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Reasoning About Games

Abstract. A mixture of propositional dynamic logic and epistemic logic that we call $\mathsf{PDL}+\mathsf{E}$ is used to give a formalization of Artemov's knowledge based reasoning approach to game theory, (KBR), [4, 5, 6, 7]. Epistemic states of players are represented explicitly and reasoned about formally. We give a detailed analysis of the Centipede game using both proof theoretic and semantic machinery. This helps make the case that $\mathsf{PDL}+\mathsf{E}$ can be a useful basis for the logical investigation of game theory.

Keywords: game theory, epistemic logic, propositional dynamic logic, centipede

1. Background

Game theory trys to predict or explain behavior of agents under a sequence of interactions. What agents do often depends on reasoning abilities and information about other agents. An agent may behave one way if the agent knows another agent is rational, yet behave differently if the agent doesn't know that. Sergei Artemov has developed a *knowledge-based* approach to games (KBR), [4, 5, 6, 7]. (These are not strictly linear—some things in earlier reports are overridden in later ones.) At the heart of Artemov's work is the idea that the game tree is only a partial specification of the game—*epistemic states* of players have a fundamental role too. Epistemic conditions are generally stated more informally, but are no less significant. Assuming common knowledge of player rationality, or merely mutual knowledge of player rationality, can lead to different outcomes with the same game tree.

We present a formal treatment of game tree + epistemic machinery. There is a semantics with epistemic states explicitly present, and a proof theory for reasoning about them. As a case study we apply the machinery to the well-known Centipede game previously analyzed by Aumann [8], and others. In order to make this paper relatively self-contained, we sketch what is needed from the combination of epistemic and dynamic logics. In particular, completeness results from [16, 17] will not be needed, only soundness results, and these are straightforward. The game theory side is a different thing, however, and it is recommended that [4, 5, 6, 7] be consulted for the motivation.

Presented by **Name of Editor**; *Received* December 1, 2005

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Papers in game theory are mixtures of formal and informal reasoning, with a certain amount of hand-waving as in most mathematics. When asserting something is so, a semi-formal argument is generally convincing. When asserting something is not so, more care may be needed because informal counter-examples might hide a crucial detail. We provide formal machinery, and apply it formally. We do not advocate that all arguments in this area must be formal—it would kill the subject. But it is a truism that a correct mathematical proof is one that *can be* formalized. For this to be applicable, formal machinery must exist. It is such formal machinery that this paper concentrates on, for knowledge based reasoning applied to game theory.

Our approach falls in the area known as *dynamic epistemic logic*, combining epistemic logics with logics of action. A standard reference in this area is [19], but the logic we use here does not appear in that book. We use a simple combination of two traditional logics, propositional dynamic logic, [10, 14] and epistemic logic [13]. This combination was applied to games in [18], but our approach is rather different. In [18], game states are also epistemic states, which is appropriate for questions investigated there. We examine knowledge based reasoning, and this leads us to think of a game state as made up of possibly many epistemic states. This provides machinery for modeling player uncertainty. We call the logics used here $\mathsf{PDL}+\mathsf{E}$. It is a family, rather than a single logic, because assumptions of varying strengths can be made concerning player knowledge, and also various 'cross' axioms can be assumed as to how the epistemic and the dynamic operators relate to each other. The interaction axioms were thoroughly investigated in [16, 17].

This paper derives from a technical report, [11]. In order to keep things at a reasonable size, most formal proofs have been omitted here, but all can be found in the report.

2. Logics

We begin with a brief discussion of epistemic logic, and of propositional dynamic logic. We then discuss their combination, which we call $\mathsf{PDL}+\mathsf{E}$.

2.1. Epistemic Logic

There is a finite set of *agents*, A, B, C, ... For each agent i there is a modal operator K_i, with K_iX read as: agent i knows X. Axiomatically each K_i is a normal modal operator, so there is a *knowledge necessitation* rule, from X conclude K_iX, and $K_i(X \supset Y) \supset (K_iX \supset K_iY)$ is valid. In addition there may be some or all of the following as axiom schemes.

E-1 $K_A X \supset X$, Factivity.

E-2 $K_A X \supset K_A K_A X$, Positive Introspection.

E-3 $\neg K_A X \supset K_A \neg K_A X$, Negative Introspection.

We will explicitly note when any of these three axiom schemes are needed, and will note their absence if that is significant.

Semantics is the familiar Kripke/Hintikka possible world version. A model is a structure $\langle \mathcal{G}, \mathcal{R}_A, \ldots, \Vdash \rangle$, where $\mathcal{G}$ is a non-empty set of epistemic states, $\mathcal{R}_A$ is a binary relation on $\mathcal{G}$ for each agent A, and $\Vdash$ is a relation between states and formulas: "at this state the formula is true." The $\Vdash$ relation meets the usual condition, $\Gamma \Vdash K_A X$ iff $\Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R}_A \Delta$. Factivity for agent A corresponds to requiring that $\mathcal{R}_A$ be reflexive, Positive Introspection corresponds to transitivity, and Factivity, Positive Introspection and Negative Introspection together corresponds to an equivalence relation. Other conditions are often considered, but they will not be needed here.

2.2. Propositional Dynamic Logic

Propositional dynamic logic (PDL) is a logic of non-deterministic *actions*—the action 'go to a store and buy milk' could be executed in many ways since a choice of store is unspecified. The formula $[\alpha]X$ is intended to express that X will be true after action α is executed, no matter how this is done. If α cannot be executed, $[\alpha]X$ is automatically true. The dual, $\langle \alpha \rangle X$, is read: there is at least one way of executing action α that leaves X true. If α cannot be executed $\langle \alpha \rangle X$ is false. For background and further information the on-line treatments [1, 9] are recommended. Also strongly recommended is [12]. Here is a brief summary of PDL.

There is a collection of *actions*. Actions are built up from arbitrarily given atomic actions. Suppose α and β are actions and A is a formula. Then $(\alpha; \beta)$ is a formula, representing action α followed by action β; $(\alpha \cup \beta)$ is a formula, representing a non-deterministic choice of α or β; α^* is an action, representing α repeated an arbitrary number of times, possibly 0; $A?$ is an action, representing a test for A.

Formulas are built up from atomic formulas in the usual way, with the additional condition: if α is an action and X is a formula then $[\alpha]X$ is a formula. $\langle \alpha \rangle X$ will be taken as an abbreviation for $\neg[\alpha]\neg X$.

A standard PDL axiom system is as follows. For rules there are *modus ponens* and *action necessitation*, from X conclude $[\alpha]X$. Then there are the following axiom schemes.

PDL-1 All tautologies (or enough of them)

PDL-2 $[\alpha](X \supset Y) \supset ([\alpha]X \supset [\alpha]Y)$

PDL-3 $[\alpha;\beta]X \equiv [\alpha][\beta]X$

PDL-4 $[\alpha \cup \beta]X \equiv ([\alpha]X \wedge [\beta]X)$

PDL-5 $[\alpha^*]X \equiv (X \wedge [\alpha][\alpha^*]X)$

PDL-6 $[A?]X \equiv (A \supset X)$

Finally, a version of induction can be captured by either an axiom scheme or a rule of inference—they are interderivable. They are as follows.

PDL-7 $[\alpha^*](X \supset [\alpha]X) \supset (X \supset [\alpha^*]X)$

PDL-8 From $X \supset [\alpha]X$ infer $X \supset [\alpha^*]X$

A semantics for PDL is a multi-modal Kripke structure with some additional conditions. A model is a structure $\langle \mathcal{G}, \mathcal{R}_\alpha \ldots, \Vdash \rangle$ where $\mathcal{G}$ is a non-empty set of states; $\mathcal{R}_\alpha$ is a binary accessibility relation between states, for each action α; and $\Vdash$ is a relation between states and formulas. The $\Vdash$ relation must satisfy the familiar Kripkean condition: $\Gamma \Vdash [\alpha]X$ iff $\Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R}_\alpha \Delta$. The special PDL machinery imposes relationships between the accessibility relations.

- $\mathcal{R}_{\alpha;\beta}$ is the relation product $\mathcal{R}_\alpha \circ \mathcal{R}_\beta$.
- $\mathcal{R}_{\alpha\cup\beta}$ is the relation $\mathcal{R}_\alpha \cup \mathcal{R}_\beta$.
- $\Gamma \mathcal{R}_{A?} \Gamma$ just in case $\Gamma \Vdash A$.
- $\mathcal{R}_{\alpha^*}$ is the reflexive and transitive closure of R_α (with respect to $\circ$).

2.3. PDL + E

Mono-modal versions of the axiomatics and semantics described in this section come from [16, 17]. Additional axioms special to particular games will be added later. These game axioms will not be axiom schemes, and so the resulting logics will not be closed under substitution—they will not be normal. This makes for difficulties when completeness and decidability are concerned, a primary consideration of [16, 17], but not a relevant issue here. Logics will not be investigated as such, but rather their applicability to particular problems. Soundness, relative to a semantics having to do with the game in question, will be easy to establish, and that is all that is needed for present purposes.

$$\begin{tikzcd} 1 \arrow[r, dashed, "\alpha"] \arrow[d, "i"'] & 2 \arrow[d, dashed, "i"] \\ 3 \arrow[r, "\alpha"'] & 4 \end{tikzcd}$$

(a) No Learning

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, dashed, "i"'] & 2 \arrow[d, "i"] \\ 3 \arrow[r, dashed, "\alpha"'] & 4 \end{tikzcd}$$

(b) Perfect Recall

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, "i"'] & 2 \arrow[d, dashed, "i"] \\ 3 \arrow[r, dashed, "\alpha"'] & 4 \end{tikzcd}$$

(c) Reasoning Ability

Figure 1. Commutative Diagrams

We start with the fusion of the logics from Sections 2.1 and 2.2. Axiomatically one simply combines the machinery of epistemic logic with that of PDL. Semantically one uses possible world models in which there are accessibility relations for each PDL action, and an accessibility relation for each agent. It is assumed that the PDL relations meet the conditions of Section 2.2, and each knowledge relation meets the general epistemic conditions of Section 2.1, and whichever combination of the reflexive, symmetric, transitive conditions are desired.

In addition, interaction conditions between PDL and epistemic machinery may be imposed. In [16, 17] three are considered. The first is a No Learning condition, given by the following axiom scheme, where i is any agent.

PDLE-1 $[\alpha]K_iX \supset K_i[\alpha]X$ (No Learning)

This says that if an agent knows X after action α is performed, the agent already knew that X would be true after α, so executing the action brought no new knowledge. Semantically, the axiom corresponds to the diagram given in Figure 1(a). The diagram should be read as follows. Assume there are arbitrary states 1, 3, and 4, with 3 accessible from 1 under the epistemic K_i accessibility relation, and 4 accessible from 3 under the PDL α accessibility relation. Then there is a state 2, accessible from 1 under the PDL α relation, with 4 accessible from 2 under the epistemic K_i relation. In short, given states 1, 3, and 4 meeting the accessibility conditions shown by the two solid arrows, there is a state 2 with accessibilities shown by the two dashed arrows, so that the diagram commutes.

The next connecting condition is that of Perfect Recall. Axiomatically it is given as follows.

PDLE-2 $K_i[\alpha]X \supset [\alpha]K_iX$ (Perfect Recall)

The corresponding semantic condition is shown in Figure 1(b), which is read similarly to the previous case.

The third condition is called *Church-Rosser* in [16, 17], but we prefer to call it the Reasoning Ability condition. Axiomatically it is the following.

PDLE-3 $\langle\alpha\rangle K_i X \supset K_i \langle\alpha\rangle X$ (Reasoning Ability)

Informally this says that if an agent could know X after action α, the agent is able to figure that out and so knows now that X could be the case after α. The semantic condition corresponding to this is given in figure Figure 1(c).

The exact connections between the three axiomatic conditions just discussed, and the corresponding semantic conditions, is a bit tricky. Much of it comes down to whether one takes all conditions as axiom schemes or as particular axioms—equivalently, whether or not there is closure under substitution. Consider Perfect Recall, PDLE-2, as a representative example. We will *not* assume it as a general scheme, but for a particular action α. As such, it is simple to verify that Perfect Recall for α evaluates to true at every world of any model meeting the Perfect Recall semantic condition for α. This is all that is needed for our intended applications. If one takes the axioms as schemes and asks about completeness issues, things become complicated. It is shown in [16, 17] that if the epistemic part is strong enough, S5, the PDL + E combination collapses to the PDL part. It is also shown that the heart of the problem is the PDL notion of *test*. If one does not have tests, or if tests are restricted to atomic, this collapse does not happen. In fact, tests in game formulations will not be used here, and all work will be with particular versions of the conditions considered above and so there will be no closure under substitution. Completeness considerations are not important, only soundness ones, and these hold with routine verifications.

2.4. Common Knowledge

A common knowledge operator, $\mathcal{C}$, is introduced in one of the usual ways. There is an 'everybody knows' operator: EX abbreviates $K_A X \wedge K_B X \wedge K_C X \wedge \ldots$. And then there are the common knowledge assumptions.

CK-1 (Axiom Scheme) $\mathcal{C}X \supset E(X \wedge \mathcal{C}X)$

CK-2 (Rule) $\dfrac{X \supset E(Y \wedge X)}{X \supset \mathcal{C}Y}$

This formulation serves for both common knowledge and common belief. One can prove $\mathcal{C}X \supset X$ if agents satisfy Factivity, E-1, but some useful items are provable without assumptions of Factivity, Positive Introspection, or Negative Introspection, and so apply even to very weak versions of belief.

These include $\mathcal{C}X \supset \mathcal{C}\mathcal{C}X$, $\mathcal{C}X \supset \mathcal{C}K_iX$, $\mathcal{C}(X \supset Y) \supset (\mathcal{C}X \supset \mathcal{C}Y)$, and the fact that $\mathcal{C}$ itself obeys the Necessitation Rule, from X conclude $\mathcal{C}X$. There are also connections with the special assumptions given earlier.

Proposition 2.1.

1. *If each agent satisfies* Reasoning Ability, PDLE-*3, for action* α *then analogous results hold for* E *and* $\mathcal{C}$*.*
   (a) $\langle\alpha\rangle EX \supset E\langle\alpha\rangle X$
   (b) $\langle\alpha\rangle\mathcal{C}X \supset \mathcal{C}\langle\alpha\rangle X$
2. *If each agent satisfies* No Learning, PDLE-*1 for action* α *then analogous results hold for* E *and* $\mathcal{C}$*.*
   (a) $[\alpha]EX \supset E[\alpha]X$
   (b) $[\alpha]\mathcal{C}X \supset \mathcal{C}[\alpha]X$
3. *If each agent satisfies* Perfect Recall, PDLE-*2 for action* α *then we have the following, for every* n*.*
   (a) $E^n[\alpha]X \supset [\alpha]E^nX$
   (b) $\mathcal{C}[\alpha]X \supset [\alpha]E^nX$

The extension of *3b* from E^n to $\mathcal{C}$ does not seem to follow (though we have no proof of this). E^n suffices for results about particular finite games, but for results about families of games one needs $\mathcal{C}$ and so, when appropriate, we will assume the following as an additional condition.

CK-3 $\mathcal{C}[\alpha]X \supset [\alpha]\mathcal{C}X$ (Extended Perfect Recall)

3. PDL + E For Games

In this section we give axioms general enough to apply to many games; semantics is in Section 5. Following Artemov, [4, 5, 6, 7], *Knowledge of the Game Tree* includes possible moves, payoffs, etc., all of which should be common knowledge. We leave payoffs until later, and concentrate now on 'general structure.' To this end we adopt some *specific axioms*—these are not axiom schemes since they refer to particular players. It is assumed that our necessitation rules apply to these axioms, necessitation both for knowledge operators and for action operators.

3.1. General Game Tree Knowledge

Games can have any number of players. The ones in this paper have two, and so things are formulated for this situation only, though generalizations are straightforward. The two propositional letters, A and B, have the intended meaning that A is true if it is the turn of agent A to move, and B is true if it is the turn of agent B to move. In addition to general $\mathsf{PDL} + \mathsf{E}$ axiom schemes and rules there are the following specific axioms. They say that exactly one player is to move, and everybody knows whose move it is. They are in a family of axioms representing *knowledge of the game*; accordingly they are numbered in a KG sequence.

KG-1 $A \vee B$	KG-4 $B \supset K_B B$
KG-2 $\neg(A \wedge B)$	KG-5 $A \supset K_B A$
KG-3 $A \supset K_A A$	KG-6 $B \supset K_A B$

Each player has a choice of moves, say these are represented by propositional letters m_1, m_2, …, m_k. At each turn the appropriate player picks exactly one move, so there are the following assumptions.

KG-7 $\mathsf{m}_1 \vee \mathsf{m}_2 \vee \ldots \vee \mathsf{m}_k$

KG-8 $\neg(\mathsf{m}_1 \wedge \mathsf{m}_2)$, $\neg(\mathsf{m}_1 \wedge \mathsf{m}_3)$, $\neg(\mathsf{m}_2 \wedge \mathsf{m}_3)$, …

Each m_i represents a decision by a player. In addition there are *transitions* from one state of the game to another state. Some choices by players end the game, other choices trigger these transitions. In this paper transitions are dynamic operators, distinct from any of the m_i, though there is certainly a connection between player choices and transitions—choices that do not end play trigger transitions. Let us assume α_1, α_2, …, α_n are the atomic transitions available, represented by atomic dynamic operators.

It is assumed players alternate, and so a transition to a new active state of the game switches the player whose turn it is to move. For each game transition α_i we assume the following.

KG-9 $A \supset [\alpha_i]B$

KG-10 $B \supset [\alpha_i]A$

We assume that each player knows if transition α_i is possible or not. The formula $\langle\alpha_i\rangle\top$ asserts that an α_i transition is possible ($\top$ is truth), while the formula $[\alpha_i]\bot$, equivalently $\neg\langle\alpha_i\rangle\top$, asserts that an α_i transition is impossible ($\bot$ is falsehood).

$\mathsf{KG\text{-}11}$ $[\alpha_i]\bot \supset K_A[\alpha_i]\bot$ $\mathsf{KG\text{-}13}$ $\langle\alpha_i\rangle\top \supset K_A\langle\alpha_i\rangle\top$

$\mathsf{KG\text{-}12}$ $[\alpha_i]\bot \supset K_B[\alpha_i]\bot$ $\mathsf{KG\text{-}14}$ $\langle\alpha_i\rangle\top \supset K_B\langle\alpha_i\rangle\top$

Some game states may be *terminal* in the sense that all plays end the game—no transitions to further states are possible. Then $\langle\alpha_1 \cup \ldots \cup \alpha_n\rangle\top$ asserts that some atomic transition is possible, while $[\alpha_1 \cup \ldots \cup \alpha_n]\bot$ asserts that no atomic transition can be made—one is at a terminal game state. We refer to a game state as terminal, or not, throughout the paper.

PROPOSITION 3.1. *The following are provable.*

1. $[\alpha_1 \cup \ldots \cup \alpha_n]\bot \supset K_a[\alpha_1 \cup \ldots \cup \alpha_n]\bot$, *for* $a = A$ *and* $a = B$
2. $\langle\alpha_1 \cup \ldots \cup \alpha_n\rangle\top \supset K_a\langle\alpha_1 \cup \ldots \cup \alpha_n\rangle\top$, *for* $a = A$ *and* $a = B$

Incidentally, KG-11 and KG-12 follow easily from the No Learning condition, PDLE-1, and KG-13 and KG-14 likewise follow from the Reasoning Ability condition, PDLE-3. Since one doesn't always want to assume these powerful conditions, it is reasonable to take them as separate assumptions.

3.2. Rationality Considerations

The last of our general game principles is the only one that is non-trivial. While it appears in one form or another in the works of a number of authors, it was given special emphasis in [4, 5, 6, 7]. It says that a player who is rational and who knows what his or her best move is, given the limitations imposed by the knowledge the player possesses, will play that best move. This presupposes that best moves must exist. Artemov has shown that under very broad conditions each player in a game, when it is his turn to move, must have a move that is best possible given what the player knows at that point of the game. This is called the *best known move*, and is represented formally by a propositional letter. Suggestively, this propositional letter is written in a special format, $\mathsf{kbest}_A(\mathsf{m})$, intended to express that move m is the best known move for player A. Rationality for a player is also represented by a propositional letter: ra_A or ra_B, for rationality of A or B respectively. The fundamental *rationality conditions* assert that a player who is rational and is aware of what his or her best known move is, will play it.

$\mathsf{RC\text{-}A}$ $(A \wedge K_A\mathsf{kbest}_A(\mathsf{m}_i) \wedge \mathsf{ra}_A) \supset \mathsf{m}_i$

$\mathsf{RC\text{-}B}$ $(B \wedge K_B\mathsf{kbest}_B(\mathsf{m}_i) \wedge \mathsf{ra}_B) \supset \mathsf{m}_i$

3.3. Backward Induction

So-called *backward induction* plays a central role in the analysis of a number of games. Backward induction concludes X is true throughout a game by showing X is true at terminal game states, and also showing X is true at a state provided some transition from that state takes the game to another state at which X is true. Thus one works backward from terminal states to encompass the entire game tree. We give a simple schematic version of backward induction, using the machinery introduced so far.

Theorem 3.2 (Backward Induction Derived Rule). *Let $\alpha_1, \ldots, \alpha_n$ be all the atomic game transitions, and let X be some formula. Assume the following.*

1. $\langle(\alpha_1 \cup \ldots \cup \alpha_n)^*\rangle[\alpha_1 \cup \ldots \cup \alpha_n]\bot$
2. $[\alpha_1 \cup \ldots \cup \alpha_n]\bot \supset X$
3. $\langle\alpha_1 \cup \ldots \cup \alpha_n\rangle X \supset X$

Then X follows.

Recall that $[\alpha_1 \cup \ldots \cup \alpha_n]\bot$ asserts one is at a terminal game state, so condition 1 asserts a terminal state can be reached through some sequence of atomic transitions. Condition 2 asserts the formula X is true at terminal states. Finally condition 3 asserts that if some atomic transition takes us to a state at which X is true, then X is true at the original state. These are the conditions for backward induction, stated less formally earlier.

Proof. By condition 3, $\neg X \supset [\alpha_1 \cup \ldots \cup \alpha_n]\neg X$. By the inference rule PDL-8, we have $\neg X \supset [(\alpha_1 \cup \ldots \cup \alpha_n)^*]\neg X$. By standard modal reasoning, and using conditions 1 and 2, we have the following.

$$\begin{aligned}
[(\alpha_1 \cup \ldots \cup \alpha_n)^*]\neg X \supset\ & [(\alpha_1 \cup \ldots \cup \alpha_n)^*]\neg X \\
& \quad \wedge \langle(\alpha_1 \cup \ldots \cup \alpha_n)^*\rangle[\alpha_1 \cup \ldots \cup \alpha_n]\bot \\
\supset\ & \langle(\alpha_1 \cup \ldots \cup \alpha_n)^*\rangle\,(\neg X \wedge [\alpha_1 \cup \ldots \cup \alpha_n]\bot) \\
\supset\ & \langle(\alpha_1 \cup \ldots \cup \alpha_n)^*\rangle(\neg X \wedge X) \\
\supset\ & \bot
\end{aligned}$$

This combines with the result above to give us $\neg X \supset \bot$, or X. ∎

4. Formal $\mathsf{PDL} + \mathsf{E}$ Proofs

Payoffs have not entered into the discussion so far. We do not formalize these directly since it can be complicated mixing a modal logic with elementary arithmetic. Instead we assume payoffs induce general strategy principles, and we formulate these strategy principles using $\mathsf{PDL}+\mathsf{E}$ machinery. It is simplest to discuss specific games, and so we turn to the well-known Centipede game. We begin with a standard presentation, giving the extensive form diagram and an informal analysis. The usual conclusion is that if there is common knowledge of player rationality, the first move in the game will be down. Not surprisingly, this will come out of the present formalization too, but the analysis of Centipede is only part of the point here. We present a methodology that should be applicable to other games as well; consequently we proceed in gradual stages to make things as transparent as possible.

4.1. The Centipede Game Tree

Figure 2 displays the extensive form diagram for a five-move version of the Centipede game, a game which in its 100 move version first appeared in [15]. Play starts at the upper left, alternating between A and B. Each player can choose to move right or move down. Payoffs shown are for A and B in that order. The payoffs are arranged so that if a player does not terminate the game (by moving down), and the other player ends the game on the next move, the first player receives a little less than if that player had simply terminated the game directly.

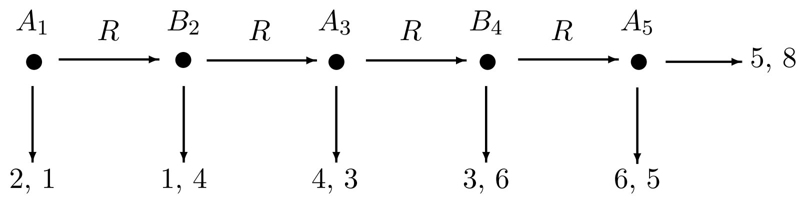


Figure 2. Five-Move Centipede

The game tree does not take knowledge into consideration. Nowhere is it represented that agent A knows, or does not know, that B is rational, for instance. It can be seen as a kind of PDL model in which transitions to the right are shown, taking players from one game state to another, but without any explicit representation of knowledge it is not a model for $\mathsf{PDL}+\mathsf{E}$. We postpone the semantic introduction of epistemic states, and work entirely proof-theoretically for this section.

We can reason informally about the game as follows. If the game were to start at the right-hand node, it is obvious that A is best off moving down, so if A is rational, the move will be down. If the game were to start at the node second from the right, B could reason: at the next node, if A is rational the move will be down, so I am better off moving down right now since I will get 6 instead of 5. Therefore at this node, if B is rational and if B knows A is rational, B will move down. This reasoning can be repeated, as a *backward induction*, leading to the conclusion that if the rationality of everybody is common knowledge (or at least sufficiently so), then A will move down at the start. The game shown in Figure 2 is one of a family—the pattern can be continued to arbitrary length, and we should keep the whole family in mind in what follows.

4.2. Centipede Game Tree Knowledge

There are two players, A and B, and axioms KG-1 – KG-6 from Section 3.1 are directly adopted, but are relabeled KGcent-1 – KGcent-6 to be uniform with axioms introduced below. At each node one of two moves can be selected by a player, right and down. Instead of m_1 and m_2, we represent these more suggestively by ri and do. Then KG-7 and KG-8 specialize to the following.

KGcent-7 $\mathsf{ri} \vee \mathsf{do}$

KGcent-8 $\neg(\mathsf{ri} \wedge \mathsf{do})$

There is one atomic transition in the Centipede game, denoted by R, and representing a transition to the next active node to the right. Now KG-9 and KG-10 specialize to the following.

KGcent-9 $A \supset [R]B$

KGcent-10 $B \supset [R]A$

In a similar way KG-11 – KG-14 specialize to the following.

KGcent-11 $[R]\bot \supset K_A[R]\bot$	KGcent-13 $\langle R\rangle\top \supset K_A\langle R\rangle\top$
KGcent-12 $[R]\bot \supset K_B[R]\bot$	KGcent-14 $\langle R\rangle\top \supset K_B\langle R\rangle\top$

In Figure 2 it is obvious that no matter at which of the five nodes one is, a sequence of transitions is possible that will take one to node 5, the terminal state, namely a sequence of moves to the right. Recall, the Centipede game displayed is one of many, since the length need not be 5 but could be anything. An axiom is needed to cover the general situation—a terminal state is always possible to reach.

KGcent-15 $\langle R^*\rangle[R]\bot$

4.3. Rationality and Strategy Considerations

Rationality conditions $\mathsf{RC\text{-}A}$ and $\mathsf{RC\text{-}B}$ from Section 3.2 now specialize.

$\mathsf{RCcent\text{-}A}$ $(A \wedge K_A\mathsf{kbest}_A(\mathsf{do}) \wedge \mathsf{ra}_A) \supset \mathsf{do}$
$(A \wedge K_A\mathsf{kbest}_A(\mathsf{ri}) \wedge \mathsf{ra}_A) \supset \mathsf{ri}$

$\mathsf{RCcent\text{-}B}$ $(B \wedge K_B\mathsf{kbest}_B(\mathsf{do}) \wedge \mathsf{ra}_B) \supset \mathsf{do}$
$(B \wedge K_B\mathsf{kbest}_B(\mathsf{ri}) \wedge \mathsf{ra}_B) \supset \mathsf{ri}$

We also assume that players do not abruptly become irrational, rationality persists.

$\mathsf{RPcent\text{-}A}$ $\mathsf{ra}_A \supset [R]\mathsf{ra}_A$

$\mathsf{RPcent\text{-}B}$ $\mathsf{ra}_B \supset [R]\mathsf{ra}_B$

The logical machinery is inadequate to represent numerical payoffs. Instead we extract from the game formulation general statements that follow from the payoff information, but that can be formulated in logical terms. For the Centipede game we have been displaying a five-move version, but the game could be of any length and certain strategy assumptions would still apply. The first pair of conditions below say that if the game has reached the terminal node (node A_5 in Figure 2), down is the best move for the active player, and this is obvious so it is also the best *known* move. In the five-move game the last player to play is A, but could be either depending on the length of the game. Recall that $[R]\bot$ distinguishes the terminal node in the Centipede game. We call these *endgame strategy* axioms.

$\mathsf{EScent\text{-}A}$ $A \supset ([R]\bot \supset \mathsf{kbest}_A(\mathsf{do}))$

$\mathsf{EScent\text{-}B}$ $B \supset ([R]\bot \supset \mathsf{kbest}_B(\mathsf{do}))$

The final conditions we call *midgame strategy* axioms. These conditions apply to the cases where play is not at the terminal node, and so a transition to the right is possible, after which there is still at least one more move. Inspection of Figure 2 shows that at any non-terminal node N, if the play is right and then down, the active player at node N will receive less than if down were played directly. So if the active player at node N somehow knows that a play of down might be made if he plays right, then the player's known best move at N is down. Likewise if the play is right and then right, the active player will receive more, no matter how the play goes afterward, than would be the case if he played down. So if the player whose turn it is at node N knows that a move of right must be made if he moves right, then right is his known best move at N. This gives us the next two axiom pairs.

MScent-A $A \supset (K_A\langle R\rangle\mathsf{do} \supset \mathsf{kbest}_A(\mathsf{do}))$
$(A \wedge \langle R\rangle\top) \supset (K_A[R]\mathsf{ri} \supset \mathsf{kbest}_A(\mathsf{ri}))$

MScent-B $B \supset (K_B\langle R\rangle\mathsf{do} \supset \mathsf{kbest}_B(\mathsf{do}))$
$(B \wedge \langle R\rangle\top) \supset (K_B[R]\mathsf{ri} \supset \mathsf{kbest}_B(\mathsf{ri}))$

A remark about the two axioms having to do with moves to the right. A 'pre-condition' of $\langle R\rangle\top$ is included because $[R]\mathsf{ri}$ is trivially true at a terminal node, but the conclusions of the two axioms conflict with those of EScent-A and EScent-B. The present axioms are meant to be applicable at midgame nodes only, and the presence of $\langle R\rangle\top$ restricts things to these nodes. There isn't a corresponding condition in the two axioms having to do with moves down because it isn't needed. $\langle R\rangle\mathsf{do}$ implies $\langle R\rangle\top$ and so, within a knowledge condition, terminal nodes are implicitly ruled out.

4.4. Consequences

Let ra abbreviate $\mathsf{ra}_A \wedge \mathsf{ra}_B$, so that ra is a general rationality assertion. There is a formal proof of $\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do}$, for Centipede of any length—common knowledge that both players are rational implies common knowledge that the move is down. We present the general outline, with only a few proofs displayed. Full proofs can be found in [11]. The assumptions that will be used in this section are the following:

- The general PDL + E axiom schemes and rules from Section 2.3;
- The Centipede game tree knowledge axioms KGcent-1 – KGcent-15 from Section 4.2;
- The Centipede rationality and strategy axioms from Section 4.3, RCcent-A, RCcent-B, RPcent-A, RPcent-B, EScent-A, EScent-B, MScent-A, and MScent-B.

The cross-conditions PDLE-3, Reasoning Ability, and CK-3, Extended Perfect Recall, play a role, and are mentioned explicitly. Remarkably, E-1, Factivity, E-2, Positive Introspection, and E-3, Negative Introspection are never needed.

The overall plan is to establish results that allow application of Backward Induction, Theorem 3.2. Our first result says that if the game is at a terminal node, and both players are rational, the move will be down. It follows easily that this result is common knowledge.

Proposition 4.1. *$[R]\bot \supset (\mathsf{ra} \supset \mathsf{do})$, and hence also $[R]\bot \supset (\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do})$.*

If the game is at an intermediate node, everybody is rational, and it is common knowledge that after a transition right the next move might be down, then a down move will be made.

PROPOSITION 4.2. *The following are provable.*

1. $(A \wedge \mathsf{ra}_A \wedge \mathcal{C}\langle R\rangle \mathsf{do}) \supset \mathsf{do}$
2. $(B \wedge \mathsf{ra}_B \wedge \mathcal{C}\langle R\rangle \mathsf{do}) \supset \mathsf{do}$
3. $(\mathsf{ra} \wedge \mathcal{C}\langle R\rangle \mathsf{do}) \supset \mathsf{do}$

Next, a few utility items whose proof uses Proposition 4.2.

PROPOSITION 4.3.

1. *Assume CK-3, Extended Perfect Recall. Then* $\mathcal{C}\mathsf{ra} \supset [R]\mathcal{C}\mathsf{ra}$.
2. *Assume PDLE-3, Reasoning Ability. Then* $\mathcal{C}\mathsf{ra} \supset (\langle R\rangle \mathcal{C}\mathsf{do} \supset \mathcal{C}\mathsf{do})$.

And now what amounts to the induction step of Backwards Induction.

PROPOSITION 4.4. *Assume CK-3, Extended Perfect Recall, and PDLE-3, Reasoning Ability. Then* $\langle R\rangle(\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do}) \supset (\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do})$.

PROOF.

$$\begin{array}{rcll}
\langle R\rangle(\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do}) & \equiv & \langle R\rangle(\neg\mathcal{C}\mathsf{ra} \vee \mathcal{C}\mathsf{do}) & \\
 & \equiv & \langle R\rangle\neg\mathcal{C}\mathsf{ra} \vee \langle R\rangle\mathcal{C}\mathsf{do} & \\
 & \equiv & \neg[R]\mathcal{C}\mathsf{ra} \vee \langle R\rangle\mathcal{C}\mathsf{do} & \\
 & \supset & \neg\mathcal{C}\mathsf{ra} \vee \langle R\rangle\mathcal{C}\mathsf{do} & \text{Proposition 4.3, part 1} \\
 & \equiv & \mathcal{C}\mathsf{ra} \supset \langle R\rangle\mathcal{C}\mathsf{do} & \\
 & \supset & \mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do} & \text{Proposition 4.3, part 2}
\end{array}$$

■

Finally, the result we have been aiming at.

THEOREM 4.5. *Assume CK-3, Extended Perfect Recall, and PDLE-3, Reasoning Ability. Then* $\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do}$.

PROOF. Axiom KGcent-15 says $\langle R^*\rangle[R]\bot$. Proposition 4.1 says $[R]\bot \supset (\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do})$. And Proposition 4.4 says $\langle R\rangle(\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do}) \supset (\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do})$. Since R is the only atomic transition, Theorem 3.2 yields $\mathcal{C}\mathsf{ra} \supset \mathcal{C}\mathsf{do}$. ■

E-1, Factivity, was not used. This means the result holds under the assumption that one is modeling player belief rather than knowledge. This should not be surprising; after all, play is based on what one believes to be the case—what the full situation 'actually' is may be unobtainable. E-2, Positive Introspection, and E-3, Negative Introspection, were also not used, which is somewhat curious given the standard assumption of S5 knowledge. Also PDLE-1, No Learning, was not used, but this is of lesser significance.

5. $\mathsf{PDL} + \mathsf{E}$ Semantics

Extensive form game trees have no machinery to keep track of what players know or do not know at various states. It is time to bring this in. Whimsically expressed, nodes of a game tree are not featureless dots, but are composed of the epistemic states of the players. From here on, *game nodes* or *game states* are nodes as seen in the usual extensive form game diagrams, while *epistemic states* are possible worlds in the Hintikka/Kripke epistemic sense of Section 2.3. Each game node has associated with it an epistemic model, and we refer to the epistemic states of this model as being *of* or *in* the game node. A game tree with its epistemic states displayed is an *augmented* game tree.

5.1. Augmented Game Tree Examples

Suppose a game tree contains the fragment shown in Figure 3, where only two game nodes are shown. On the left A is to play, on the right B, and there is a transition α, from left to right.

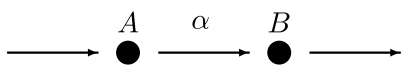
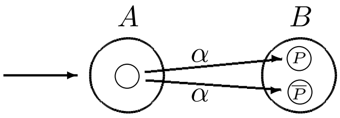


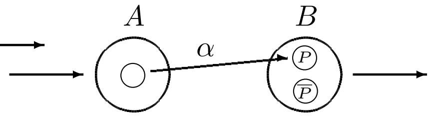
Figure 3. Game Tree Fragment, GTF

Suppose that at the left node A has no uncertainty—for every formula Z either $K_A Z$ or $K_A \neg Z$. Suppose also that at the right node B is uncertain of the status of some proposition P. In Figure 4(a) we have expanded the 'dots' of Figure 3 to reflect these conditions. In this *augmented game tree* the left game node contains one epistemic state and the right two. On the left we have only displayed things appropriate to A and on the right we have only displayed things appropriate to B. More could be shown, but it makes diagrams hard to read and is not relevant for this example. It is assumed that the epistemic state of the left game node is accessible from itself, with respect to A's accessibility relation, and for the right game node the two epistemic states are mutually accessible, including from themselves, with respect to B's accessibility relation. (That is, $\mathsf{S5}$ knowledge is assumed in both cases.) In one of B's epistemic states P is true, and in one P is false, reflecting the uncertainty B has concerning P. The lack of uncertainty possessed by A is reflected by the single epistemic state in the left game node.

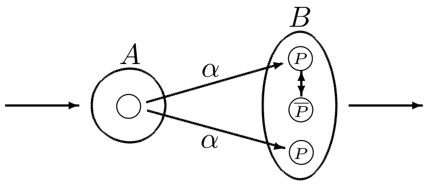
Note that there are two transition arrows labeled α in Figure 4(a). This



(a) First Augmentation



(b) Second Augmentation



(c) Third Augmentation

Figure 4. GTF With Epistemic Augmentations

does not mean two different moves are available from the left game node, both called α. Rather, one move α is available, but there is some question about what epistemic state we will find B in after the move is made.

We now follow the Section 2 rules for evaluation of formulas at states of Figure 4(a). At both of the epistemic states for B in the right game node, $K_B P$ is false, since there is an epistemic state in which P itself is false, and hence B does not know P. Likewise $K_B \neg P$ is false in both states. Hence at both states of the right game node we have $\neg K_B P$ and $\neg K_B \neg P$. Then since all α transitions from the single epistemic state in the left node lead to states in which both of these formulas are true, in the epistemic state of the left game node we have the truth of $[\alpha]\neg K_B P$ and $[\alpha]\neg K_B \neg P$. And since there is only one epistemic state for A in the left game node, we have $K_A[\alpha]\neg K_B P$ and $K_A[\alpha]\neg K_B \neg P$ both true at it. On the other hand, we also have $\langle\alpha\rangle P$ and $\langle\alpha\rangle\neg P$ true at the epistemic state in the left game node, and hence also $K_A\langle\alpha\rangle P$ and $K_A\langle\alpha\rangle\neg P$. More colloquially, at the epistemic state in the left game node A knows that a transition of α could leave P true or leave it false, and so A is uncertain of the effect of the transition on P. But also A knows that after a transition of α, B will be uncertain about P.

Figure 4(b) shows a modification of Figure 4(a), one of the transition arrows is missing. In this, at the epistemic state in the left game node, it is still the case that $K_A[\alpha]\neg K_B P$ and $K_A[\alpha]\neg K_B \neg P$, and it is the case that $K_A\langle\alpha\rangle P$, but it is no longer true that $K_A\langle\alpha\rangle\neg P$. Instead $K_A[\alpha]P$.

Figure 4(c) shows yet another augmentation of Figure 3. Now the right game node contains three epistemic states for B, with the top two mutually accessible, and the bottom one not accessible from either of the top two.

At both of the top two epistemic states for B, $\neg K_B P$ is true, while at the bottom one $K_B P$ is true. We leave it to you to check that at the epistemic state in the left game node, $K_A[\langle\alpha\rangle K_B P \wedge \langle\alpha\rangle\neg K_B P \wedge \neg\langle\alpha\rangle K_B\neg P]$ is true.

5.2. Centipede Augmented

The examples of Section 5.1 are not related to any particular game of interest. They are based on the game tree fragment of Figure 3, which is quite generic. Now we make use of semantic machinery to prove that, in Theorem 4.5, mutual knowledge of rationality is not enough. To keep things simple we use a three-move version of the game, shown in Figure 5. An

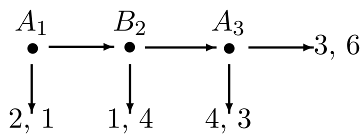


Figure 5. Three-Move Centipede

augmented version of this game is shown in Figure 6, with epistemic states shown for both players. Since the diagram is rather complicated, we explain how to read it. First, the three game nodes of Figure 5 are expanded, and labeled G1, G2, and G3. Each game node now has internal structure, each with three epistemic states labeled E1, E2, and E3. We will, in effect, use coordinates to refer to particular epistemic states, as in (G1, E3) for game node G1, epistemic state E3, for instance. The epistemic states for each game node have accessibility relations defined on them, for each player. These are represented by ellipses, and happen to be the same for each game node. Thus, epistemic state E1 is related only to itself for B. Epistemic states E1 and E2 are related to each other and to themselves for A. Similarly E2 and E3 are related to each other and to themselves for B. Finally E3 is related only to itself for A. Accessibility relations for each player are equivalence relations, and hence S5 knowledge is assumed for each player.

Transitions to the right are shown by arrows. The reader may check that the diagram satisfies all the conditions discussed in Section 2.3, No Learning, Perfect Recall, and Reasoning Ability, for each player. For instance, for player A there is an epistemic arrow from (G1, E1) to (G1, E2), and a game arrow from (G1, E2) to (G2, E2). But there is also a game arrow from (G1, E1) to (G2, E1) and an epistemic arrow from (G2, E1) to (G2, E2), filling out the square shown in Figure 1(a) for the No Learning condition.

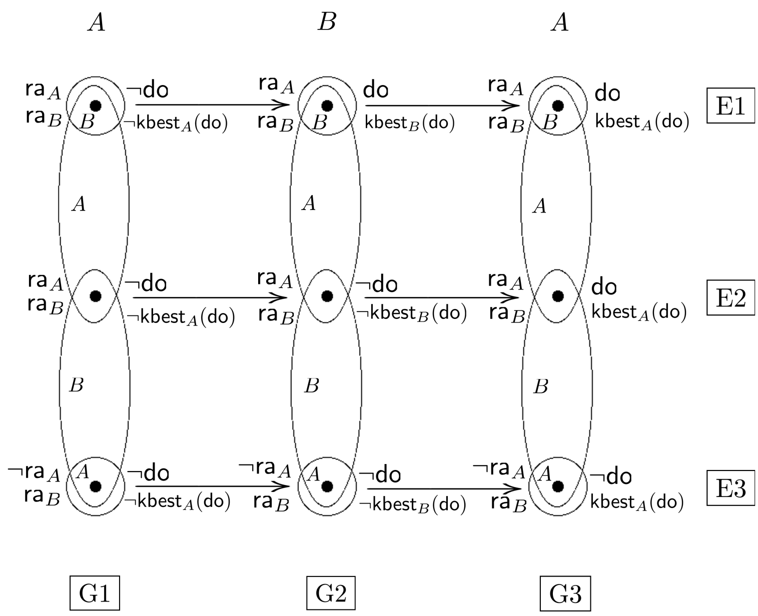


Figure 6. An Augmented Three-Move Centipede

At each epistemic state truth or falsity for various atomic propositions is shown. For instance, at epistemic state (G1, E1), ra_A and ra_B are both true, both players are rational at this state. Also do is false, that is, player A does not choose to move down. Finally $\mathsf{kbest}_A(\mathsf{do})$ is also false. It is easy to verify that at this state A knows the rationality of both A and B, as does B, but A does not know that B knows A is rational. This latter is the case because, at epistemic state (G1, E3) A is not rational; B cannot distinguish between (G1, E2) and (G1, E3) so at (G1, E2) B does not know A is rational, and A cannot distinguish between (G1, E1) and (G1, E2), so at (G1, E1) A does not know that B knows A is rational.

We assume A is true at each epistemic state of G1 and of G3, and B is true at each epistemic state of G2. It follows easily that each of the axioms KGcent-1 through KGcent-6 is true at each of the nine epistemic states of the model, that is, these axioms are *valid* in the model.

It is also assumed that ri is true at exactly the epistemic nodes where $\neg\mathsf{do}$ is displayed. It follows that both of the axioms KGcent-7 and KGcent-8 are valid in the model.

Each of the arrows shown is implicitly labeled R, and validity of axioms KGcent-9 through KGcent-15 is easily checked.

It is assumed in the diagram that $\mathsf{kbest}_A(\mathsf{ri})$ is true at exactly the nodes at which $\mathsf{kbest}_A(\mathsf{do})$ is false, and similarly for $\mathsf{kbest}_B(\mathsf{ri})$. It follows that each of RCcent-A and RCcent-B is valid. Consider, for example, $(A \wedge K_A \mathsf{kbest}_A(\mathsf{do}) \wedge \mathsf{ra}_A) \supset \mathsf{do}$. It is true at each of (G3, E1) and (G3, E2) because do is true, and at (G3, E3) because ra_A is false. It is true at each of (G2, E1), (G2, E2) and (G2, E3) because A is false at each. And it is true at each of (G1, E1), (G1, E2) and (G1, E3) because $K_A \mathsf{kbest}_A(\mathsf{do})$ is false.

Axioms RPcent-A and RPcent-B are easily seen to be valid, as are EScent-A and EScent-B. Finally there are axioms MScent-A and MScent-B. At all epistemic states of G3 MScent-A is true because $\langle R \rangle \mathsf{do}$ is false and hence so is $K_A \langle R \rangle \mathsf{do}$. At all epistemic states of G2 MScent-A is trivially true because A is false. Finally we consider G1. At state (G1, E2) $\langle R \rangle \mathsf{do}$ is false, because do is false at (G2, E2), and so $K_A \langle R \rangle \mathsf{do}$ is false at both epistemic states (G1, E1) and (G1, E2), which are indistinguishable for A. Likewise $\langle R \rangle \mathsf{do}$ is false at (G1, E3) because do is false at (G2, E3), and so $K_A \langle R \rangle \mathsf{do}$ is false at (G1, E3). It follows that MScent-A is true at the three epistemic states of G1, and hence is valid in the model. Validity of MScent-B is checked similarly.

Thus every axiom for Centipede, given in Section 4.2, is valid in the model of Figure 6, as are the No Learning, Perfect Recall, and Reasoning Ability conditions from Section 2.3. In addition, since all epistemic accessibility relations are equivalence relations, Factivity, Positive Introspection, and Negative Introspection are valid for both players.

Now consider node (G1, E1). $\mathcal{C}\mathsf{ra}$ is not true at this epistemic state because if it were, ra would have to be true at every epistemic state reachable from here, but ra_A is not true at the reachable node (G1, E3). On the other hand, $E\mathsf{ra}$ is true at (G1, E1)—this abbreviates a formula equivalent to $K_A \mathsf{ra}_A \wedge K_A \mathsf{ra}_B \wedge K_B \mathsf{ra}_A \wedge K_B \mathsf{ra}_B$ and expresses that everybody knows that all players are rational. At (G1, E1) ra_A and ra_B are true, and hence so are $K_B \mathsf{ra}_A$ and $K_B \mathsf{ra}_B$. Likewise ra_A and ra_B are true at both (G1, E1) and (G1, E2), and hence $K_A \mathsf{ra}_A$ and $K_A \mathsf{ra}_B$ are true at (G1, E1). Since do is not true at (G1, E1), this establishes that *$E\mathsf{ra} \supset \mathsf{do}$ is not derivable from our axioms for Centipede*, together with additional strong knowledge assumptions.

6. Considering Irrationality

We conclude with one more Centipede-based example, in which the game tree is the same but the epistemic assumptions are substantially different They are asymmetric—the two players are not interchangeable, so to speak.

What does it mean to be rational? We have taken an operational view here, though the wording sometimes obscures this. Axioms RC-A and RC-B are necessary conditions for a *play* to be rational—a rational move must be in accordance with what is most advantageous to the player, within the limits of what the player knows at the moment of play. Rationality is not treated as a predisposition or psychological state; rather it is moves of the player that are rational. A player is considered rational if the player only makes rational moves. Centipede game axioms $\mathsf{RCcent\text{-}A}$ and $\mathsf{RCcent\text{-}B}$, when combined with axioms $\mathsf{RPcent\text{-}A}$ and $\mathsf{RPcent\text{-}B}$, amount to rationality assumptions about *players*—there will be rational play throughout.

We now consider *irrational play*—play that is not in the best interests of the player. In investigating strategy, rationality or irrationality is not what actually matters. Rather it is what players *know* about these things that counts. For this there are three, not two, possibilities. A player might know another player is rational, or he might know another player is irrational, or he might not know either the rationality or the irrationality of the other player. We do not investigate this third possibility here—the state of ignorance concerning rationality. We only examine the consequences of knowing a player will act irrationally.

We add two new axioms, counterparts of $\mathsf{RC\text{-}A}$ and $\mathsf{RC\text{-}B}$. For each move m_i of a game, we assume the following *irrationality conditions.*

$\mathsf{IC\text{-}A}$ $(A \wedge K_A \mathsf{kbest}_A(\mathsf{m}_i) \wedge \neg \mathsf{ra}_A) \supset \neg \mathsf{m}_i$

$\mathsf{IC\text{-}B}$ $(B \wedge K_B \mathsf{kbest}_A(\mathsf{m}_i) \wedge \neg \mathsf{ra}_B) \supset \neg \mathsf{m}_i$

A player plays irrationally if the player does not choose the known best move. These axioms, combined with $\mathsf{RC\text{-}A}$ and $\mathsf{RC\text{-}B}$, give us simple equivalences.

$$(A \wedge K_A \mathsf{kbest}_A(\mathsf{m}_i)) \supset (\mathsf{ra}_A \equiv \mathsf{m}_i)$$
$$(B \wedge K_B \mathsf{kbest}_A(\mathsf{m}_i)) \supset (\mathsf{ra}_B \equiv \mathsf{m}_i)$$

6.1. Centipede Again

To make the discussion concrete we now examine the Centipede game under the assumption that one of the players is irrational—that is, always makes irrational moves. Figure 2 displayed a five move version of Centipede. Now a bigger game is more illustrative, so a nine move version is shown in Figure 7.

For the game tree of Figure 7 our epistemic conditions are these: assume the terminal node player, A, plays irrationally while the other player plays rationally. In the figure we have displayed information about which player is to move at each game state, and we have numbered these states. The numbering starts from the end, rather than from the beginning as is more usual. This simplifies the discussion.

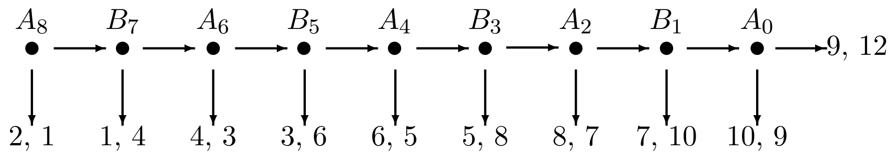


Figure 7. Nine Move Centipede

One can reason informally as follows. If the game were to start at state 0 the best move for A would be down, but since A plays irrationally, A would play right. At state 1, B should be able to duplicate our reasoning and so would know the best move for it would be right and, being rational, B would move right at state 1. At state 2, A can also reason as we just did, and so would know that B would move right at state 1, and so would know that its best move is right. But playing irrationally, A would move down at state 2. And so on. Informally, one can show (for games of arbitrary finite length) moves are right at 0, 4, 8, ..., also at 1, 5, 9, ..., while moves are down at 2, 6, 10, ..., and at 3, 7, 11, ... The task is to formalize this informal discussion.

An inspection of the informal reasoning above shows that moves repeat repeat in patterns of 4. We take R^4 to abbreviate $R;R;R;R$, and R^{4*} for $(R^4)^*$. Moves should be right at nodes 0, 4, 8, ... Node 0 is terminal, so at it $[R]\bot$ is true. At node 4 we have $\langle R^4\rangle[R]\bot$, and so on. In brief, the move should be right at game nodes where $\langle R^*\rangle[R]\bot$ is true. Similarly for the other patterns. This leads us to the following abbreviations.

$$\begin{aligned}
F_0 &= \langle R^{4*}\rangle[R]\bot \supset \mathsf{ri}\\
F_1 &= \langle R\rangle\langle R^{4*}\rangle[R]\bot \supset \mathsf{ri}\\
F_2 &= \langle R;R\rangle\langle R^{4*}\rangle[R]\bot \supset \mathsf{do}\\
F_3 &= \langle R;R;R\rangle\langle R^{4*}\rangle[R]\bot \supset \mathsf{do}\\
F &= F_0 \wedge F_1 \wedge F_2 \wedge F_3\\
\mathsf{ra}' &= \neg\mathsf{ra}_A \wedge \mathsf{ra}_B
\end{aligned}$$

Our goal is to show formally that there is a proof of the following, analogous to our earlier Theorem 4.5.

$$\mathcal{C}\mathsf{ra}' \supset \mathcal{C}F$$

6.2. Beginning Formalization

Axioms RPcent-A and RPcent-B assert that rationality persists in Centipede. We now say a similar thing about irrationality, thus positing that it is a *player* who is irrational because he always makes irrational, moves.

IPcent-A $\neg \mathsf{ra}_A \supset [R]\neg \mathsf{ra}_A$

IPcent-B $\neg \mathsf{ra}_B \supset [R]\neg \mathsf{ra}_B$

The irrationality conditions IC-A and IC-B specialize to Centipede as follows, taking KGcent-7 and KGcent-8 into account.

ICcent-A $(A \wedge K_A \mathsf{kbest}_A(\mathsf{do}) \wedge \neg \mathsf{ra}_A) \supset \mathsf{ri}$
$(A \wedge K_A \mathsf{kbest}_A(\mathsf{ri}) \wedge \neg \mathsf{ra}_A) \supset \mathsf{do}$

ICcent-B $(B \wedge K_B \mathsf{kbest}_B(\mathsf{do}) \wedge \neg \mathsf{ra}_B) \supset \mathsf{ri}$
$(B \wedge K_B \mathsf{kbest}_B(\mathsf{ri}) \wedge \neg \mathsf{ra}_B) \supset \mathsf{do}$

We assume A plays last, that is, has the turn at the terminal node.

Last-A $[R]\bot \supset A$

6.3. Additional Structural Assumptions

The Section 4.2 axioms embody structural information about the Centipede game. We now propose two further semantic assumptions, which lead to a formula we can use to complete our discussion of Centipede with an irrational player. Both of these assumptions hold in the diagram of Figure 6.

The game tree for Centipede is obviously linear, so each move to the right takes us from a game state to a *unique* next game state. The game is one of perfect information, so there is no ambiguity as to which state we are in. This should be reflected from the game tree to *augmented* game trees for Centipede, where epistemic states within game states are shown.

Linearity Assumption: Suppose there is an R transition from an epistemic state e in Centipede game state g_1 to some epistemic state in game state g_2. Then every R transition from e must be to an epistemic state of g_2.

Different epistemic states in the same game state should be capable of affecting each other. There should be no isolated epistemic states. This is a useful proposal, but we admit it needs more thought and exploration.

Reachability Assumption: Suppose g is a Centipede game state and e_1 and e_2 are two different epistemic states of g. Then e_2 is reachable from e_1 via a path of epistemic states in which each is accessible from its predecessor via the accessibility relation for player A or for player B.

We have characterized terminal nodes by the truth of $[R]\bot$, but truth where? A game state g might contain many epistemic states—could $[R]\bot$ be true at some while false at others. The Reachability Assumption rules this out. Suppose some epistemic state e of g has $[R]\bot$ true at it. It follows from KGcent-11 and KGcent-12 that $[R]\bot$ is common knowledge (belief) at e, and then CK-1 implies $[R]\bot$ will be true at every other epistemic state of g.

Assume the Linearity Assumption and the Reachability Assumption. Suppose $\langle R\rangle\mathcal{C}X$ is true at epistemic state e of game state g_1. There must be an R transition to an epistemic state of some game node g_2, at which $\mathcal{C}X$ is true. Using the Reachability Assumption and CK-1, $\mathcal{C}X$ must be true at every epistemic state of g_2. Using the Linearity Assumption, every R transition from e must be to an epistemic state of g_2, where $\mathcal{C}X$ is true. It follows that $[R]\mathcal{C}X$ is true at e. Thus our final Centipede axiom.

Lin + Reach $\langle R\rangle\mathcal{C}X \supset [R]\mathcal{C}X$

Now we can formally prove the following. The argument is rather long, but can be found in full in [11]. Once again, backward induction is involved.

THEOREM 6.1. *Assume No Learning, PDLE-1, Reasoning Ability, PDLE-3, and Lin + Reach. Then* $\mathcal{C}\mathsf{ra}' \supset \mathcal{C}F$.

7. Conclusion

The Centipede analysis strongly suggests that PDL + E is a natural tool for reasoning about games in which an epistemic component is central, a point also made by [18]. PDL + E provides machinery to establish derivability, and non-derivability, of various statements—there is both a proof theory and a model theory. Nonetheless, more remains to be done.

Completeness was not essential here since we have been concerned with specific games. Still, for example, could one characterize games of perfect information? We need examples of PDL+E applied to other games. Assumptions MScent-A, MScent-B, EScent-A, EScent-B, and Lin + Reach would need to be replaced with other conditions , and this is partly why completeness is a side issue. Similarly, what is the general applicability of the interaction conditions, No Learning, PDLE-1, Perfect Recall, PDLE-2, and Reasoning Ability, or PDLE-3? Also what, exactly, is the status of Extended Perfect Recall, CK-3? Finally, *Justification Logic*, [2, 3], has become an important field. These are epistemic logics in which one can track reasoning—one knows for explicit reasons. The cross axioms of Section 2.3 suggest there may be operations on justifications that have not been investigated yet.

This material is based upon work supported by the National Science Foundation under Grant No. 0830450.

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