphi_n\sigma$ is a $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ proof for it. ∎

Corollary 9.8 *Under the conditions of Theorem 9.7, if S is $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ consistent then S is $\langle \mathbf{W}_3, \mathbf{W}_4 \rangle$ consistent.*

Proof Take φ to be $\perp$ in Theorem 9.7. ∎

Now we are ready to introduce the things that will function as possible worlds in our varying domain canonical models.

Definition 9.9 We say $\langle S, \mathbf{W}_1, \mathbf{W}_2 \rangle$ is *world-like* if:

1. $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ is a specification pair (Definition 9.2) where $\mathbf{W}_2$ omits countably many witness variables,
2. S is a set of closed $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ formulas (Definition 9.1),
3. S is $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ consistent (Definition 9.6),
4. if S' is a set of closed $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ formulas that is $\langle \mathbf{W}_1, \mathbf{W}_2 \rangle$ consistent, and $S \subseteq S'$ then $S = S'$ (that is, S is maximal),
5. S contains witnesses from $\mathbf{W}_1$, meaning that if $\neg\forall x\varphi(x) \in S$ then $\neg\varphi(w) \in S$ for some $w \in \mathbf{W}_1$.

Here is the key item, an elaboration of the usual Henkin construction.

Theorem 9.10 (Extension Theorem) *Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be disjoint sets of witness variables and let S be a set of closed formulas whose witness variables are all in $\mathbf{W}_2$. Assume $\mathbf{W}_1$ is countable, $\mathbf{W}_1 \cup \mathbf{W}_2$ omits countably many witness variables, and S is $\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ consistent. Then S can be extended to a set S' so that $\langle S', \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ is world-like.*

Proof Assume the hypotheses. Enumerate the countable set of closed $\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ formulas, as $\varphi_1, \varphi_2, \ldots$, and enumerate the members of $\mathbf{W}_1$, as $w_1, w_2, \ldots$. Construct a sequence of

$\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ consistent sets as follows.

$$S_1 = S$$
$$S_{n+1} = \begin{cases} S_n & \text{if } S_n \cup \{\varphi_n\} \text{ is not } \langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle \text{ consistent} \\ S_n \cup \{\varphi_n\} & \text{if } S_n \cup \{\varphi_n\} \text{ is } \langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle \text{ consistent} \\ & \text{and } \varphi_n \text{ is not a negated universal} \\ S_n \cup \{\varphi_n\} \cup \{\neg\psi(w)\} & \text{if } S_n \cup \{\varphi_n\} \text{ is } \langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle \text{ consistent} \\ & \text{and } \varphi_n = \neg\forall x \psi(x) \\ & \text{and } w \text{ is the first term in the sequence } w_1,\, w_2,\, \ldots \\ & \text{that does not occur in } S_n \cup \{\varphi_n\} \end{cases}$$

$S_1 = S$ involves no members of $\mathbf{W}_1$. It follows that every S_n involves only a finite number of members of $\mathbf{W}_1$, so in the third clause of the definition of S_{n+1} some w_i must be available.

Each member of S_1, S_2, S_3, ... will be consistent. Two of the cases are simple; the only one that needs checking is the negative universal case. To keep notation less cluttered we write $\vdash$ for $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle}$ throughout the following paragraph.

Suppose $S_n \cup \{\varphi_n\}$ is $\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ consistent, $\varphi_n = \neg\forall x \psi(x)$, w does not occur in $S_n \cup \{\varphi_n\}$, and $S_{n+1} = S_n \cup \{\varphi_n\} \cup \{\neg\psi(w)\}$ is not $\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ consistent. We derive a contradiction. If S_{n+1} is not $\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ consistent, for some finite subset F of $S_n \cup \{\varphi_n\}$, $\vdash \bigwedge F \supset \psi(w)$. Now w is a witness variable and cannot be quantified, but we can use Corollary 9.4. For some basic variable y that does not occur in $\bigwedge F \supset \psi(w)$, $\vdash \forall y(\bigwedge F \supset \psi(w))(w/y)$. Since w was a new witness variable when introduced, it does not occur in F, so we have $\vdash \forall y(\bigwedge F \supset \psi(y))$ and hence also $\vdash \bigwedge F \supset \forall y \psi(y)$. But $F \subseteq S_n$ and $S_n \cup \{\neg\forall x \psi(x)\}$ is consistent. This is a contradiction in classical logic.

Of course $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots$. Define $S = S_1 \cup S_2 \cup S_3 \cup \ldots$. Familiar arguments, which we omit, show that S is $\langle \mathbf{W}_1, \mathbf{W}_1 \cup \mathbf{W}_2 \rangle$ consistent, appropriately maximal, and we have just shown that there are witnesses in $\mathbf{W}_1$ for negated universals. ■

Definition 9.11 Let S be a set of closed formulas allowing witness variables. By $S^\sharp$ we mean $\{\vec{\forall}\varphi \mid \Box_{\mathsf{v}(\varphi)}\varphi \in S\}$, where $\vec{\forall}$ indicates universal closure and v is from Definition 9.1.

Now we construct our quantified varying domain canonical model. It is here that we use what we called world-like sets in Definition 9.9.

Definition 9.12 (Varying Domain Canonical Model) The *binding modality varying domain canonical model* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ is defined as follows.

(1) $\mathcal{G}$ is the collection of all sets $\langle S, \mathbf{W}_1, \mathbf{W}_2 \rangle$ that are world-like.

(2) Let $\Gamma, \Delta \in \mathcal{G}$, where $\Gamma = \langle S, \mathbf{W}_1, \mathbf{W}_2 \rangle$ and $\Delta = \langle T, \mathbf{W}_3, \mathbf{W}_4 \rangle$. $\Gamma \mathcal{R} \Delta$ iff $\mathbf{W}_2 \subseteq \mathbf{W}_4$ and $S^\sharp \subseteq T$.

(3) Let $\Gamma = \langle S, \mathbf{W}_1, \mathbf{W}_2 \rangle \in \mathcal{G}$. $\mathcal{D}(\Gamma) = \mathbf{W}_1$.

(4) Let $\Gamma = \langle S, \mathbf{W}_1, \mathbf{W}_2 \rangle \in \mathcal{G}$. For an n-place relation symbol P, $\mathcal{I}(P, \Gamma)$ is the set of all $\vec{a}$ where $\vec{a} \in \mathbf{W}_2$ and $P(\vec{a}) \in S$.

We have a somewhat more complicated version of the familiar Truth Lemma than usual.

Theorem 9.13 (Truth Lemma) *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be the binding modality varying domain canonical model, $\Gamma = \langle S, \mathbf{W}_1, \mathbf{W}_2 \rangle \in \mathcal{G}$, and φ be a closed formula whose witness variables are all in $\mathbf{W}_2$. Then we have the following.*

$$\mathcal{M}, \Gamma \Vdash \varphi \textit{ if and only if } \varphi \in S \tag{$*$}$$

Proof The proof is by induction on formula complexity. All the cases but one are straightforward and are left to the reader. We consider only the modal case. Assume $\varphi = \Box_X \psi(\vec{x})$ where φ is a closed model formula and $\vec{x}$ are all the free basic variables of ψ. As our induction hypothesis we assume $(*)$ is known for arbitrary possible worlds and for formulas simpler than φ, in particular for closed instances of $\psi(\vec{x})$ involving witness variables. Let $\Gamma = \langle S, \mathbf{W}_1, \mathbf{W}_2 \rangle \in \mathcal{G}$ and assume all witness variables of φ are in $\mathbf{W}_2$. We show $(*)$ for φ itself.

Suppose $\varphi \in S$, that is, $\Box_X \psi(\vec{x}) \in S$. Since all witness variables in φ are in $\mathbf{W}_2$, the same is true of ψ. Now let $\Delta \in \mathcal{G}$ be arbitrary, with $\Gamma \mathcal{R} \Delta$. Δ must be $\langle T, \mathbf{W}_3, \mathbf{W}_4 \rangle$ for some T, $\mathbf{W}_3$, and $\mathbf{W}_4$, and by definition of $\mathcal{R}$, $S^\sharp \subseteq T$ and $\mathbf{W}_2 \subseteq \mathbf{W}_4$. Then $\forall \vec{x} \psi(\vec{x}) \in T$ and all witness variables in ψ are in $\mathbf{W}_4$. Since T meets maximally condition 4 from Definition 9.9, using FOBMK–A-2, $\psi(\vec{a}) \in T$ for all $\vec{a} \in \mathbf{W}_3$. Then by the induction hypothesis $(*)$ we have $\mathcal{M}, \Delta \Vdash \psi(\vec{a})$ for all $\vec{a} \in \mathcal{D}(\Delta)$, and so $\mathcal{M}, \Delta \Vdash \psi(\vec{x})$. Since Δ was arbitrary we have $\mathcal{M}, \Gamma \Vdash \Box_X \psi(\vec{x})$, that is, $\mathcal{M}, \Gamma \Vdash \varphi$.

Now suppose $\varphi \notin S$ so by maximal consistency $\neg\varphi \in S$, that is, $\neg\Box_X \psi(\vec{x}) \in S$. Let $\mathbf{W}_0$ be a countable set of witness variables disjoint from $\mathbf{W}_2$ chosen so that $\mathbf{W}_2 \cup \mathbf{W}_0$ still omits countably many witness variables.

We first show the set $S^\sharp \cup \{\neg\forall \vec{x} \psi(\vec{x})\}$ is $\langle \mathbf{W}_0, \mathbf{W}_0 \cup \mathbf{W}_2 \rangle$ consistent. Suppose not. Then using Definition 9.11 there are $\forall \vec{y_1} G_1(\vec{y_1}), \ldots, \forall \vec{y_n} G_n(\vec{y_n}) \in S^\sharp$ such that $\vdash_{\langle \mathbf{W}_0, \mathbf{W}_0 \cup \mathbf{W}_2 \rangle} (\forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \forall \vec{y_n} G_n(\vec{y_n}) \wedge \neg\forall \vec{x} \psi(\vec{x})) \supset \bot$, where for each $i = 1, \ldots, n$, $\Box_{\mathsf{v}(G_i)} G_i(\vec{y_i}) \in S$. Equivalently, $\vdash_{\langle \mathbf{W}, \mathbf{W}_2 \cup \mathbf{W} \rangle} (\forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \forall \vec{y_n} G_n(\vec{y_n})) \supset \forall \vec{x} \psi(\vec{x})$. Let $Z = \mathsf{v}(\psi) \cup \mathsf{v}(G_1) \cup \ldots \cup \mathsf{v}(G_n)$. Note that Z is the set of witness variables appearing in $(\forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \forall \vec{y_n} G_n(\vec{y_n})) \supset \forall \vec{x} \psi(\vec{x})$. Of course the witness variables in Z must be in $\mathbf{W}_2$. But in a $\langle \mathbf{W}_0, \mathbf{W}_0 \cup \mathbf{W}_2 \rangle$ proof of $(\forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \forall \vec{y_n} G_n(\vec{y_n})) \supset \forall \vec{x} \psi(\vec{x})$, witness variables introduced by instances of axiom FOBMK–A-2 must come from $\mathbf{W}_0$. Since $\mathbf{W}_0$ and $\mathbf{W}_2$ are disjoint, in any such proof no witness variable introduced by FOBMK–A-2 can be in Z. so by Corollary 9.5, $\vdash_{\langle \mathbf{W}_0, \mathbf{W}_0 \cup \mathbf{W}_2 \rangle} \Box_Z[((\forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \forall \vec{y_n} G_n(\vec{y_n})) \supset \forall \vec{x} \psi(\vec{x})]$. But then, by Theorem 9.7, $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2 \rangle} \Box_Z[((\forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \forall \vec{y_n} G_n(\vec{y_n})) \supset \forall \vec{x} \psi(\vec{x})]$. It follows that $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2 \rangle} (\Box_Z \forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \Box_Z \forall \vec{y_n} G_n(\vec{y_n})) \supset \Box_Z \forall \vec{x} \psi(\vec{x})$. Using FOBMK–A-4, $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2 \rangle} (\Box_{\mathsf{v}(G_1)} \forall \vec{y_1} G_1(\vec{y_1}) \wedge \ldots \wedge \Box_{\mathsf{v}(G_n)} \forall \vec{y_n} G_n(\vec{y_n})) \supset \Box_X \forall \vec{x} \psi(\vec{x})$, so by Theorem 6.3, $\vdash_{\langle \mathbf{W}_1, \mathbf{W}_2 \rangle} (\Box_{\mathsf{v}(G_1)} G_1(\vec{y_1}) \wedge \ldots \wedge \Box_{\mathsf{v}(G_n)} G_n(\vec{y_n})) \supset \Box_X \psi(\vec{x})$. But for each i, $\Box_{\mathsf{v}(G_i)} G_i(\vec{y_i}) \in S$, and so $\Box_X \psi(\vec{x}) \in S$, contradicting the consistency of S. This contradiction establishes that $S^\sharp \cup \{\neg\forall \vec{x} \psi(\vec{x})\}$ is $\langle \mathbf{W}_0, \mathbf{W}_0 \cup \mathbf{W}_2 \rangle$ consistent.

Now we apply Theorem 9.13; $S^\sharp \cup \{\neg\forall \vec{x} \psi(\vec{x})\}$ extends to S' so that $\langle S', \mathbf{W}_0, \mathbf{W}_0 \cup \mathbf{W}_2 \rangle$ is world-like, and hence is in $\mathcal{G}$. Call this Δ. We have $\Gamma \mathcal{R} \Delta$ because $S^\sharp \subseteq S'$ and $\mathbf{W}_2 \subseteq \mathbf{W}_0 \cup \mathbf{W}_2$. By item 5 of Definition 9.9, for some $\vec{w} \in \mathbf{W}_0 = \mathcal{D}(\Delta)$, $\neg\psi(\vec{w}) \in S'$, so by consistency, $\psi(\vec{w}) \notin S'$. Then by the induction hypothesis, $\mathcal{M}, \Delta \not\Vdash \psi(\vec{w})$. Hence $\mathcal{M}, \Gamma \not\Vdash \Box_{\mathsf{v}(\psi)} \psi(\vec{x})$ and, using FOBMK–A-4, $\mathcal{M}, \Gamma \not\Vdash \Box_X \psi(\vec{x})$, that is, $\mathcal{M}, \Gamma \not\Vdash \varphi$. ∎

Now strong completeness follows by the usual argument, somewhat fancied up.

Theorem 9.14 (Completeness) *Let S be a set of closed formulas and φ be a single closed formula, all in the basic language. If $S \not\vdash \varphi$ in the basic language then, in the binding modality varying domain canonical modality $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$, there is some $\Gamma \in \mathcal{G}$ such that $\mathcal{M}, \Gamma \Vdash \psi$ for every $\psi \in S$ but $\mathcal{M}, \Gamma \not\Vdash \varphi$.*

Proof Suppose $S \not\vdash \varphi$ in the basic language. Then using the terminology of Definitions 9.2 and 9.6, $S \cup \{\neg\psi\}$ is $\langle\emptyset, \emptyset\rangle$ consistent. Let $\mathbf{W}_0$ be a countable set of witness variables that also omits countably many witness variables. By Corollary 9.8, $S \cup \{\neg\psi\}$ is $\langle\mathbf{W}_0, \mathbf{W}_0 \cup \emptyset\rangle$ consistent. Then by Theorem 9.13, S extends to a set S' so that $\Gamma = \langle S', \mathbf{W}_0, \mathbf{W}_0 \cup \emptyset\rangle$ is world-like. $\Gamma \in \mathcal{G}$ and, by Theorem 9.13, $\mathcal{M}, \Gamma \Vdash \psi$ for every $\psi \in S$ but $\mathcal{M}, \Gamma \not\Vdash \varphi$. ■

10 Conclusion

Binding modalities machinery has a plausible interpretation, is straightforward to work with, and helps elucidate the *de re/de dicto* dichotomy. What remains is to say something about the connection between binding modalities and the the familiar necessity operator and, in particular, with the axiomatic approach that Kripke took in [13].

As noted earlier, Kripke used an axiom system that did not allow free variable occurrences in formulas appearing in a proof. To do this he worked with his version of the "closure" of a formula, defined as follows.

> "If A is a formula containing free variables, we define a *closure* of A to be any formula without free variables obtained by prefixing universal quantifiers and necessity signs, in any order, to A."

He then presented a standard classical first-order axiomatization using schemas, allowing free variable occurrences, except that universal instantiation had the form $\forall y(\forall x\varphi(x) \supset \varphi(y))$. Of course he also adopted appropriate modal axiom schemes, again allowing free variables. But all this was actually preliminary to his official axiom system. For this he took *all closures of instances of the schemes just mentioned*, together with modus ponens.

It is not hard to see that Kripke's use of his notion of closures corresponds to our two versions of axiom necessitation. As an illustrative example, consider the following, where $\varphi(x_1, x_2, x_3)$ is an instance of one of our axiom schemes other than FOBMK–A-2, and all free variables are shown. Using FOBMK–R-3, Axiom Necessitation 1, the following might occur in a proof in FOBMK.

$$\Box_{\{x_1\}}\Box_{\{x_1,x_2\}}\varphi(x_1, x_2, x_3)$$

Using Quantifier Absorption, Corollary 6.3, this is equivalent to the following.

$$\Box_{\{x_1\}}\forall x_2\Box_{\{x_1,x_2\}}\forall x_3\varphi(x_1, x_2, x_3)$$

And by Universal Generalization and Universal Instantiation, it is further equivalent to the following, which has no free variables.

$$\forall x_1\Box_{\{x_1\}}\forall x_2\Box_{\{x_1,x_2\}}\forall x_3\varphi(x_1, x_2, x_3)$$

Suppose we write $\Box\psi$ for $\Box_X\psi$ where X consists of all the variables occurring free in ψ. Then the formula above rewrites to the following.

$$\forall x_1\Box\forall x_2\Box\forall x_3\varphi(x_1, x_2, x_3)$$

In this the prefix has exactly the form of Kripke's closure.

We call attention to our rule FOBMK–R-4, which we called Axiom Necessitation 2. It covers formulas of the form $\forall x\varphi(x) \supset \varphi(y)$, which are exactly the ones that Kripke singled out as the source of provability problems. In our necessitation rule formulation we can conclude

$\Box_{X_1}\Box_{X_2}\ldots\Box_{X_n}(\forall x\varphi(x) \supset \varphi(y))$, *provided* $y \notin X_n$. Kripke didn't allow the free variable version of universal instantiation we use, but replaced it with $\forall y(\forall x\varphi(x) \supset \varphi(y))$. The prefixed universal quantifier corresponds directly to our condition that $y \notin X_n$.

As was noted in [6], the binding modalities formulation "is as expressive as traditional FOS4 since the binding modality $\Box_X F$ can be semantically encoded by $\Box\forall\vec{y}F$ where $\Box$ is the FOS4 modality and $\vec{y}$ is the list of all free variables of F that are not in X." Exact connections with Kripke's axiomatization require some delicacy however. In [13] Kripke observed that "all the laws of quantification theory—modified to admit the empty domain—hold." The empty domain must be brought in since $\forall x\varphi(x) \supset \varphi(y)$ is disallowed in favor of its closure. Such details need care, but are not fundamentally significant. Our point is, simply, that relationships between the binding modalities system and Kripke's system are easily seen. The important difference is that in Kripke's approach the complexities he adds are essentially proof theoretic devices to avoid the possibility of proving something undesirable, while with binding modalities the extra machinery has a direct semantic interpretation, enhances the natural expressiveness of the system, and relates to issues of long-standing philosophical interest.

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