Proving Completeness for Nested Sequent Calculi[1]

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ABSTRACT. Proving the completeness of classical propositional logic by using maximal consistent sets is perhaps the most common method there is, going back to Lindenbaum (though not actually published by him). It has been extended to a variety of logical formalisms, sometimes combined with the addition of Henkin constants to handle quantifiers. Recently a deep-reasoning formalism called *nested sequents* has been introduced by Kai Brünnler, able to handle a larger variety of modal logics than are possible with standard Gentzen-type sequent calculi. In this paper we sketch how yet another variation on the maximality method of Lindenbaum allows one to prove completeness for nested sequent calculi. It is certainly not the only method available, but it should be entered into the record as one more useful tool available to the logician.

1 Introduction

Recently Kai Brünnler introduced a version of deep reasoning called *nested sequents*, [1], with systems for a number of common modal logics including some that lack cut-free sequent calculi in the ordinary sense. Proving completeness for such systems is most commonly done by what might be called a *systematic backward proof search.* One starts with the desired goal and works backward until either a proof is found or enough information is produced to generate a counter-model. In [3] it was shown that there are close connections between nested sequents and prefixed tableaus, and this allows one to transfer completeness results from prefixed tableau calculi to nested sequent systems. In addition, syntactic proof of cut-elimination can be given, see [1], and this allows transferal of completeness results from axiomatic formulations as well.

In this paper we sketch another approach to proving completeness for nested sequent systems of modal logic. It makes use of a suitably generalized maximal consistency construction, combined with a version of the introduction of Henkin witnesses as in first-order completeness arguments. This is not really a new approach. It has been applied to standard sequent calculi and to tableau systems, including prefixed ones. For nested sequents the form the construction takes is somewhat peculiar. A maximality requirement must be met by every nested subsequent, and Henkin witnesses are themselves nested sequents. We believe the construction, while fairly simple, has interest so we are presenting it to the logic community.

[1]This paper is dedicated to Walter Carnielli on the occasion of his sixtieth birthday. Thank you Walter, and I wish you many more years and theorems.

We begin with a discussion about nested sequent calculi for the logic K, with some minor modifications to the original formulation. Then we give our completeness proof for it. Finally we make some brief remarks about how the work can be extended to other modal logics.

2 Formulas and Uniform Notation

Formuas are built up from atomic formulas, propositional letters, P, Q, ... They are built up using propositional connectives $\wedge$, $\vee$, $\neg$, $\supset$, and modal operators $\Box$ and $\Diamond$, in the usual way.

We use the common grouping of formulas into classes that behave alike—α, β, ν, π. It is often referred to as *uniform notation*. Compound formulas and their negations are grouped into those that behave conjunctively, α formulas, and those that behave disjunctively, β formulas. For each, *components* are defined, α_1 and α_2 for α formulas, and β_1 and β_2 for β formulas. These are given in the following tables.

α	α_1	α_2
$X \wedge Y$	X	Y
$\neg(X \vee Y)$	$\neg X$	$\neg Y$
$\neg(X \supset Y)$	X	$\neg Y$

β	β_1	β_2
$X \vee Y$	X	Y
$\neg(X \wedge Y)$	$\neg X$	$\neg Y$
$X \supset Y$	$\neg X$	Y

In a similar way there are the necessary formulas, ν, and possible formulas, π, and their components. These are given in the following tables.

ν	ν_0
$\Box X$	X
$\neg\Diamond X$	$\neg X$

π	π_0
$\Diamond X$	X
$\neg\Box X$	$\neg X$

3 Nested Sequents

We sketch a modal nested sequent system for the logic K. There are some changes from the original formulation in [1], making our work somewhat easier. The changes are as follows. We do not assume formulas are in negation normal form. The paper [1] works with multisets, assuming contraction and weakening rules. We use sets in place of multisets, and drop all structural rules. We do not allow the empty sequent. And finally, we make use of uniform notation, α, β, ν, and π. The definition of nested sequent that we use is a recursive one.

DEFINITION 1. A *nested sequent* is a non-empty finite set of formulas and nested sequents.

Nested sequents generalize *Tait* or *one-sided* sequents. All formulas have been moved to the right sides of arrows, and the arrows deleted. In effect, they are disjunctions. Nested sequents for modal logic iterate this idea; nesting corresponds to necessitation. More formally, let $\Gamma = \{X_1, \ldots, X_n, \Delta_1, \ldots, \Delta_k\}$ be a nested sequent, where each X_i is a formula and each Δ_j is a nested sequent. Then one defines a translation to ordinary formulas; think of $\Gamma^\dagger$ as the 'meaning' of Γ.

$$\Gamma^\dagger = X_1 \vee \ldots \vee X_n \vee \Box\Delta_1^\dagger \vee \ldots \vee \Box\Delta_k^\dagger$$

Axioms $\Gamma(A, \neg A)$,

A a propositional letter

Double Negation Rule $\dfrac{\Gamma(X)}{\Gamma(\neg\neg X)}$

α **Rule** $\dfrac{\Gamma(\alpha_1) \qquad \Gamma(\alpha_2)}{\Gamma(\alpha)}$

β **Rule** $\dfrac{\Gamma(\beta_1, \beta_2)}{\Gamma(\beta)}$

ν **Rule** $\dfrac{\Gamma([\nu_0])}{\Gamma(\nu)}$

π **Rule** $\dfrac{\Gamma(\pi, [\pi_0, \ldots])}{\Gamma(\pi, [\ldots])}$

Figure 1. Nested Sequent Rules for K

A proof of the soundness of nested sequent systems can be based on this translation. In this present brief paper we omit discussion of soundness issues.

There are standard notational conventions for nested sequents. Enclosing outer set brackets are often omitted. A nested sequent that is a member of another nested sequent has its members listed in square brackets, and is called a *boxed sequent*. For example, $A, B, [C, [D, E], [F, G]]$, is the conventional way of writing $\{A, B, \{C, \{D, E\}, \{F, G\}\}\}$. For this, the 'meaning' defined above is $A \vee B \vee \Box(C \vee \Box(D \vee E) \vee \Box(F \vee G))$. We systematically use Γ, Δ, … for nested sequents, boxed or top level.

DEFINITION 2. *Subsequents* are defined as follows.

1. Γ is a subsequent of Γ.

2. If $\Delta \in \Gamma$, any subsequent of Δ is a subsequent of Γ.

Suppose Γ is a nested sequent in which propositional letter P occurs once—we write $\Gamma(P)$ for this. Subsequently, $\Gamma(X)$ is the result of replacing P in Γ with X. Similarly for $\Gamma(X, Y)$, $\Gamma(\Delta)$, and so on. Using this convention, Figure 1 displays the nested sequent rules for K, from [1] (extended to allow arbitrary formulas and not just those in negation normal form). Assume $\Gamma(P)$ is some nested sequent with one occurrence of propositional letter P, implicit behind the formulation of the rules displayed. Also we use $[\ldots]$ to stand for a *non-empty* nested sequent, and $[Z, \ldots]$ is the same sequent but with Z added.

Sequent proofs start with axioms and end with the nested sequent being proved. Proof of a formula is a derivative notion: a proof of the nested sequent consisting of just the formula X is taken to be a proof of X itself. Figure 2 contains an example of a nested sequent K proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Diamond(P \vee R), [\neg P, P, R], [\neg Q]}{\Diamond(P \vee R), [\neg P, P \vee R], [\neg Q]}}{\Diamond(P \vee R), [\neg P], [\neg Q]} \qquad \dfrac{\dfrac{\Diamond(Q \vee S), [\neg P], [\neg Q, Q, S]}{\Diamond(Q \vee S), [\neg P], [\neg Q, Q \vee S]}}{\Diamond(Q \vee S), [\neg P], [\neg Q]}}{\Diamond(P \vee R) \wedge \Diamond(Q \vee S), [\neg P], [\neg Q]}}{\Diamond(P \vee R) \wedge \Diamond(Q \vee S), \neg\Diamond Q, [\neg P]}}{\Diamond(P \vee R) \wedge \Diamond(Q \vee S), \neg\Diamond P, \neg\Diamond Q}}{\neg(\Diamond P \wedge \Diamond Q), \Diamond(P \vee R) \wedge \Diamond(Q \vee S)}}{(\Diamond P \wedge \Diamond Q) \supset (\Diamond(P \vee R) \wedge \Diamond(Q \vee S))}$$

Figure 2. K Proof Example

4 Dual-Consistency

When proving propositional completeness axiomatically, a set $\{X_1, \ldots, X_n\}$ is called consistent if it is not the case that $\neg X_1 \wedge \ldots \wedge \neg X_n$ is provable. One shows that a consistent set extends to a maximal consistent one. Then if one calls a member of a maximal consistent set true, and a non-member false, it is verified that this is a standard truth-functional assignment. As it happens, the introduction of negation is complex to deal with when nested sequents are being used. We avoid the problem by dualizing everything, both at the beginning of the argument and at the end. The end game will be discussed in Section 6 but anticipating, we will call members (never mind members of what for now) false instead of true. Here at the beginning we make use of the following, which does not bring negation into the picture.

DEFINITION 3. Call a nested sequent Γ *dual-consistent* if Γ is not provable in the sequent calculus for K given in Figure 1 of Section 3.

In the usual treatments of logic, subsets of consistent sets are consistent. Here is the analog of this for the present setting.

PROPOSITION 4. *Suppose Δ is a subsequent of $\Gamma(\Delta)$, sequent Δ^* is such that $\Delta \subseteq \Delta^*$, and $\Gamma(\Delta^*)$ is like $\Gamma(\Delta)$ but with Δ replaced by Δ^*. If $\Gamma(\Delta^*)$ is dual-consistent, so is $\Gamma(\Delta)$.*

Proof. We just sketch the basic idea. Suppose $\Gamma(\Delta)$ is not dual-consistent, that is, $\Gamma(\Delta)$ is provable. In the final line of the proof of $\Gamma(\Delta)$, enlarge Δ to Δ^* getting $\Gamma(\Delta^*)$, and then 'propagate upward' throughout the proof the addition of formulas and nested sequents that effected this enlargement. Rule applications remain rule applications, and axioms remain axioms. This converts the proof of $\Gamma(\Delta)$ into one for $\Gamma(\Delta^*)$, so $\Gamma(\Delta^*)$ is not dual-consistent. ■

We also need a version of maximality, and we need it relativized it to a specified set of formulas. What is maximal now is a subsequent, in a sequent. Note that formulas *and their negations* are taken into account. The following terminology addresses this.

DEFINITION 5. For a set S of formulas, we say a formula X is S *determinate* if X belongs to S or is the negation of a formula belonging to S.

LEMMA 6. *Suppose S is a set of formulas that is closed under subformulas. If α is S determinate so are α_1 and α_2; if β is S determinate so are β_1 and β_2; if $\neg\neg X$ is S determinate so is X.*

Proof. We show the result for α formulas—the argument is similar in the β case and simpler for double negations.

Suppose first that $\alpha \in S$. There are three cases to consider. (1) If $\alpha = X \wedge Y$ then $X \in S$ and $Y \in S$ because of closure under subformulas, that is, $\alpha_1 \in S$ and $\alpha_2 \in S$, so these are S determinate. (2) If $\alpha = \neg(X \vee Y)$ then $X \in S$ and $Y \in S$, again because of subformula closure. Then $\alpha_1 = \neg X$ is S determinate because it is the negation of a member of S, and similarly for α_2. (3) $\alpha = \neg(X \supset Y)$. This case is a mixture of (1) and (2).

Next suppose that α is the negation of a member of S. Then α cannot be $X \wedge Y$, so there are only two cases to consider. (1) $\alpha = \neg(X \vee Y)$ where $X \vee Y \in S$. Then both X and Y are in S by subformula closure, so $\alpha_1 = \neg X$ and $\alpha_2 = \neg Y$ are S determinate since they are negations of members of S. (2) $\alpha = \neg(X \supset Y)$. This is similar to case (1). ■

Now we have the central notion of maximal dual-consistency.

DEFINITION 7. Let $\Gamma(\Delta)$ be a nested sequent that is dual-consistent, with Δ as a subsequent. Also let S be a set of formulas that is closed under subformulas. We say Δ is *maximal* in $\Gamma(\Delta)$ with respect to S provided, for each X that is S determinate, if $X \notin \Delta$, the addition of X to Δ makes $\Gamma(\Delta)$ dual-inconsistent.

THE LINDENBAUM CONSTRUCTION: Let $\Gamma(\Delta)$ be a dual-consistent sequent with a subsequent Δ, and let S be a *finite* subformula closed set. It is straightforward to extend Δ to Δ^* so that Δ^* is maximal in $\Gamma(\Delta^*)$ with respect to S. A standard Lindenbaum-style construction will do—we omit the details. We restrict S to being finite because otherwise we would be forced to deal with infinite nested sequents, and we would rather not do so.

Maximality has something like the usual properties one expects from axiomatic completeness proofs, but dualized. For example in the α case below one might think the conclusion should involve "and" since α formulas are conjunctive, but in fact the case involves "or."

PROPOSITION 8. *Suppose S is a finite set of formulas that is closed under subformulas. Let $\Gamma(\Delta)$ be dual-consistent, with a subsequent Δ that is maximal with respect to the set S.*

1. *If A is atomic, not both A and $\neg A$ are in Δ.*
2. *If $\neg\neg X$ is S determinate and $\neg\neg X \in \Delta$ then $X \in \Delta$.*
3. *If α is S determinate and $\alpha \in \Delta$ then $\alpha_1 \in \Delta$ or $\alpha_2 \in \Delta$.*
4. *If β is S determinate and $\beta \in \Delta$ then $\beta_1 \in \Delta$ and $\beta_2 \in \Delta$.*

Proof. The cases are as follows.

1. If both A and $\neg A$ are in Δ, then $\Gamma(\Delta)$ is an axiom and hence provable.

2. Assume that $\neg\neg X$ is S determinate, and hence so is X by Lemma 6. Now we proceed contrapositively. Suppose $X \notin \Delta$. Using maximality, $\Gamma(\Delta \cup \{X\})$ is not dual-consistent, and hence is provable. It follows by the Double Negation Rule that $\Gamma(\Delta \cup \{\neg\neg X\})$ is also provable, and thus is not dual-consistent. Since $\Gamma(\Delta)$ is dual-consistent, it follows that $\neg\neg X \notin \Delta$.

3. Assume that α is S determinate, and hence both α_1 and α_2 also are by Lemma 6. Again the argument is contrapositive. Suppose $\alpha_1 \notin \Delta$. Maximality implies $\Gamma(\Delta \cup \{\alpha_1\})$ is not dual-consistent, and hence is provable. Similarly suppose $\alpha_2 \notin \Delta$; then $\Gamma(\Delta \cup \{\alpha_2\})$ is provable. Using the α Rule, $\Gamma(\Delta \cup \{\alpha\})$ is provable and so not dual-consistent. It follows that $\alpha \notin \Delta$.

4. Similar to the preceding case.

■

5 Henkin Witnesses

Dual-consistency and maximality will allow us to take care of propositional connectives. We still need machinery for the modal operators. What is presented in this section is an analog of a familiar step in standard axiomatic completeness proofs in modal logic.

HENKIN-LINDENBAUM EXPANSION: Let $\Gamma(\Delta)$ be a dual-consistent nested sequent with Δ as a subsequent. We define a *Henkin-Lindenbaum expansion* of Δ in $\Gamma(\Delta)$ to be a nested sequent $\Gamma(\Delta^*)$, where Δ^* is constructed in the following way.

1. First some definitions. Suppose $\nu \in \Delta$. The *Henkin witness for ν in Δ* is the nested sequent $[\nu_0, \pi_0^1, \ldots, \pi_0^n]$ where $\pi^1, \ldots, \pi^n$ are all the π formulas in Δ. A *Henkin witness* in Δ is a Henkin witness for ν in Δ, for some ν. We say $[\nu_0, \pi_0^1, \ldots, \pi_0^n]$ is *new to* Δ if it is not a subset of any nested sequent that is a member of Δ. We call the set S of all subformulas of $\nu_0, \pi_0^1, \ldots, \pi_0^n$ the *foundation set* of $[\nu_0, \pi_0^1, \ldots, \pi_0^n]$.
   Let Δ' be the result of adding to Δ all Henkin witnesses that are new to Δ. $\Gamma(\Delta')$ is also dual-consistent (shown below in Lemma 10).
2. For each Henkin witness $[\nu_0, \pi_0^1, \ldots, \pi_0^n]$ that was added to Δ to get Δ', enlarge it to a subsequent that is maximal in Γ with respect to its foundation set, using the Lindenbaum construction as sketched in Section 4. Let Δ^* be the result of thus expanding every Henkin witness in Δ'.
3. The outcome is a dual-consistent nested sequent, $\Gamma(\Delta^*)$ where $\Delta \subseteq \Delta^*$, Δ and Δ^* contain the same formulas, but Δ^* also contains, for each Henkin witness that is new to Δ, a nested sequent extending that Henkin witness, maximally dual-consistent in Γ with respect to the foundation set of that Henkin witness.

EXAMPLE 9. Suppose $\Delta = [\Box B, \Box C, \Diamond D, \Diamond E, F]$ and $\Gamma(\Delta) = A, [\Box B, \Box C, \Diamond D, \Diamond E, F]$. Then Δ has two Henkin witnesses, $[B, D, E]$ and $[C, D, E]$, in Γ. The outcome of step 1 above is the nested sequent $A, [\Box B, \Box C, \Diamond D, \Diamond E, F, [B, D, E], [C, D, E]]$. Then the outcome of step 2 is the nested sequent $A, [\Box B, \Box C, \Diamond D, \Diamond E, F, [B, D, E]^*, [C, D, E]^*]$ where $[B, D, E]^*$ is some extension of $[B, D, E]$ that is maximally dual-consistent in Γ with respect to the set of all subformulas of B, D, E, and similarly for $[C, D, E]^*$.

In step 1 of the Henkin-Lindenbaum Expansion process we said certain nested sequents would be dual-consistent. We now show this.

LEMMA 10. *Suppose* $\Gamma(\Delta)$ *is dual-consistent and* Δ' *is the result of adding to* Δ *all Henkin witnesses that are new to* Δ*. Then* $\Gamma(\Delta')$ *is also dual-consistent.*

Proof. We show the result of adding a single Henkin witness preserves dual-consistency. The Proposition then follows by iterating this.

Assume that $\nu \in \Delta$, and $\pi^1, \ldots, \pi^n$ are all the π formulas in Δ. Suppose $\Gamma(\Delta \cup \{[\nu_0, \pi^1_0, \ldots, \pi^n_0]\})$ were not dual-consistent. Then it would be provable. By repeated application of the π Rule, $\Gamma(\Delta \cup \{[\nu_0]\})$ would be provable. Then by the ν Rule, $\Gamma(\Delta)$ would be provable, and hence not dual-consistent. It follows that if $\Gamma(\Delta)$ is dual-consistent, so is the result of adding one Henkin witness to Δ. ■

6 Completeness

With the basic work out of the way, we can now prove completeness itself. Let X be a formula that is fixed for the rest of this section, and suppose X is not provable in the nested sequent system for K. Using the unprovability of X we describe a process for generating a sequence of more and more elaborate nested sequents $\Gamma_1, \Gamma_2, \ldots$. Every subsequent of each Γ_i will have an associated foundation set, as specified in the Henkin-Lindenbaum Expansion of Section 5, and will be maximally dual-consistent in Γ_i with respect to its foundation set. As a bookkeeping device, certain subsequents of each Γ_i will be marked as *finished*.

Since X is not provable, the nested sequent $\{X\}$ is dual-consistent. This sequent is treated a little differently from later ones in the process since it is not a subsequent of anything else. Take the set of all subformulas of X as the foundation set for $\{X\}$, enlarge $\{X\}$ to a set that is maximally dual-consistent, with respect to its foundation set, and call the result Γ_1.

Next suppose Γ_n has been defined, every subsequent of it has an associated foundation set and is maximally dual-consistent with respect to it. If every subsequent of Γ_n is marked as finished, the construction stops. Otherwise choose a subsequent, Δ, that is not marked as finished, and consider $\Gamma_n(\Delta)$. Let $\Gamma_n(\Delta^*)$ be a Henkin-Lindenbaum expansion of Δ in $\Gamma_n(\Delta)$, as specified in Section 5, and set $\Gamma_{n+1} = \Gamma_n(\Delta^*)$. Mark Δ^* itself as finished in Γ_{n+1}, and also mark as finished any subsequents that were carried over unchanged from Γ_n and were marked as finished there. Note that if $\Gamma_n = \Gamma_1$, it is its only unfinished subsequent. The process just described still applies, but Γ_1 is, in effect, in an empty context.

This process must stop after a finite number of steps, for the following reason. The modal degree of each member of a Henkin witness in Δ must be less (by 1) than the modal degree of some member of Δ. Conseqently, if $\Gamma_{n+1} = \Gamma_n(\Delta^*)$, the maximal modal degrees of the foundation sets of members of Δ^* must be less than the maximal modal degree of the foundation set of Δ^* itself. When modal degrees reach 0 the process stops. Let us say Γ_∞ is the final member of the sequence just described.

Now we construct a a Kripke model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ from Γ_∞, as follows.

Let the set of possible worlds, $\mathcal{G}$, be the collection of all subsequents of Γ_∞.

Next we specify the accessibility relation, $\mathcal{R}$. For $\Delta, \Omega \in \mathcal{G}$, let $\Delta \mathcal{R} \Omega$ provided $\Omega \in \Delta$.

Finally we have the truth-at-a-world relation. For an atomic formula A, set $\mathcal{M}, \Delta \Vdash A$ just when $A \notin \Delta$. Note that this condition is dual to the one usually seen in completeness proofs.

We now have a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$. For it we have a kind of *dual truth lemma*, Proposition 12. In proving this we make use of structural induction, but it is not quite the usual version based on complexity of formulas as measured by degree. Instead the version we use is the following—it is reasonably intuitive, and a formal proof that it works can be found in [2], as Theorem 2.6.3.

PROPOSITION 11. *Every formula has property **Q** provided:*

1. *every atomic formula and its negation has property **Q**;*
2. *if X has property **Q** so does $\neg\neg X$;*
3. *if α_1 and α_2 have property **Q** so does α;*
4. *if β_1 and β_2 have property **Q** so does β;*
5. *if ν_0 has property **Q** so does ν;*
6. *if π_0 has property **Q** so does π.*

Now here is our *dual truth lemma*, proved using Proposition 11.

PROPOSITION 12. *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ be the model constructed above, let $\Delta \in \mathcal{G}$, and let S be the foundation set of Δ. For each formula Z that is S determinate: $Z \in \Delta \Longrightarrow \mathcal{M}, \Delta \nVdash Z$;*

Proof. By induction on complexity we show that, if Z is S determinate where S is the foundation set of Δ, and $Z \in \Delta$, then $\mathcal{M}, \Delta \nVdash Z$. There are several cases.

Atomic Case: Suppose A is atomic and $A \in \Delta$. Then $\mathcal{M}, \Delta \nVdash A$ by definition of the model.

Negated Atomic Case: Suppose A is atomic and $\neg A \in \Delta$. Then $A \notin \Delta$ by Proposition 8 part 1, so $\mathcal{M}, \Delta \Vdash A$, again by definition of the model, and hence $\mathcal{M}, \Delta \nVdash \neg A$.

Double Negation Case: Suppose the result is known for Z, $\neg\neg Z$ is S determinate where S is the foundation set of Δ, and $\neg\neg Z \in \Delta$. By Lemma 6, Z is S determinate, and by Proposition 8 part 2, $Z \in \Delta$. Then by the induction hypothesis, $\mathcal{M}, \Delta \not\Vdash Z$. It follows that $\mathcal{M}, \Delta \not\Vdash \neg\neg Z$.

α *Case:* Suppose α is S determinate where S is the foundation set of Δ, the result is known for α_1 and α_2, and $\alpha \in \Delta$. By Lemma 6 both α_1 and α_2 are S determinate, and $\alpha_1 \in \Delta$ or $\alpha_2 \in \Delta$, by Proposition 8 part 3. By the induction hypothesis, $\mathcal{M}, \Delta \not\Vdash \alpha_1$ or $\mathcal{M}, \Delta \not\Vdash \alpha_2$. In either case, $\mathcal{M}, \Delta \not\Vdash \alpha$.

β *Case:* Similar to the previous case.

ν *Case:* Suppose ν is S determinate where S is the foundation set of Δ, the result is known for ν_0, and $\nu \in \Delta$. In the process of constructing Γ_∞, at some point a Henkin witness in Δ, $[\nu_0, \pi_0^1, \ldots, \pi_0^n]$ has been expanded to produce a member, Ω of Δ. Since ν_0 must be in the foundation set of Ω, by the induction hypothesis, $\mathcal{M}, \Omega \not\Vdash \nu_0$. And since $\Omega \in \Delta$, $\Delta \mathcal{R} \Omega$. Then $\mathcal{M}, \Delta \not\Vdash \nu$.

π *Case:* Suppose π is S determinate where S is the foundation set of Δ, the result is known for π_0, and $\pi \in \Delta$. In this case π_0 must be in every Henkin witness in Δ. It follows that π_0 must belong to every subsequent that is a member of Δ, and must be a member of its foundation set. Then by the induction hypothesis, $\mathcal{M}, \Omega \not\Vdash \pi_0$ for every $\Omega \in \mathcal{G}$ with $\Delta \mathcal{R} \Omega$, and so $\mathcal{M}, \Delta \not\Vdash \pi$.

■

Since the construction of Γ_∞ begins with a sequent containing X, then X must be a member of Γ_∞ itself. By the Proposition above, $\mathcal{M}, \Gamma_\infty \not\Vdash X$, and so X is not K-valid.

7 Other Modal Logics

There are extensions of the nested sequent system described in Section 3 for many standard modal logics, see [1; 3]. We do not state the rules here. Completeness for K is easiest to establish, with that for T and D a close second. Things become harder when transitivity is involved because the construction process described in Section 6, when appropriately adapted to these logics, need not terminate. There are two solutions, at least, for this problem.

First, we can simply accept the fact that the construction process goes on forever. Then we need to define an appropriate notion of limit for the sequence $\Gamma_1, \Gamma_2, \ldots$, and this is not difficult. Conceptually the limit would be a nested sequent that allowed infinitely deep nesting. This is not 'legal' given the way we have defined nested sequents, as sets, because it violates well-foundedness. It is, however, an intuitively plausible thing, and the notion of direct limit, from category theory, is a formal substitute. We then carry out the construction of a model using this limit, instead of using the last term of the sequence as we did above.

Second, since all formulas are subformulas of the formula we are trying to prove, if a construction goes on forever there must be repetition. One can terminate work on a subsequent when it duplicates one of its 'ancestors.' In this way work can be forced to halt, as it did for K, after a finite number of steps. Unfortunately, this method won't extend to admit quantifiers, though the one with limits, described above, can be made to work.

We do not go into details of these more complex constructions here. In this paper we merely wanted to show how the maximal consistent set construction, familiar from classical propositional logic, could be extended to nested sequent calculi, and enough has now been said to give the general idea.

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